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Laplacian Spectral Radius and *k*-Connected Graphs

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Abstract. Using Laplacian spectral radius, we in this note present a sufficient condition for a graph to be k-connected.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. We use $\delta = d_1 \leq d_2 \leq, \ldots, \leq d_n = \Delta$ to denote the degree sequence of a graph. The eigenvalues of a graph G are defined as the eigenvalues of its adjacency matrix $A(G)$. The largest eigenvalue of a graph G is called the spectral radius of G. The Laplacian eigenvalues of a graph G are defined as the eigenvalues of the matrix $L(G) := D(G) - A(G)$, where $D(G)$ is the diagonal matrix $diag(d_1, d_2, \ldots, d_n)$ and $A(G)$ is the adjacency matrix of G. The largest Laplacian eigenvalue of a graph G, denoted $\mu(G)$, is called the Laplacian spectral radius of G. The signless Laplacian eigenvalues of a graph G are defined as the eigenvalues of the matrix $Q(G) := D(G) + A(G)$, where $D(G)$ is the diagonal matrix $diag(d_1, d_2, \ldots, d_n)$ and $A(G)$ is the adjacency matrix of G. The largest signless Laplacian eigenvalue of a graph G is called the signless Laplacian spectral radius of G.

In [3], Li obtained spectral conditions which are based on the spectral radius or the signless Laplacian spectral radius for a graph to be k-connected. Using similar ideas as the ones in [3], we will present a sufficient condition which is based on Laplacian spectral radius for a graph to be k - connected. The result is as follows.

Theorem 1. Let G be a connected graph of order $n \geq 2$ and let $1 \leq k \leq n - 1$. If

$$\mu > \frac{(2\delta+1)+\sqrt{(2\delta+1)^2+4(f(n,k)-2\delta(e+1))}}{2},$$

then G is k - connected, where $f(n, k) = ((n - k + 1)(n + k - 3)^2 + 8(n - k)(n - 2)^2 + 8(k - 1)(n - 1)^2)/8$.

In order to prove Theorem 1, we need the following results as our lemmas.

Lemma 1. ([1]) Let G be a graph of order $n \geq 2$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ and let $1 \leq k \leq n - 1$. If $1 \leq i \leq \left\lfloor \frac{n-k+1}{2} \right\rfloor$, $d_i \leq i + k - 2 \Rightarrow d_{n-k+1} \geq n - i$, then G is k - connected.

Lemma 2. ([4]) Let G be a connected graph of order n with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Then

$$\mu(G) \leq d_1 + \frac{1}{2} + \sqrt{\left(d_1 - \frac{1}{2}\right)^2 + \sum_{i=1}^{n} d_i(d_i - d_1)},$$

the equality holds if and only if G is a regular bipartite graph.

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Suppose that G is not k - connected. Then, from Lemma 1, there exists an integer j such that $1 \leq j \leq \left\lfloor \frac{n-k+1}{2} \right\rfloor \leq \frac{n-k+1}{2}$, $d_j \leq j + k - 2$, and $d_{n-k+1} \leq n - j - 1$. Obviously, $d_j \geq 1$. Then, from Lemma 2, we have that

$$\mu \leq d_1 + \frac{1}{2} + \sqrt{\left(d_1 - \frac{1}{2}\right)^2 + \sum_{i=1}^{n} d_i(d_i - d_1)},$$

Thus

$$\mu^2 - \mu(2\delta + 1) + 2\delta(1 + e) \leq \sum_{i=1}^{n} d_i^2.$$

Notice that

$$\sum_{i=1}^{n} d_i^2 \leq j(j + k - 2)^2 + (n - k - j + 1)(n - j - 1)^2 + (k - 1)(n - 1)^2$$

$$\leq \left(\frac{n-k+1}{2}\right)\left(\frac{n-k+1}{2} + k - 2\right)^2 + (n - k)(n - 2)^2 + (k - 1)(n - 1)^2$$

$$= \frac{(n-k+1)(n+k-3)^2 + 8(n-k)(n-2)^2 + 8(k-1)(n-1)^2}{8}.$$

Set

$$f(n,k) := \frac{(n-k+1)(n+k-3)^2 + 8(n-k)(n-2)^2 + 8(k-1)(n-1)^2}{8}.$$

Hence

$$\mu^2 - \mu(2\delta + 1) + 2\delta(1 + e) - f(n,k) \leq 0.$$

By solving the inequality, we have that

$$\mu \leq \frac{(2\delta+1) + \sqrt{(2\delta+1)^2 + 4(f(n,k) - 2\delta(e+1))}}{2},$$

which is a contradiction.

This completes the proof of Theorem 1.

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