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A Survey on Energy of Graphs

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Abstract. Let G be a simple graph with n vertices and m edges. The ordinary energy of the graph is defined as the sum of the absolute values of the Eigen values of its adjacency matrix .This graph invariant is very closely connected to a chemical quantity known as the total π- electron energy of conjugated hydro carbon molecules .In recent times analogous energies are being considered, based on eigen values of a variety of other graph matrices. We briefly survey this energy of simple graphs. Here we present some basic definitions and techniques used to study energy.

Keywords: Graph energy, adjacency spectrum, Laplacian spectrum, matrices

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

Several matrices can be associated to a graph such as the adjacency matrix (denoted by A) or the Laplacian matrix L = D-A where D is the diagonal matrix of degrees. Some structural properties can be deduced from their spectrum but in general we can't determine a graph from its adjacency or Laplacian spectrum.

A concept related to the spectrum of a graph is that of energy .As its name suggests, it is inspired by energy in chemistry. In 1978, Gutman defined energy mathematically for all graphs [13]. Energy of graphs has many mathematical properties which are being investigated. Energy of different graphs including regular, nonregular, circulant and random graphs is also under study.

2. Preliminaries

Let G be a simple graph with n vertices and m edges . Adjacency matrix of the graph g is given by

$$A\ (G),(a_{ij}) = \begin{cases} 1\ if\ \ v_i\ is\ adjacency\ tov_j \\ o \qquad\qquad\qquad otherwise. \end{cases}$$

The characteristic polynomial of the above adjacency matrix is given by P_G (X)
The zeros of the polynomial are given by $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots \lambda_n$ which are eigen values of G.

Here $\lambda_1^{\mu_1}, \lambda_2^{\mu_2} \ldots\ldots, \lambda_n^{\mu_n}$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots\ldots \geq \lambda_n$ and multiplicities $\mu_1, \mu_2, \ldots\ldots \mu_n$ are called spectrum of A. The spectrum of A is called spectrum of G.

Definition 2.1. Energy of a simple graph G =(V,E) with adjacency matrix A is defined as the sum of absolute values of eigen values of A. It is denoted by E(G). E(G) = ,Where λ_i is an eigen values of A. i = 1,2…n.

Suppose K eigen values are positive then E(G) = 2

Also

The energy of a graph is zero if and only if it is trivial. Using rational root theorem, the energy of nontrivial graph is an even number, if it is rational.

Applying Cauchy Schwartz inequality for (1,1,… 1) and (

E(G) ≤

3. Comparative study of graph energies

3.1. Adjacency spectrum [8,10,11]

Let G be a graph possessing n vertices and m edges .Let $v_1, v_2 \ldots . v_n$ be the vertices of G. Then the adjacency matrix A = A(G) of the graph is the square matrix of order n.

$(a_{ij}) =$

Consider the graph K_4.

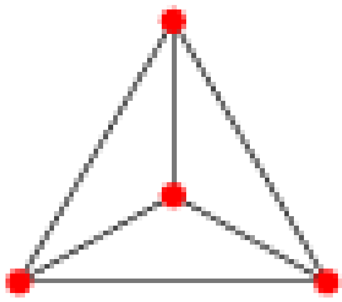

Figure 1:

Adjacency matrix is given by A (k_4) =

Characteristic Polynomial $P(K_4,x) = \det(xI_4 - A(k_4)) = (x+1)^3(x-3)$.

Spec (K_4) = and the energy of the graph E(G) = 6.

3.2. Laplacian energy [14,13]

Laplacian matrix L = L(G) of (n,m)graph is defined via its matrix elements as

$$l_{ij} = \begin{cases} -1\ if\ i \neq j\ v_i\ \ and\ v_j\ are\ adjacent \\ 0 if\ i \neq j\ v_i\ \ and\ v_j\ are\ not\ adjacent \\ d_i\ if\ i = j \end{cases},$$

where d_i is the degree of the i^{th} vertex of G.

To find the eigen values, we can solve for v, det (VI –L(G)) = 0

The eigen values of V are given by $\mu_1, \mu_2, \mu_3, \ldots. \mu_n$

Laplacian spectrum version of graph energy = $\sum_{i=1}^{n} |\mu_i|$.

Instead the Laplacian energy can be represented as LE = LE (G) = $\sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$

This definition is adjusted so that for regular graphs, LE(G) = E(G).

It was proved that LE(G) ≥ E(G holds for all bipartite graphs and for almost all the graphs.

For disconnected graph G consisting of (disjoint) components G_1 and G_2 the equality $LE(G) = LE(G_1) + LE(G_2)$ is not generally valid which is the drawback of Laplacian energy concept[14].

Some bounds of laplacian matrix

1. For the graphs with number of graphs of componenets p ≤ 2M $\left(\frac{2m}{n}\right)^{-2}$.

The upper bound increases wiiith decreasing p.Hence for such graphs,

LE(G) ≤ $\frac{2m}{n} + \sqrt{(n-1)2m - \frac{2m^2}{n}}$

2. Let G be a connected graph of order n with m edges amd maximum degree Δ .Then

LE(G) ≥ 2 (Δ +1 - $\frac{2m}{n}$).

3.3. Signless Laplacian energy [1]

The signless Laplacian matrix $L^+ = L^+(G)$ is defined as

$$L_{ij}^+ = \begin{cases} +1\ \ if\ i \neq j\ v_i\ \ and\ v_j\ are\ adjacent \\ 0\ \ \ if\ i \neq j\ v_i\ \ and\ v_j\ are\ not\ adjacent \\ d_i\ \ \ \ if\ i = j \end{cases}$$

Let $\mu_1^+, \mu_2^+, \mu_3^+, \ldots. \mu_n^+$ be the eigen values of L^+ . we define,

$LE^+ = LE^+(G) = \sum_{i=1}^{n} \left|\mu_i{}^+ - \frac{2m}{n}\right|$

To find the eigen values ,we can solve for v, det (VI –L^+) = 0

Also in this case, for regular graphs , $LE^{+(}G) = E(G)$.

For bipartite graph $LE^+ = LE$. For non – bipartite graphs the relation between LE^+ and LE is not known, but seems to be not simple.

3.4. Q-Laplacian energy [13,14,16]

Q- Laplacian matrix of graph G denoted by QE(G). The Q – Laplacian matrix of G(n,m) defined by Q(G) = D(G) +A(G) is the sum of the diagonal matrix of vertex degrees and the adjacency matrix . Let $q_1 \geq q_2 \geq, \ldots \geq q_n \geq 0$ be the Q – Laplacian spectrum of G. Then we define the Q-Laplacian energy of G as QE(G) = $\sum_{i=1}^{n} \left|q_{i-}\ \frac{2m}{n}\right|$.

Bounds of Q –Laplacian energy

Theorem 1. QE(G) ≥ 0 while the inequality holds if and only if A= 0.

Theorem 2. QE(G) $\leq \sqrt{2nM}$ for any constant k or Q - $\frac{t}{n} I$ can be orthogonally diagonalized to the block matrix $\begin{pmatrix} q_1 I_k & 0 \\ 0 & -q_1 I_k \end{pmatrix}$ where $q_1 = \sqrt{\frac{s - t^2}{n}} = \sqrt{\frac{2m}{n}}$ is real and n = 2k.

3.5. Seidel energy [15,16]

The modified adjacency matrix S(G) = S_{ij} is called Seidel matrix of graph G is defined by the following way.

$$S_{ij} = \begin{cases} -1, \; if\ i\ and\ j\ are\ adjacent\ i \neq j \\ 1\ , if\ i\ and\ j\ are\ non\ adjacent, i \neq j \\ 0, \qquad\qquad otherwise \end{cases}$$

Obviously, S = J – I – 2A, where J denotes square matrix all of whose entries are equal. SE(G) = $\sum_{i=1}^{n} |s_i|$. Obviously Q(G) and S(G) are Hermitian matrices.

Bounds of Seidel energy

1. SE(G)$\leq \sqrt{n(n^2 - n)}$ = n$\sqrt{n-1}$ while the equality holds if and only if either n = 1 or S(G) can be orthogonally diagonalized to the block matrix $\begin{pmatrix} s_1 I_k & 0 \\ 0 & -s_1 I_k \end{pmatrix}$ where $S_1 = \sqrt{\frac{S - \frac{t^2}{n}}{n}} = \sqrt{n-1}$ is real and n = 2k.
2. SE(G) ≥ 0, equality holds if and only if n = 1.

3.6. Common – neighborhood energy [5]

Let G be a simple graph with vertex set V(G) ={ $v_1, v_2, \ldots v_n$ }. For $i \neq j$, the common neighbourhood of the vertices v_i and v_j denoted by $\boldsymbol{\Gamma}(v_i, v_j)$ is the set of vertices different from v_i and v_j. The common neighborhood of G is

$$CN(G) = \begin{cases} \boldsymbol{\Gamma}\left(\mathbf{v_i}, v_j\right) \text{ if } i \neq j \\ 0\ othewise \end{cases}$$

Trace of CN (G) is zero, the sum of its eigen values is also equal to zero.

Bounds of common –neighborhood energy:

1. If a graph G consists of (disconnected) components $G_1, G_2, \ldots\ldots G_p$, then $E_{CN}(G) = E_{CN}(G_1) + E_{CN}(G_2) + \ldots\ldots + E_{CN}(G_p)$
2. E_{CN} (G) = 0 if and only no component of G possesses more than two vertices

3.7. Laplacian–energy like [13,14,15]

Recently another Laplacian spectrum based energy and called it **Laplacian – energy like** invariant (LEL).

It is defined as LEL(G) = $\sum_{i=1}^{n} \sqrt{\mu_i}$.

Bounds of LEL

1. LEL(G) $\leq \sqrt{2M\,(N-1)}$
2. $\sqrt{2m+(n-1\,)(n-2)(n\Theta)^{\frac{1}{n}-1}} \;\leq$ LEL $\leq \sqrt{2m(n-2)+(n-1\,)(n\Theta)^{\frac{1}{n}-1}}$
   for disconnected graph , the bound $\sqrt{2m} \leq$ LEL $\leq \sqrt{2m(n-2)}$

3.8. Distance energy [7]
Let G be a connected graph on n vertices are $v_1,v_2,\ldots\ldots v_n$. The distance matrix of G is the square matrix of order n whose (i,j) th entry is the distance (length of the shortest path) between the vertices v_i and v_j.

Let $\rho_1,\rho_{2,}\ldots\ldots\rho_n$ be the eigen values of the distance matrix of G. DE=DE(G) = $\sum_{i=1}^{n}|\rho_i|$. Only some elementary properties are established now.

Bounds of distance energy
1. Let G be a connected n vertex graph . Let $\sum_{i=1}^{n}\mu_i{}^2 = 2\sum_{i<j}(d_{ij})^2$
2. Let G be a connected n- vertex graph and Δ be the absolute value of the determinant of the distance matrix D(G).Then

$$\sqrt{2\sum_{i<j}(d_{ij})^2+n\,(n-1)\Delta^{2/n}} \leq E_D\,(G) \leq \sqrt{2(n-1)\sum_{i<j}(d_{ij})^2+n\,\Delta^{\frac{2}{n}}}$$

3.9. Randic energy [8,21]
It is the energy of the Randic matrix whose (i, j) element is
$R_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & if\ i \neq j\ v_i\ \ and\ v_j\ are\ adjacent \\ 0\ otherwise \end{cases}$, Where d_i stands for the degree of the i- th vertex.
The General Randic energy is defined as RE(G) = $\sum_{i=1}^{n}\left|\rho_i{}^{(\alpha)}\right|$.

Some bounds of Randic energy
Theorem 3. Let G be a simple connected graph with n vertices and R_α (G) be its general Randic index then $\rho_i{}^{(\alpha)} \geq 2\,\frac{R_\alpha\,(G)}{n}$. The equality holds in above if and only if $R_1{}^{(\alpha)} = R_2{}^{(\alpha)} \ldots. = R_n{}^{(\alpha)}$, where α = -1/2.

Theorem 4. Let G be a connected graph of order n .Suppose G has minimum vertex degree. Then R_{-1}(G) $\leq$ n/2 δ. Equality occurs if and only if G is regular.

3.10. Sum – connectivity energy [23]
It is the energy of the matrix whose (i,j) th element is

$$SC_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_i+d_j}} & if\ i \neq j\ v_i\ \ and\ v_j\ are\ adjacent \\ 0 & otherwise \end{cases}$$

The Sum –connectivity energy, SE(G) is the sum of absolute values of eigen values.

Some bounds of sum connectivity energy

Theorem 5. Let G be a graph with n vertices . then SE(G) $\leq \sqrt{2n \sum_{i,j} \frac{1}{d_i + d_j}}$ with equality if and only if G is an empty graph or regular graph of degree one.

Theorem 6. Let G be a semi regular graph of degrees r≥1and s≥1 then
$\sqrt{r+s\,SE(G)}$ = E(G).

3.11. Maximum degree energy [4]
It is the energy of the matrix whose (i,j) element is

$$MD_{ij} = \begin{cases} \max\{d_i, d_j\} \; if\ i \neq j\ v_i\ and\ v_j\ are\ adjacent \\ 0 \qquad otherwise \end{cases}$$

The Characteristic polynomial of the maximum degree M(G) is det(μI – M(G))
Here $\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots . \geq \mu_n$

Some bounds of maximum degree energy

Theorem 7. If $\mu_1, \mu_2, \mu_3, \ldots \ldots \mu_n$ are the maximum degree eigen values of a graph G. Then $\sum_{i=1}^{n} {\mu_i}^2$ = -2 C_2.

Theorem 8. If G is a graph of order n then for any maximum degree eigen value μ_j we have $|\mu_j| \leq (n-1)^2$

3.12. Haray energy [9,18]
It is the energy of Harary matrix,whose (i,j) –elements is

$$H_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} \; if\ i \neq j\ v_i\ and\ v_j\ are\ adjacent \\ 0 \qquad if\ i = j \end{cases}$$

,where d(v_i, v_j) stands for the distance between the vertices v_i and v_j. We can define Harary energy as HE(G) = $\sum_{i=1}^{n} |\rho_i|$, where $\rho_1, \rho_2, \ldots \ldots \rho_n$ are the eigen values of the Harary mattrix.

Some bounds of harary energy

Theorem 9. If G is connected (n,m)graph then $\sqrt{2\sum_{1\leq i<j<n}\left(\frac{1}{d_{ij}}\right)^2} \leq$ HE (G) $\leq \sqrt{2n\sum_{1\leq i<j<n}\left(\frac{1}{d_{ij}}\right)^2}$.

Theorem 10. If Gis connected (n,m) graph then

$$HE(G) \leq \frac{2}{n} \sum_{1\leq i<J<n}\left(\frac{1}{d_{ij}}\right)^2 + \sqrt{(n-1)(2\sum_{1\leq i<j\leq n}\left(\frac{1}{d_{ij}}\right)^2 - \left\{\frac{2}{n}\sum_{1\leq i<j\leq n}\left(\frac{1}{d_{ij}}\right)^2\right\}^2}$$

3.13. Hyper energetic graph
An n vertex graph G is said to be Hyper energetic if E(G) >E(K_n).The energy of

complete K_n is equal to 2(n-1).

The following table shows the various energies for graph K_4.

Name of the energy	**Characteristic polynomial**	**Eigen values**	**Energy of the graph**
Adjacency energy	$(x+1)^3(x-3)$	3,-1,-1,-1	6
Laplacian energy	$x^4-12x^3+48x^2-6x$	4,0,4,4	12
Signless energy	$x^4-12x^3+48x^2-80x+48$	6,2,2,2	12
Q- Laplacian energy	$x^4-12x^3+48x^2-80x+48$	6,2,2,2	12
Seidel energy	x^4-6x^2+8x-3	-3,1,1,1	6
Common – neighborhood energy	$x^4-24x^2-64x-48$	6,-2,-2,-2	12
Laplacian Energy like	$x^4-12x^3+48x^2-6x$	4,0,4,4	6
Distance energy	x^4-6x^2-8x-3	3,-1,-1,-1	6
Randic energy	$x^4-2/3\,x^2-8/27\,x-1/27$	1, -0.3333,-0.3333, -0.3333	1.9999
Sum-connectivity energy	$x^4-x^2-0.5443x-0.0833$	1.2247,-0.4083,-0.4082,-0.4082	2.4494
Maximum degree energy	$x^4-54x^2-216x-243$	9,-3,-3,-3	15
Harary energy	x^4-6x^2-8x-3	3,-1,-1,-1	6

3.14. Energies of digraphs [2,3,19,20]
Adiga and Smitha defined the skew laplacian energy for a simple digraph G as
$SLE_k(G) = \sum_{i=1}^{n} {\mu_i}^2$, where $\mu_1, \mu_2, \mu_3, \ldots\ldots \mu_n$ are the eigen values of the skew Laplacian matrix SL(G) = D(G) – S(G) of G.

Bounds
1. For any simple digraph G on n vertices whose degrees are $d_1,d_2,\ldots..d_n$
$SLE_k(G) = \sum_{i=1}^{n} d_i\,(d_i-1)$
2. For any connected simple digraph G on n $\geq$ 2 vertices ,$2n-4 \leq SLE_k(G) \leq n(n-1)(n-2)$ where left inequality holds if and only if G is directed path n vertices and the right equality holds if and only if G is complete digraph on n vertices.

3.15. New skew Laplacian energy

Given a simple digraph G with vertex set V(G) = $\{v_1, v_2, \ldots v_n\}$. The skew Laplacian energy of G is defined as SLE(G) $=\sum_{i=1}^{n}|\mu_i|$ where $\mu_1, \mu_2, \mu_3, \ldots\ldots \mu_n$. $\widetilde{SL}$(G)= $\widetilde{D(G)}$ - S(G).

Example: Let P_4 be directed complete graph on 4 vertices with arc set {(1,2),(2,3)(3,4)}

$$\widetilde{SL}(p_4) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Eigenvaues are $i\sqrt{2}$,$-i\sqrt{2}$, 0,0 . New skew Laplacian energy = $2\sqrt{2}$.

4. Conclusion

Graph energy has so many application in the field of chemistry,physics and mathematics also. Some types of graph energies are studied in this paper. In graph K_4 Lapalcian energy,distance energy,Harary energy are equal. Among the specified energies Maximum degree energy dominates the other energies. This can be generalized for K_n.. Further study on energy and spectra of graphs may reveal more analogous results of these kind and will be discussed in the forthcoming papers.

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