Annals of Pure and Applied Mathematics
Vol. 8, No. 2, 2014, 51-58
ISSN: 2279-087X (P), 2279-0888(online)
Published on 17 December 2014
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Matrix Representation of Double Layered Fuzzy Graph and its Properties

***T.Pathinathan*[1] and *J.Jesintha Rosline*[2]**

Department of Mathematics, Loyola College, Chennai – 34, India
[1]Email: pathimathsloyola@gmail.com; [2]Email: jesi.simple@gmail.com

Received 10 September 2014; accepted 21 November 2014

Abstract. Uncertainties in a problem are represented as fuzzy matrices using fuzzy principles. Recent days fuzzy matrices have become very famous. In this paper unlike the usual matrix representation of a fuzzy graph with respect to vertices, a new matrix representation with edge membership values as rows and columns is introduced. The relationship between the double layered fuzzy graph and the given fuzzy graph whose crisp graph is a cycle are analyzed.

Keywords: fuzzy graph, strong fuzzy graph, double layered fuzzy graph, matrix representation of double layered fuzzy graph, edge matrix representation of fuzzy graph

AMS Mathematics Subject Classification (2010): 94D05

1. Introduction

The concept of fuzzy set was introduced by Zadeh in 1965. Fuzzy graph theory was introduced by Rosenfeld in 1975 [5]. It is well known that matrices play a major role in various areas such as mathematics, physics, statistics, engineering etc. Matrices with entries from [0, 1] and matrix operation defined by fuzzy logical operations are fuzzy matrices. Fuzzy matrices play a fundamental role in fuzzy set theory. They provide us with a logical framework within which many problems of practical applications can be formulated.

Fuzzy matrices can be successfully used when fuzzy uncertainty occurs in a problem. Fuzzy matrix has been proposed to represent fuzzy relation in a system based on fuzzy set theory [1]. Fuzzy matrices were introduced first time by Thomson [2], who discussed the convergence of powers of fuzzy matrices. Two new operations in fuzzy graphs were introduced by Shayamal and Pal [4]. The determinant and adjoint of a square fuzzy matrix are introduced by Ragab and Emam [13]. Pathinathan and Jesintha Rosline had defined the double layered fuzzy graph [12]. In this Paper the edge matrix representation is defined, using it the matrix representation of the double layered fuzzy graph is given. The relationship between the DLFG and the given fuzzy graph are given as propositions and simple examples are presented with verification.

2. Preliminaries

Definition 2.1. [5] A fuzzy graph G is a pair of functions G:(σ,μ) where σ is a fuzzy subset of a non empty set S and μ is a symmetric fuzzy relation on σ . The underlying crisp graph of G:(σ,μ) is denoted by $G^* : (\sigma^*, \mu^*)$

Definition 2.2. [8] Let $G(\sigma,\mu)$ be a fuzzy graph, the order of G is defined as $O(G) = \sum_{u \in V} \sigma(u)$

Definition 2.3. [8] Let $G(\sigma,\mu)$ be a fuzzy graph, the size of G is defined as $S(G) = \sum_{u,v \in V} \mu(u,v)$

Definition 2.4. [10] Let G be a fuzzy graph, the degree of a vertex u in G is defined as $d(u) = \sum_{\substack{v \neq u \\ v \in V}} \mu(u,v)$ and is denoted as $d_G(u)$.

Definition 2.5. [12] Let $G:(\sigma,\mu)$ be a fuzzy graph with the underlying crisp graph $G^* : (\sigma^*, \mu^*)$. The pair $DL(G) : (\sigma_{DL}, \mu_{DL})$ is defined as follows. The node set of $DL(G)$ be $\sigma^* \cup \mu^*$. The fuzzy subset σ_{DL} is defined as $\sigma_{DL} = \begin{cases} \sigma(u) \text{ if } u \in \sigma^* \\ \mu(uv) \text{ if } uv \in \mu^* \end{cases}$

The fuzzy relation μ_{DL} on VUE is defined as

$$\mu_{DL} = \begin{cases} \mu(uv) \text{ if } u,v \in \sigma^* \\ \mu(e_i) \wedge \mu(e_j) \text{ if the edge } \mathrm{e}_i \text{ and } \mathrm{e}_j \textit{ have } \text{a node in common between them} \\ \mu(u_i) \wedge \mu(e_i) \text{ if } u_i \in \sigma^* \text{ \& } \mathrm{e}_i \in \mu^* \text{ and each } \mathrm{e}_i \text{ incident with single } \mathrm{u}_i \text{ only} \\ \qquad\qquad \text{either clockwise or anticlockwise.} \\ 0 \text{ otherwise.} \end{cases}$$

By definition, $\mu_{DL}(u,v) \leq \sigma_{DL}(u) \wedge \sigma_{DL}(v)$ ∀ u,v in $\sigma^* \mathrm{U} \mu^*$. Here μ_{DL} is a fuzzy relation on the fuzzy subset σ_{DL}. Hence the pair $DL(G) : (\sigma_{DL}, \mu_{DL})$ is a fuzzy graph and is termed as Double Layered Fuzzy Graph.

Definition 2.6. A fuzzy graph G:(σ,μ) with the fuzzy relation μ to be reflexive and symmetric is completely determined by the fuzzy matrix M_G, where

$$(\mathrm{M}_G)_{ij} = \begin{cases} \mu(v_i, v_j) & \text{if } \mathrm{i} \neq \mathrm{j} \\ \sigma(\mathrm{v}_i) & \text{if } \mathrm{i} = \mathrm{j} \end{cases}$$

If σ^* has n elements then M_G has n x n elements.

Remark 2.1. In this paper, we are named the above matrix as Matrix representation of a fuzzy graph with respect to vertices and is denoted as M_{G_σ}.

3. Matrix representation of DLFG

Consider a fuzzy graph G with n = 3 vertices.

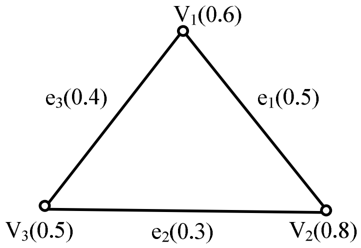

Figure 1: G:(σ,μ)

The matrix representation with respect to vertices for the fuzzy graph G is given by

$$M_{G_\sigma} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.5 & 0.4 \\ 0.5 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

The double layered fuzzy graph for G is given by

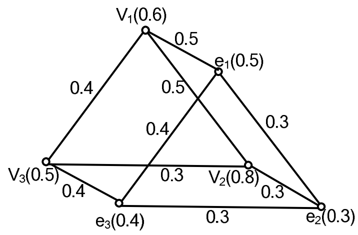

Figure 2: Double layered fuzzy graph $DL(G)$: (σ_{DL}, μ_{DL})

The matrix representation of DLFG is

$$M_{DL(G_\sigma)} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.5 & 0.4 & 0.5 & 0 & 0 \\ 0.5 & 0.8 & 0.3 & 0 & 0.3 & 0 \\ 0.4 & 0.3 & 0.5 & 0 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0.5 & 0.3 & 0.4 \\ 0 & 0.3 & 0 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0.4 & 0.4 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

3.1. Edge matrix representation of fuzzy graphs

For a fuzzy graph $G:(\sigma,\mu)$ with the fuzzy relation μ to be reflexive and symmetric , the edge matrix M_{G_μ} is defined as follows,

$$\left(\mathrm{M}_{G_\mu}\right)_{ij} = \begin{cases} \min\{\mu(\mathrm{e}_i),\mu(\mathrm{e}_j)\} \text{ if } \mathrm{v}_i \text{ is the common vertex between } \mathrm{e}_i \text{ and } \mathrm{e}_j \\ \mu(\mathrm{e}_i) \text{ if i = j} \\ 0 \text{ otherwise} \end{cases}$$

If μ^* contains 'n' elements then M_{G_μ} is a square matrix of order n.

Example 3.1 For Figure 1, the edge matrix representation is given by

$$\mathrm{M}_{G_\mu} = \begin{array}{c c} & \begin{array}{ccc} \mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 \end{array} \\ \begin{array}{c} \mathrm{e}_1 \\ \mathrm{e}_2 \\ \mathrm{e}_3 \end{array} & \begin{bmatrix} 0.5 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

Thus for Figure 2, the matrix representation becomes

$$\mathrm{M}_{DL(G_\sigma)} = \begin{bmatrix} \mathrm{M}_{G_\sigma} & D_{G_\mu} \\ D_{G_\mu} & \mathrm{M}_{G_\mu} \end{bmatrix}$$

4. Theoretical concepts

Theorem 4.1. $M_{DL(G_\sigma)}$ is a symmetric matrix.

Proof: By the definition of $\mathrm{M}_{\mathrm{DL},}$ the relation μ is a symmetric relation.
Hence,

$$\begin{aligned} \left(\mathrm{M}_{\mathrm{DL}(\mathrm{G}_\sigma)}\right)_{i,j} &= \mu(\mathrm{v}_i,\mathrm{v}_j) \\ &= \mu(\mathrm{v}_j,\mathrm{v}_i) \quad \because \mu \text{ is symmetric} \\ &= \left(\mathrm{M}_{\mathrm{DL}(\mathrm{G}_\sigma)}\right)_{j,i} \end{aligned}$$

$$\therefore \mathrm{M}_{\mathrm{DL}(\mathrm{G}_\sigma)} = \mathrm{M}^{\mathrm{T}}_{\mathrm{DL}(\mathrm{G}_\sigma)}$$

Thus $M_{DL(G_\sigma)}$ is a symmetric matrix

Theorem 4.2. Trace $(\mathrm{M}_{DL(G_\sigma)})$ = Order(G) + Size(G)

Proof: Trace ($\mathrm{M}_{DL(G_\sigma)}$) = Sum of the diagonal entries in $\mathrm{M}_{DL(G_\sigma)}$.

$$= \sum_{i=1}^{n} \mu_{DL(G)}(v_i,v_i) = \sum_{v_i\in\sigma^*_{DL}} \sigma_{DL(G)}(v_i)$$

$= \sum_{\substack{v_i \in \sigma^*, \\ e_i \in \mu^*}} \sigma_G(v_i) + \mu_G(e_i)$ by the definition of node set in DLFG.

$= \sum_{v_i \in \sigma^*} \sigma_G(v_i) + \sum_{e_i \in \mu^*} \mu_G(e_i)$ = Order (G) + Size (G).

Theorem 4.3. The sum of all the entries in $M_{DL(G)}$ except the diagonal element in the row or column is degree of DLFG, i.e.,

(i) If $v_i \in \sigma^*$, then $d_{DL(G)}(v_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} (M_{G_\sigma})_{ij} + \left(M_{G_\sigma}\right)_{ik}$

(ii) if $v_i \in \mu^*$, then $d_{DL(G)}(v_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} (M_{G_\mu})_{ij} + \left(M_{G_\sigma}\right)_{ik}$,where

$$k = \begin{cases} i+1 & \text{if } i+1 \leq n \\ \operatorname{Re} m\left(\dfrac{i+1}{n}\right) & \text{if } i+1 > n \end{cases}$$

Proof: The sum of all the entries in $M_{DL(G)}$ except the diagonal element in the row or column is $\sum_{\substack{j=1 \\ i \neq j}}^{n} (M_{DL(G)})_{ij} = \sum_{\substack{j=1 \\ i \neq j}}^{n} \mu_{DL(G)}(v_i, v_j) = d_{DL(G)}(v_i)$

Case i: If $v_i \in \sigma^*$ in G , then $d_{DL(G)}(v_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \mu_{DL(G)}(v_i, v_j)$

$= \sum_{\substack{j=1 \\ i \neq j}}^{n} \mu_G(v_i, v_j) + \mu_G(v_i, v_k) = \sum_{\substack{j=1 \\ i \neq j}}^{n} (M_{G_\sigma})_{ij} + (M_{G_\sigma})_{ik}$,

where $k = \begin{cases} i+1 & \text{if } i+1 \leq n \\ \operatorname{Re} m\left(\dfrac{i+1}{n}\right) & \text{if } i+1 > n \end{cases}$

Case ii:

If $v_i \in \mu^*$ in G , then $d_{DL(G)}(v_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \mu_{DL(G)}(v_i, v_j) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \mu_G(v_i, v_j) + \mu_G(v_i, v_k)$

$= \sum_{\substack{j=1 \\ i \neq j}}^{n} (M_{G_\mu})_{ij} + (M_{G_\sigma})_{ik}$, where $k = \begin{cases} i+1 & \text{if i+1} \leq \text{ n} \\ \operatorname{Re}m(\frac{i+1}{n}) & \text{if i+1 > n} \end{cases}$

Theorem 4.4. The sum of all entries in $\mathrm{M}_{DL(G)}$ except the diagonal element is 4 size(G) $+\ 2\sum_{i=1}^{n}(\mu(e_i)\wedge\mu(e_j))$.

Proof: The sum of all entries in $\mathrm{M}_{DL(G)}$ except the diagonal element is

$$= \sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}\mu(v_i,v_j) = \sum_{i=1}^{n} d_{DL(G)}(v_i)$$

Case i:

If $v_i \in \sigma^*$, then $\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}\mu(v_i,v_j) = \sum_{i=1}^{n} d_{DL(G)}(v_i) = \sum_{\substack{j=1\\i\neq j}}^{n}(\sum_{\substack{j=1\\i\neq j}}^{n}(\mathrm{M}_{G_\sigma})_{ij} + \left(\mathrm{M}_{G_\sigma}\right)_{ik})$

$$= \sum_{i=1}^{n}(d_G(v_i) + \left(\mathrm{M}_{G_\sigma}\right)_{ik}) = \sum_{i=1}^{n} d_G(v_i) + \sum_{i=1}^{n}\left(\mathrm{M}_{G_\sigma}\right)_{ik}$$

= 2 size(G) + Size(G)= 3 size(G).

Case ii:

If $v_i \in \mu^*$, then $\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}\mu(v_i,v_j) = \sum_{i=1}^{n} d_{DL(G)}(v_i) = \sum_{i=1}^{n}(\sum_{\substack{j=1\\i\neq j}}^{n}(\mathrm{M}_{G_\mu})_{ij} + \left(\mathrm{M}_{G_\sigma}\right)_{ik})$

$$= \sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}(\mu(e_i)\wedge\mu(e_j) + \left(\mathrm{M}_{G_\sigma}\right)_{ik}) = 2\sum_{i=1}^{n}\mu(e_i)\wedge\mu(e_j) + \sum_{i=1}^{n}\left(\mathrm{M}_{G_\sigma}\right)_{ik}$$

$$= 2\sum_{i=1}^{n}(\mu(e_i)\wedge\mu(e_j)) + \text{ size(G).}$$

Thus, if $v_i \in \sigma_{DL}{}^*$, then $\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}\mu(v_i,v_j) = 2\sum_{i=1}^{n}(\mu(e_i)\wedge\mu(e_j))$+ 4 size(G)

Example 4.1. Consider the fuzzy graph with n = 4 vertices.

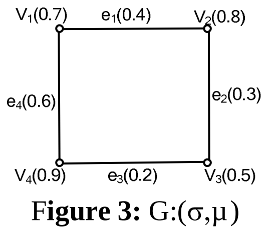

Figure 3: G:(σ,μ)

Here, Size (G) = 1.5 and $\sum_{\substack{i,j=1\\ i\neq j}}^{n} (\mu(e_i) \wedge \mu(e_j)) = 1.1$.

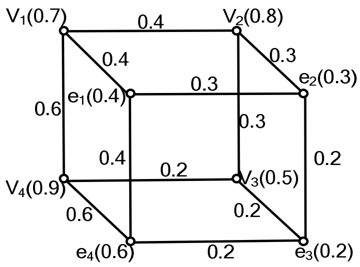

Figure 4: $DL(G): (\sigma_{DL}, \mu_{DL})$

$$M_{DL(G)} = \begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ e_1\\ e_2\\ e_3\\ e_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 0.7 & 0.4 & 0 & 0.6 & 0.4 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0.3 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0.3 & 0.5 & 0.2 & 0 & 0 & 0.2 & 0 \\ 0.6 & 0 & 0.2 & 0.9 & 0 & 0 & 0 & 0.6 \\ 0.4 & 0 & 0 & 0 & 0.4 & 0.3 & 0 & 0.4 \\ 0 & 0.3 & 0 & 0 & 0.3 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0.2 & 0.6 \end{bmatrix} \end{array}$$

Sum of all entries except the diagonal elements = 8.2

$$4\ \text{size(G)} + 2\sum_{i=1}^{n} (\mu(e_i) \wedge \mu(e_j)) = 2(1.1) + 4(1.5) = 2.2 + 6 = 8.2.$$

Thus, Sum of all entries except the diagonal elements

$$= 4\ \text{size(G)} + 2\sum_{i=1}^{n} (\mu(e_i) \wedge \mu(e_j))$$

5. Conclusion

In this paper, we have defined a new matrix representation using edge membership values. The relationship between the matrix representation of double layered fuzzy graph using vertices and given fuzzy graph whose crisp graph is found to be a cycle is examined. Numerical example is given to verify the results. Further analysis will lead to application of DLFG in different networks.

Acknowledgement

I thank MANF for the financial support provided for my research work. The authors would like to thank the anonymous reviewers for their careful reading of this article and for their helpful comments.

REFERENCES

1. S.V.Overhinnikov, Structure of fuzzy relations, *Fuzzy Sets and Systems*, 6 (1981) 169 – 195.
2. M.G.Thomson, Convergence of powers of a fuzzy matrix, *Journal of Mathematical Analysis and Application*, 57 (1977) 476 – 480.
3. M.S.Sunitha and A.Vijayakumar, Complement of a fuzzy graph, *Indian Journal of Pure and Applied Mmathematics,* 33(9) (2002) 1451-1464.
4. A.K.Shayamal and M.Pal, Two new operations on fuzzy matrices, *Journal of Applied Mathematics and Computing,* 15 (2004) 91 – 107.
5. A.Rosenfeld, Fuzzy graphs, in: L.A.Zadeh, K.S.Fu, K.Tanaka and M.Shimura, (editors), *Fuzzy sets and its application to cognitive and decision process*, Academic press, New York (1975) 77 – 95.
6. A.Nagoorgani and K.Radha, The degree of a vertex in some fuzzy graphs, *International Journal of Algorithms, Computing and Mathematics*, 2(3) (2009) 107 - 116.
7. R.T.Yeh and S.Y.Bang, Fuzzy relations, fuzzy graphs and their applications to clustering analysis, in: L.A.Zadeh, K.S.Fu, K.Tanaka and M. Shimura, (editors), *Fuzzy sets and its application to cognitive and decision process*, Academic press, New York (1975) 125 – 149.
8. A.Nagoorgani and M. Basheed Ahamed, Order and size in fuzzy graphs, *Bulletin of Pure and Applied Sciences,* 22E(1) (2003) 145 – 148.
9. J.N.Mordeson, Fuzzy line graphs, *Pattern Recognition Letter,* 14(1993) 381 – 384.
10. J.N.Mordeson and P.S.Nair, *Fuzzy Graphs and Fuzzy Hypergraphs,* Physica verlag Publication, Heidelberg 1998, second edition 2001.
11. A.Nagoorgani and J.Malarvizhi, Some aspects of neighbourhood fuzzy graph, *International Journal of Pure and Applied Sciences*, 29E, 2(2010) 327 – 333.
12. T.Pathinathan and J.Jesintha Rosline, Double layered fuzzy graph, *Annals of Pure and Applied Mathematics*, 8(1) (2014) 135-143.
13. M.Z.Ragab and E.G.Emam, The determinant and adjoint of a square fuzzy matrix, *An International Journal of Information Sciences-Intelligent Systems*, 84 (1995) 209-220.
14. S.Samanta and M.Pal, Fuzzy tolerance graphs, *International Journal of Latest Trends in Mathematics,* 1(2) (2011) 57-67.
15. Mini Tom and M.S.Sunitha, Sum Distance in Fuzzy Graphs, *Annals of Pure and Applied Mathematics*, 7(2) (2014) 73-89.
16. M.S.Sunitha and Sunil Mathew, Fuzzy Graph Theory: A Survey, *Annals of Pure and Applied Mathematics*, 4(1) (2013) 92-110.
17. Madhumangal Pal, Intersection Graphs: An Introduction, *Annals of Pure and Applied Mathematics*, 4(1) (2013) 43-91.