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A Reliable Treatment of Iterative Laplace Transform Method for Fractional Telegraph Equations

S.C. Sharma [1] **and *R.K. Bairwa*** [2*]

[1]Department of Mathematics, University of Rajasthan, Jaipur- 302004, India.
E-mail: sureshchand26@gmail.com
[2]Department of Mathematics, University of Rajasthan, Jaipur- 302004, India.
E-mail: rajendrabairwa1984@gmail.com

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Abstract. In this paper a reliable algorithm for the iterative Laplace transform method (ILTM) is presented. ILTM is a combination of Laplace transform method and Iterative method to solve space- and time- fractional telegraph equations. The fractional derivatives are considered in Caputo sense. Closed form analytical expressions are derived in terms of the Mittag-Leffler functions. An illustrative numerical case study is presented for the proposed method to show the preciseness and effectiveness of the method.

Keywords: Laplace transform, Iterative Laplace transform method, telegraph equations, Caputo fractional derivative, Mittag-Leffler function, fractional differential equation

AMS Mathematics Subject Classification (2010): 26A33, 33E12, 35R11, 44A10

1. Introduction

Fractional calculus has been rediscovered by scientists and engineers due to the increasing use in number of fields such as electromagnetism, signal Processing, Control Engineering, physics, mathematical biology, visco-elasticity and other areas of science [5,14,18,34]. Various methods are available in literature for the solution of fractional order differential equation such as fractional subequation method [28], fractional wavelet method [13,20,29,34], fractional Laplace adomian decomposition method [8,21], fractional operational matrix method [1,33], fractional variational iteration method [22,26], fractional improved homotopy perturbation method [35,36], fractional differential transform method [32] and fractional complex transform method [24] etc.

The iterative method was introduced in 2006 by Daftardar-Gejji and Jafari to solve numerically the nonlinear functional equations [9, 30]. By now, the iterative method has been used to solve many non-linear differential equations of integer and fractional order [25] and fractional boundary value problem [31]. In recent, Jafari *et al.* firstly applied Laplace transform in the iterative method to develop the iterative Laplace transform method [10] for searching numerical solutions of a system of fractional partial differential equations. The iterative Laplace transform method (ILTM) has been successfully applied to solve fractional Fokker-Planck equations [17].

In this paper, we consider the space-time fractional telegraph equations in the following form:

$$D_x^{\alpha}u(x,t)=D_t^{p\beta}u(x,t)+aD_t^{r\beta}u(x,t)+bu(x,t)+g(x,t),\quad 0<x<1,\ t>0, \tag{1}$$

where $\beta = 1/q,\ p,q,r\in N,\ 1<\alpha\le 2,\ 1<p\beta\le 2,\ 0<r\beta\le 1,\ D_t^{p\beta}\equiv D_t^{\beta}D_t^{\beta}...D_t^{\beta}$ (p times), $D_t^{r\beta}\equiv D_t^{\beta}D_t^{\beta}...D_t^{\beta}$ (r times), $D_x^{\alpha}, D_t^{\beta}$ are Caputo fractional derivatives defined by equation (2), a,b and c are constants and $g(x,t)$ is given function. In the case of $\alpha=2, q=1, p=2, r=1, g=0$, space-time fractional telegraph equation reduces to classical telegraph equation.

Further, we apply the iterative Laplace transform method (ILTM) to solve fractional telegraph equations. It is worth mentioning that this method is an elegant coupling of the Iterative Method and Laplace Transform Method. The ILTM provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear fractional equations.

2. Basic definitions of fractional calculus and Laplace transform

In this section some basic definitions and properties of fractional calculus and Laplace transform theory are given.

Definition 1. The Caputo fractional derivative [11, 16] of function $u(x,t)$ is defined as

$$\begin{aligned}D_x^{\alpha}u(x,t)&=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-\xi)^{m-\alpha-1}u^{(m)}(\xi,t)d\xi,\quad m-1<\alpha\le m, m\in N,\\ &=J_x^{m-\alpha}D^m u(x,t)\end{aligned} \tag{2}$$

here $D^m\equiv\frac{d^m}{dx^m}$ and J_x^{α} stands for the Riemann-Liouville fractional integral operator of order $\alpha>0$ [16] defined as

$$J_x^{\alpha}u(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}u(\xi,t)d\xi,\ \xi>0,\quad (m-1<\alpha\le m), m\in N. \tag{3}$$

Definition 2. The Laplace transform of a function $f(x),\ x>0$ is defined as [11, 16]

$$L[f(x)]=F(s)=\int_0^{\infty}e^{-st}f(t)dt. \tag{4}$$

Definition 3. Laplace transform of $D_x^{\alpha}u(x,t)$ is given as [11, 16]

$$L[D_t^{\alpha}u(x,t)]=L[u(x,t)]-\sum_{k=o}^{m-1}u^k(0,t)s^{\alpha-k-1},\quad m-1<\alpha\le m,\ m\in N, \tag{5}$$

where $u^k(0,t)$ is the k-order derivative of $u(x,t)$ at $x=0$.

Definition 4. The Mittag-Leffler function which is a generalization of exponential function is defined as [16]:

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+1)}(\alpha\in C,\ \mathrm{Re}(\alpha)>0). \tag{6}$$

a further generalization of (6) is given in the form [3]:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+\beta)};(\alpha,\beta\in C,\ R(\alpha)>0,R(\beta)>0) \tag{7}$$

3. Basic idea of the iterative Laplace Transform method

To illustrate the basic idea of this method [10], we consider a general fractional nonlinear non-homogeneous partial differential equation with the initial conditions of the form:

$$D_{x}^{\alpha}u(x,t)+R\,u(x,t)+N\,u(x,t)=g(x,t),\qquad m-1<\alpha\le m,\ \ m\in N, \tag{8}$$

$$u^{k}(0,t)=h_{k}(t),\ \ k=0,1,2.........,m-1 \tag{9}$$

where $D_{x}^{\alpha}u(x,t)$ is the Caputo fractional derivative of the function u(x, t), R is the linear differential operator, N represents the general nonlinear differential operator and g(x, t) is the source term. Applying the Laplace transform (denoted by L throughout the present paper) on both sides of Eq. (8), we get

$$L[D_{t}^{\alpha}u(x,t)]+L[R\,u(x,t)+Nu(x,t)]=L[g(x,t)]. \tag{10}$$

Using the property of the Laplace transform, we have,

$$L[u(x,t)]=\frac{1}{s^{\alpha}}\sum_{k=0}^{m-1}s^{\alpha-1-k}u^{k}(0,t)+\frac{1}{s^{\alpha}}L[g(x,t)]-\frac{1}{s^{\alpha}}L[R\,u(x,t)+N\,u(x,t)]. \tag{11}$$

Operating with the Laplace inverse on both sides of Eq. (11) gives

$$u(x,t)=L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1}s^{\alpha-1-k}u^{k}(0,t)+L[g(x,t)]\right)\right]-L^{-1}\left[\frac{1}{s^{\alpha}}L[Ru(x,t)+Nu(x,t)]\right], \tag{12}$$

Now we apply the Iterative method,

$$u(x,t)=\sum_{i=0}^{\infty}u_{i}(x,t) \tag{13}$$

Since R is a linear operator,

$$R\left(\sum_{i=0}^{\infty}u_{i}(x,t)\right)=\sum_{i=0}^{\infty}R(u_{i}(x,t) \tag{14}$$

and the nonlinear operator N is decomposed as

$$N\left(\sum_{i=0}^{\infty}u_{i}(x,t)\right)=N(u_{0}(x,t))+\sum_{i=1}^{\infty}\left\{N(\sum_{k=0}^{i}u_{0}(x,t))-N(\sum_{k=0}^{i-1}u_{k}(x,t))\right\} \tag{15}$$

Substituting (13), (14) and (15) in (12), we get

$$\begin{aligned}\sum_{i=0}^{\infty}u_{i}(x,t)&=L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1}s^{\alpha-1-k}u^{k}(0,t)+L[g(x,t)]\right)\right]-\\ &L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\sum_{i=0}^{\infty}R\big(u_{i}(x,t)\big)+N(u_{0}(x,t))+\sum_{i=1}^{\infty}\left\{N(\sum_{k=0}^{i}u_{k}(x,t))-N(\sum_{k=0}^{i-1}u_{k}(x,t))\right\}\right]\right],\end{aligned} \tag{16}$$

We define the recurrence relations as

$$u_0(x,t)=L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1}s^{\alpha-1-k}u^k(0,t)+L[g(x,t)]\right)\right]$$

$$u_1(x,t)=-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[R\left(u_0(x,t)\right)+N\left(u_0(x,t)\right)\right]\right] \tag{17}$$

$$u_{m+1}(x,t)=-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[R\left(u_m(x,t)\right)-\left\{N(\sum_{k=0}^{m}u_k(x,t))-N(\sum_{k=0}^{m-1}u_k(x,t))\right\}\right]\right],\quad m\geq 1$$

Therefore the m -term approximate solution of (8) - (9) in series form is given by

$$u(x,t)\cong u_0(x,t)+u_1(x,t)+u_2(x,t)+........+u_m(x,t),\quad m=1,2,... \tag{18}$$

4. Applications

In this section, we use the iterative Laplace transform method (ILTM) to solve the homogeneous and non-homogeneous fractional telegraph equations.

Example 1 Consider the following homogeneous space-time fractional telegraph equation:

$$D_x^{\alpha}u(x,t)=D_t^{p\beta}u(x,t)+D_t^{r\beta}u(x,t)+u(x,t),0<x<1,t>0, \tag{19}$$

where $\beta=1/q$, $p,q,r\in N$, $1<\alpha\leq 2$, $1<p\beta\leq 2$, $0<r\beta\leq 1$, $D_t^{p\beta}\equiv D_t^{\beta}D_t^{\beta}...D_t^{\beta}$ (p times), $D_t^{r\beta}\equiv D_t^{\beta}D_t^{\beta}...D_t^{\beta}$ (r times), D_x^{α} , D_t^{β} are Caputo fractional derivatives defined by equation (2), $p+r$ is odd and initial conditions are given by

$$u(0,t)=E_{\beta}\left(-t^{\beta}\right)\text{ and }u_x(0,t)=E_{\beta}\left(-t^{\beta}\right). \tag{20}$$

Applying the Laplace transform on the both sides of Eq. (19), subject to the initial condition (20) we have

$$L\left[D_x^{\alpha}u(x,t)\right]=L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u(x,t)\right],\ 0<x<1\quad ,t>0, \tag{21}$$

Using the property of the Laplace transform, we have

$$L[u(x,t)]=\frac{E_{\beta}\left(-t^{\beta}\right)}{s}+\frac{E_{\beta}\left(-t^{\beta}\right)}{s^2}+\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u(x,t)\right] \tag{22}$$

Operating with the Laplace inverse on both sides of Eq. (22) gives

$$u(x,t)=(1+x)E_{\beta}\left(-t^{\beta}\right)+L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u(x,t)\right]\right] \tag{23}$$

Now, applying the Iterative method,

Substituting (13) - (15) into (23) and applying (17), we obtain the components of the solution as follows:

$$u_0(x,t)=(1+x)E_{\beta}\left(-t^{\beta}\right) \tag{24}$$

$$u_1(x,t)=L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u_0(x,t)\right]\right]$$

$$=\left(\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{\alpha+1}}{\Gamma(2+\alpha)}\right)E_{\beta}\left(-t^{\beta}\right) \tag{25}$$

$$u_2(x,t)=L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)\left(u_1(x,t)+u_0(x,t)\right)\right]\right]-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u_0(x,t)\right]\right]$$

$$=\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha+1}}{\Gamma(2+2\alpha)}+\frac{x^{\alpha+1}}{\Gamma(2+\alpha)}\right)E_{\beta}\left(-t^{\beta}\right)-\left(\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{\alpha+1}}{\Gamma(2+\alpha)}\right)E_{\beta}\left(-t^{\beta}\right)$$

$$=\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{2\alpha+1}}{\Gamma(2+2\alpha)}\right)E_{\beta}\left(-t^{\beta}\right) \tag{26}$$

The solution in series form is then given by

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+.............$$

$$u(x,t)=E_{\beta}\left(-t^{\beta}\right)\left[1+x+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{\alpha+1}}{\Gamma(2+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{2\alpha+1}}{\Gamma(2+2\alpha)}+.....\right]$$

$$=\left[E_{\alpha}\left(x^{\alpha}\right)+xE_{\alpha,2}\left(x^{\alpha}\right)\right]E_{\beta}\left(-t^{\beta}\right). \tag{27}$$

The same result was obtained by Garg and Sharma [19] using ADM.

Remark 1. Setting $p=2, q=r=1$, the space-time fractional telegraph Eq. (19) reduces to space fractional telegraph equation and the solution is same as obtained by Momani [27] using ADM, Odibat and Momani [37] using GDTM, Yildirim [4] using HPM and Alawad [7] using LVIM.

Remark 2. Setting $\alpha=2$, Eq. (19) reduces to time fractional telegraph equation, with the meaning of various symbols and parameters as given with Eq. (19), as follows

$$D_x^2u(x,t)=D_t^{p\beta}u(x,t)+D_t^{r\beta}u(x,t)+u(x,t), 0<x<1, t>0, \tag{28}$$

with solution

$$u(x,t)=e^{x}E_{\beta}\left(-t^{\beta}\right). \tag{29}$$

Remark 3. Setting $\alpha=2, p=2, q=r=1$, Eq. (19) reduces to classical telegraph equation and the same solution has been obtained by Kaya [6] using ADM.

Example 2 Consider the following non-homogeneous space-time fractional telegraph equation

$$D_x^{\alpha}u(x,t)=D_t^{p\beta}u(x,t)+D_t^{r\beta}u(x,t)+u(x,t)-2E_{\alpha}\left(x^{\alpha}\right)E_{\beta}\left(-t^{\beta}\right), 0<x<1, t>0, \tag{30}$$

where $\beta=1/q$, $p,q,r\in N$, $1<\alpha\le 2$, $1<p\beta\le 2$, $0<r\beta\le 1$, $D_t^{p\beta}\equiv D_t^{\beta}D_t^{\beta}...D_t^{\beta}$ (p times), $D_t^{r\beta}\equiv D_t^{\beta}D_t^{\beta}...D_t^{\beta}$ (r times), $D_x^{\alpha}, D_t^{\beta}$ are Caputo fractional derivatives defined by Eq. (2), p and r are even and initial conditions are given by

$$u(0,t)=E_{\beta}\left(-t^{\beta}\right), u_x(0,t)=0. \tag{31}$$

Applying the Laplace transform on the both sides of Eq. (30), subject to the initial condition (31) we have

$$L\left[D_x^{\alpha}u(x,t)\right]=L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u(x,t)\right]-2L\left[E_{\alpha}\left(x^{\alpha}\right)E_{\beta}\left(-t^{\beta}\right)\right],\quad 0<x<1, t>0, \tag{32}$$

Using the property of the Laplace transform, we have

$$L[u(x,t)]=\frac{E_{\beta}\left(-t^{\beta}\right)}{s}+\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u(x,t)\right]-\frac{2}{s^{\alpha}}L\left[E_{\alpha}\left(x^{\alpha}\right)E_{\beta}\left(-t^{\beta}\right)\right] \tag{33}$$

Operating with the Laplace inverse on both sides of Eq. (33) gives

$$u(x,t)=E_{\beta}\left(-t^{\beta}\right)-L^{-1}\left[\frac{2}{s^{\alpha}}L\left[E_{\alpha}\left(x^{\alpha}\right)E_{\beta}\left(-t^{\beta}\right)\right]\right]+L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u(x,t)\right]\right] \tag{34}$$

Now, applying the Iterative method,

Substituting (13) - (15) into (34) and applying (17), we obtain the components of the solution as follows:

$$\begin{aligned}u_0(x,t)&=E_{\beta}\left(-t^{\beta}\right)-L^{-1}\left[\frac{2}{s^{\alpha}}L\left[E_{\alpha}\left(x^{\alpha}\right)E_{\beta}\left(-t^{\beta}\right)\right]\right]\\&=E_{\alpha}\left(x^{\alpha}\right)E_{\beta}\left(-t^{\beta}\right)-3E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+1)}}{\Gamma\left(\alpha(k+1)+1\right)}\end{aligned} \tag{35}$$

$$\begin{aligned}u_1(x,t)&=L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u_0(x,t)\right]\right]\\&=3\left[E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+1)}}{\Gamma\left(\alpha(k+1)+1\right)}-3E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+2)}}{\Gamma\left(\alpha(k+2)+1\right)}\right]\end{aligned} \tag{36}$$

$$\begin{aligned}u_2(x,t)&=L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)\left(u_1(x,t)+u_0(x,t)\right)\right]\right]-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u_0(x,t)\right]\right]\\&=3^2\left[E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\left(\frac{x^{\alpha(k+2)}}{\Gamma\left(\alpha(k+2)+1\right)}\right)-3E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+3)}}{\Gamma\left(\alpha(k+3)+1\right)}\right]\end{aligned} \tag{37}$$

The solution in series form is then given by

$$\begin{aligned}u(x,t)&=u_0(x,t)+u_1(x,t)+u_2(x,t)+\ldots\ldots\ldots\ldots.\\&=\left[E_{\alpha}\left(x^{\alpha}\right)E_{\beta}\left(-t^{\beta}\right)-3E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+1)}}{\Gamma\left(\alpha(k+1)+1\right)}\right]\\&\quad+3\left[E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+1)}}{\Gamma\left(\alpha(k+1)+1\right)}-3E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+2)}}{\Gamma\left(\alpha(k+2)+1\right)}\right]\\&\quad+3^2\left[E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\left(\frac{x^{\alpha(k+2)}}{\Gamma\left(\alpha(k+2)+1\right)}\right)-3E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+3)}}{\Gamma\left(\alpha(k+3)+1\right)}\right]+\ldots\ldots\ldots\ldots.\\&=E_{\alpha}\left(x^{\alpha}\right)E_{\beta}\left(-t^{\beta}\right)\end{aligned} \tag{38}$$

The same result was obtained by Garg and Sharma [19] using ADM.

Remark 1. Setting $q=2, p=4, r=2$, Eq. (30) reduces to non-homogeneous space fractional telegraph equation, with the meaning of various symbols and parameters as given with Eq. (30), as follows

$$D_x^{\alpha}u(x,t)=D_t^2u(x,t)+D_t\,u(x,t)+u(x,t)-2E_{\alpha}\left(x^{\alpha}\right)e^{-t}, 0<x<1, t>0, \tag{39}$$

with solution

$$u(x,t)=E_{\alpha}\left(x^{\alpha}\right)e^{-t}. \tag{40}$$

Remark 2. Setting $\alpha=2$, Eq. (30) reduces to non-homogeneous time fractional telegraph equation, with the meaning of various symbols and parameters as given with Eq. (30), as follows

$$D_x^2u(x,t)=D_t^{p\beta}u(x,t)+D_t^{r\beta}u(x,t)+u(x,t)-2e^xE_{\beta}\left(-t^{\beta}\right), 0<x<1, t>0, \tag{41}$$

with solution

$$u(x,t)=e^xE_{\beta}\left(-t^{\beta}\right). \tag{42}$$

Remark 3. Setting $\alpha=2, q=2, p=4, r=2$, Eq. (30) reduces to non-homogeneous telegraph equation, with the meaning of various symbols and parameters as given with Eq. (30), as follows

$$D_x^2u(x,t)=D_t^2u(x,t)+D_t\,u(x,t)+u(x,t)-2e^xE_{1/2}\left(-t^{1/2}\right), 0<x<1, \quad t>0, \tag{43}$$

with solution

$$u(x,t)=e^xE_{1/2}\left(-t^{1/2}\right). \tag{44}$$

REFERENCES

1. A.Saadatmandi and M. Dehghan, A new operational matrix for solving fractional-order differential equations, *Computers & Mathematics with Applications,* 59(3) (2010) 1326–1336.
2. A.Sevimlican, An approximation to solution of space and time fractional telegraph equations by He's variational iteration method, *Mathematical Problems in Engineering*, (2010) Article ID 290631, 10 pages.
3. A.Wiman, Uber de fundamental satz in der theorie der funktionen $E_{\alpha}(x)$, *Acta Math.* 29 (1905) 191-201.
4. A.Yıldırım, He's homtopy perturbation method for solving the space and time fractional telegraph equations. *International Journal of Computer Mathematics*, 87(13) (2010) 2998 – 3006.
5. D.Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus, vol. 3 of Series on Complexity, Nonlinearity and Chaos,* World Scientific, Singapore, 2012.
6. D.Kaya, A new approach to the telegraph equation: an application of the decomposition method, *Bulletin of the Institute of Mathematics. Academia Sinica,* 28(1) (2000) 51–57.
7. F.A. Alawad, E.A. Yousif and A.I. Arab, A New Technique of Laplace variational iteration method for solving space –time fractional Telegraph equations. *International Journal of Differential Equations,* (2013) Article ID 256593, 10 pages.

8. H. Jafari, C. M. Khalique, and M. Nazari, Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion-wave equations, *Applied Mathematics Letters*, 24 (2011) 1799–1805.
9. H. Jafari, Iterative Methods for solving system of fractional differential equations Ph.D. thesis, Pune University, 2006.
10. H. Jafari, M. Nazari, D. Baleanu, and C. M. Khalique, A new approach for solving a system of fractional partial differential equations, *Computers & Mathematics with Applications,* 66(5) (2013) 838–843.
11. I.Podlubny, *Fractional Differential Equations, vol. 198,* Academic Press, New York, NY, USA, 1999.
12. J.Biazar, H.Ebrahimi and Z. Ayati, An approximation to the solution of telegraph equation by variational iteration method, *Numerical Methods for Partial Differential Equations*, 25(4) (2009) 797–801.
13. J.L.Wu, A wavelet operational method for solving fractional partial differential equations numerically, *Applied Mathematics and Computation,* 214(1) (2009) 31–40.
14. J.Sabatier, O.P.Agrawal and J.A.Tenreiro Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering,* Springer, 2007.
15. J.H.He, S.K.Elagan and Z.B.Li, Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus, *Physics Letters A*, 376(4) (2012) 257– 259.
16. K.S.Miller and B.Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* John Wiley & Sons, New York, USA, 1993.
17. L.Yan, Numerical Solutions of Fractional Fokker-Planck Equations Using Iterative Laplace Transform Method, *Abstract and Applied Analysis*, (2013) Article ID 465160, 7 pages.
18. M.D.Ortigueira, *Fractional calculus for scientists and engineers,* Springer 2011.
19. M.Garg and A.Sharma, Solution of space-time fractional telegraph equation by Adomian decomposition method, *Journal of Inequalities and Special Functions*, 2(1) (2011) 1–7.
20. M.Rehman and R.Ali Khan, The Legendre wavelet method for solving fractional differential equations, *Communications in Nonlinear Science and Numerical Simulation,* 16(11) (2011) 4163–4173.
21. M.Y.Ongun, The Laplace adomian decomposition method for solving a model for HIV infection of CD4+ cells, *Mathematical and Computer Modelling,* 53(5-6) (2011) 597–603.
22. N.H.Sweilam, M.M.Khader and R.F.Al-Bar, Numerical studies for a multi-order fractional differential equation, *Physics Letters A,* 371(1-2) (2007) 26–33.
23. R.Hilfer, *Applications of Fractional Calculus in Physics,* World Scientific, Singapore, 2000.
24. R.W.Ibrahim, Fractional complex transforms for fractional differential equations, *Advances in Difference Equations,* (2012) article 192.
25. S.Bhalekar and V.Daftardar-Gejji, Solving evolution equations using a new iterative method, Numerical *Methods for Partial Differential Equations,* 26(4) (2010) 906–916.

26. S.Das, Analytical solution of a fractional diffusion equation by variational iteration method, *Computers & Mathematics with Applications*, 57(3) (2009) 483–487.
27. S.Momani, Analytic and approximate solutions of the space- and time-fractional telegraph equations, *Applied Mathematics and Computation*, 170(2) (2005) 1126–1134.
28. S. Zhang and H.Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, Physics *Letters A*, 375(7) (2011) 1069–1073.
29. U.Lepik, Solving fractional integral equations by the Haar wavelet method, *Applied Mathematics and Computation*, 214(2) (2009) 468–478.
30. V.D.-Gejji and H.Jafari, An iterative method for solving nonlinear functional equations, *J. Mathematical Analysis and Applications*, 316(2) (2006) 753–763.
31. V.Daftardar-Gejji and S. Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, *Computers &Mathematics with Applications*, 59(5) (2010) 1801–1809.
32. V.S.Erturk and S.Momani, Solving systems of fractional differential equations using differential transform method, *Journal of Computational and Applied Mathematics*, 215(1) (2008) 142–151.
33. Y.Li and N.Sun, Numerical solution of fractional differential equations using the generalized block pulse operational matrix, *Computers & Mathematics with Applications*, 62(3) (2011)1046–1054.
34. Y. Li, Solving a nonlinear fractional differential equation using Chebyshev wavelets, *Communications in Nonlinear Science and Numerical Simulation*, 15(9) (2010) 2284–2292.
35. Y.Liu, Approximate solutions of fractional nonlinear equations using homotopy perturbation transformation method, *Abstract and Applied Analysis*, (2012), Article ID 752869, 14 pages.
36. Y.Liu, Variational homotopy perturbation method for solving fractional initial boundary value problems, *Abstract and Applied Analysis*, (2012) Article ID 727031, 10 pages.
37. Z.Odibat and S.Momani, A generalized differential transform method for linear partial differential equations of fractional order. *Applied Mathematics Letters* 21 (2008), 194–199.