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Spectral Radius and Some Hamiltonian Properties of Graphs

Rao Li
Department of Mathematical Sciences
University of South Carolina Aiken
Aiken, SC 29801, USA. Email: raol@usca.edu

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Abstract. Using upper bounds for the spectral radius of graphs established by Cao, we in this note present sufficient conditions which are based on the spectral radius for some Hamiltonian properties of graphs.

Keywords: Spectral radius, Hamiltonian property

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use n, e, δ, and Δ to denote its order $|V|$, size $|E|$, minimum degree, and maximum degree, respectively. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. A graph G is called Hamilton-connected if for each pair of vertices in G there is a Hamiltonian path between them. The eigenvalues of a graph are defined as the eigenvalues of its adjacency matrix. The largest eigenvalue of a graph G, denoted $\rho(G)$, is called the spectral radius of G. If G and H are two vertex-disjoint graphs, we use $G \vee H$ to denote the join of G and H. We use $C(n,r)$ to denote the number of r-combinations of a set with n elements.

Spectral invariants have been used to study a variety of properties of graphs. See, for instance, [9] [5] [6] [7]. In this note, we will present sufficient conditions which are based on the spectral radius for some Hamiltonian properties of graphs. The results are as follows.

Theorem 1. Let G be a connected graph of order $n \geq 3$. If G is not $K_1 \vee (K_1 + K_{n-2})$ or $K_2 \vee (3K_1)$ and

$$\rho > \sqrt{n^2 - 3n + 2 - \delta(n-1) + (\delta - 1)\Delta},$$

then G is Hamiltonian.

Theorem 2. Let G be a connected graph of order $n \geq 4$. If G is not $K_1 \vee (K_{n-3} + (2K_1))$, $K_2 \vee ((3K_1) + K_2)$, or $K_4 \vee (6K_1)$ and

$$\rho > \sqrt{n^2 - 5n + 8 - \delta(n-1) + (\delta-1)\Delta},$$

then G is traceable.

Theorem 3. Let G be a connected graph of order $n \geq 3$. If

$$\rho > \sqrt{n^2 - 3n + 6 - \delta(n-1) + (\delta-1)\Delta},$$

then G is Hamilton-connected.

Theorem 4. Let G be a connected graph of order n with $n \geq 6\delta$. If

$$\rho > \sqrt{(n-\delta)(n-\delta-1) + 2\delta^2 - \delta(n-1) + (\delta-1)\Delta},$$

then G is Hamiltonian.

Theorem 5. Let G be a 2-connected graph of order $n \geq 12$. If G is not $K_2 \vee ((2K_1) + K_{n-4})$ and

$$\rho > \sqrt{n^2 - 5n + 12 - \delta(n-1) + (\delta-1)\Delta},$$

then G is Hamiltonian.

Theorem 6. Let G be a 3-connected graph of order $n \geq 18$. If G is not $K_3 \vee ((3K_1) + K_{n-6})$ and

$$\rho > \sqrt{n^2 - 7n + 28 - \delta(n-1) + (\delta-1)\Delta},$$

then G is Hamiltonian.

Theorem 7. Let G be a k-connected graph of order $n \geq 3$. If

$$\rho > \sqrt{n^2 - n - (k+1)(n-k-1) - \delta(n-1) + (\delta-1)\Delta},$$

then G is Hamiltonian.

Notice that $-\delta(n-1) + (\delta-1)\Delta \leq -\delta(n-1) + (\delta-1)(n-1) = -n+1$. Thus Theorems $1 - 7$ have the following Corollaries 1 - 7, respectively.

Corollary 1. Let G be a connected graph of order $n \geq 3$. If G is not $K_1 \vee (K_1 + K_{n-2})$ or $K_2 \vee (3K_1)$ and

$$\rho > \sqrt{n^2 - 4n + 3},$$

then G is Hamiltonian.

Corollary 2. Let G be a connected graph of order $n \geq 4$. If G is not $K_1 \vee (K_{n-3} + (2K_1))$, $K_2 \vee ((3K_1) + K_2)$, or $K_4 \vee (6K_1)$ and

$$\rho > n - 3,$$

then G is traceable.

Corollary 3. Let G be a connected graph of order $n \geq 3$. If

$$\rho > \sqrt{n^2 - 4n + 7},$$

then G is Hamilton-connected.

Corollary 4. Let G be a connected graph of order n with $n \geq 6\delta$. If

$$\rho > \sqrt{(n-\delta)(n-\delta-1)+2\delta^2-n+1},$$

then G is Hamiltonian. \

Corollary 5. Let G be a 2-connected graph of order $n \geq 12$. If G is not $K_2 \vee ((2K_1) + K_{n-4})$ and

$$\rho > \sqrt{n^2-6n+13},$$

then G is Hamiltonian.

Corollary 6. Let G be a 3-connected graph of order $n \geq 18$. If G is not $K_3 \vee ((3K_1) + K_{n-6})$ and

$$\rho > \sqrt{n^2-8n+29},$$

then G is Hamiltonian.

Corollary 7. Let G be a k-connected graph of order $n \geq 3$. If

$$\rho > \sqrt{(n-1)^2-(k+1)(n-k-1)},$$

then G is Hamiltonian.

2. Some results

In order to prove the theorems above, we need the following results as our lemmas. The following lemma is Corollary 4 on Page 60 in [2].

Lemma 1. Let G be a graph of order $n \geq 3$. If

$$e \geq C(n-1,2)+1,$$

then G is Hamiltonian unless G is $K_1 \vee (K_1 + K_{n-2})$ or $K_2 \vee (3K_1)$.

Lemma 2. ([8]) Let G be a connected graph of order $n \geq 4$. If

$$e \geq C(n-2,2)+2,$$

then G is traceabl unless G is $K_1 \vee (K_{n-3} + (2K_1))$, $K_2 \vee ((3K_1) + K_2)$, or $K_4 \vee (6K_1)$.

The following lemma is Theorem 17 on Page 220 in [1].

Lemma 3. Let G be a graph of order $n \geq 3$. If

$$e \geq C(n-1,2)+3,$$

then G is Hamilton-connected.

The following lemma is Exercise 4.2.8 on Page 61 in [2].

Lemma 4. Let G be a graph of order n with $n \geq 6\delta$. If $e > C(n-\delta,2)+\delta^2$, then G is Hamiltonian.

Lemma 5. ([3]) Let G be a 2-connected graph of order $n \geq 12$. If $e \geq C(n-2,2)+4$, then G is Hamiltonian or $G = K_2 \vee ((2K_1) \cup K_{n-4})$.

Lemma 6. ([3]) Let G be a 3-connected graph of order $n \geq 18$. If $e \geq C(n-3,2)+9$,

then G is Hamiltonian or $G = K_3 \vee ((3K_1) \cup K_{n-6})$.

Lemma 7. ([3]) Let G be a k-connected graph of order n. If $e \geq C(n,2) - \frac{(k+1)(n-k-1)}{2} + 1$, then G is Hamiltonian.

Lemma 8. ([4]) Let G be a graph of order n and size e with minimum degree $\delta \geq 1$ and maximum degree Δ. Then

$$\rho \leq \sqrt{2e - \delta(n-1) + (\delta - 1)\Delta}$$

with equality if and only if G is regular, a star plus copies of K_2, or a complete graph plus a regular graph with smaller degree of vertices.

3. Proofs of Theorems

Next we will prove Theorems $1 - 7$.

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Suppose that G is not Hamiltonian. Then, from Lemma 1, we have that $e \leq C(n-1,2)$. By Lemma 8, we have

$$\rho \leq \sqrt{2e - \delta(n-1) + (\delta - 1)\Delta} = \sqrt{n^2 - 3n + 2 - \delta(n-1) + (\delta - 1)\Delta},$$

which is a contradiction. This completes the proof of Theorem 1. ■

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 2. Suppose that G is not traceable. Then, from Lemma 2, we have that $e \leq C(n-2,2) + 1$. By Lemma 8, we have

$$\rho \leq \sqrt{2e - \delta(n-1) + (\delta - 1)\Delta} = \sqrt{n^2 - 5n + 8 - \delta(n-1) + (\delta - 1)\Delta},$$

which is a contradiction. This completes the proof of Theorem 2. ■

Proof of Theorem 3. Let G be a graph satisfying the conditions in Theorem 3. Suppose that G is not Hamilton-connected. Then, from Lemma 3, we have that $e \leq C(n-1,2) + 2$. By Lemma 8, we have

$$\rho \leq \sqrt{2e - \delta(n-1) + (\delta - 1)\Delta} = \sqrt{n^2 - 3n + 6 - \delta(n-1) + (\delta - 1)\Delta},$$

which is a contradiction. This completes the proof of Theorem 3. ■

Proof of Theorem 4. Let G be a graph satisfying the conditions in Theorem 4. Suppose that G is not Hamiltonian. Then, from Lemma 4, we have that $e \leq C(n-\delta,2) + \delta^2$. By Lemma 8, we have

$$\rho \leq \sqrt{2e - \delta(n-1) + (\delta - 1)\Delta}$$

$$= \sqrt{(n-\delta)(n-\delta-1) + 2\delta^2 - \delta(n-1) + (\delta - 1)\Delta},$$

which is a contradiction. This completes the proof of Theorem 3. ■

Proof of Theorem 5. Let G be a graph satisfying the conditions in Theorem 4. Suppose that G is not Hamiltonian. Then, from Lemma 5, we have that $e \leq C(n-2,2) + 3$. By Lemma 8, we have

$$\rho \le \sqrt{2e - \delta(n-1) + (\delta-1)\Delta} = \sqrt{n^2 - 5n + 12 - \delta(n-1) + (\delta-1)\Delta},$$

which is a contradiction. This completes the proof of Theorem 5. ■

Proof of Theorem 6. Let G be a graph satisfying the conditions in Theorem 5. Suppose that G is not Hamiltonian. Then, from Lemma 6, we have that $e \le C(n-3,2) + 8$. By Lemma 8, we have

$$\rho \le \sqrt{2e - \delta(n-1) + (\delta-1)\Delta} = \sqrt{n^2 - 7n + 28 - \delta(n-1) + (\delta-1)\Delta},$$

which is a contradiction. This completes the proof of Theorem 6. ■

Proof of Theorem 7. Let G be a graph satisfying the conditions in Theorem 6. Suppose that G is not Hamiltonian. Then, from Lemma 7, we have that $e \le C(n,2) - \frac{(k+1)(n-k-1)}{2}$. By Lemma 8, we have

$$\begin{aligned}\rho &\le \sqrt{2e - \delta(n-1) + (\delta-1)\Delta} \\ &\le \sqrt{n^2 - n - (k+1)(n-k-1) - \delta(n-1) + (\delta-1)\Delta},\end{aligned}$$

which is a contradiction. This completes the proof of Theorem 7. ■

REFERENCES

1. C.Berge, *Graphs and Hypergraphs*, the second edition, American Elsevier Publishing Company, 1976.
2. J.A.Bondy and U.S.R.Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York, 1976.
3. W.Byer and D.Smeltzer, Edge bounds in nonhamiltonian k-connected graphs, *Discrete Math.*, 307 (2007) 1572 – 1579.
4. D.Cao, Bounds on eigenvalues and chromatic numbers, *Linear Algebra Appl.*, 270 (1998) 1 – 13.
5. M.Fiedler, Algebraic connectivity of graphs, *Czechoslovak Math. J.*, 23 (1973) 298-305.
6. M.Fiedler and V.Nikiforov, Spectral radius and Hamiltonicity of graphs, *Linear Algebra Appl.*, 432 (2010) 2170 - 2173.
7. R.Li, Spectral conditions for a graph to be k-connected, *Annuls of Pure and Applied Math.*, 8 (2014) 11 – 14.
8. M.Lu, H.Liu and F.Tian, Spectral radius and Hamiltonian graphs, *Linear Algebra Appl.*, 437 (2012) 1670 – 1674.
9. H.Wilf, The eigenvalues of a graph and its chromatic numbers, *J. London Math. Soc.*, 42 (1967) 330 – 332.