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Relation Between Convexity Number and Independence Number of a Graph

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Abstract. The convexity number denoted by k in a connected graph G is the maximum cardinality of a proper convex set in G. Here in this paper graphs for which the independence number $\beta_0(G)$ of a graph G where $\beta_0(G) = k$, $\beta_0(G) < k$ and $\beta_0(G) > k$ are completely characterised. Also graphs for which $\omega(G) = k$ are characterised. Construction of graphs with prescribed $\beta_0(G)$ and k are presented.

Keywords: Convex set, Convexity number, Independence number

AMS Mathematics Subject Classification (2010): 05C12

1. Introduction

By a graph we mean undirected graph without loops or multiple edges.For terminology and notation not given here,the reader may refer to **Error! Reference source not found.**.

For a connected graph G a subset S of vertices of G is said to be a convex set if for any two vertices u, v of S, S contains all the vertices of every $u - v$ shortest path in G. The convexity number of G is the maximum cardinality of a proper convex set of G. For a graph $G = (V, E)$, a subset T of V is independent if no two vertices in T are adjacent. The independence number $\beta_0(G)$ is the maximum cardinality of an independent set in G. Convexity number of a graph have been studied in [2,3,4,6]. Clique number of a graph is the maximum cardinality of a clique in G denoted by $\omega(G)$.

Example 1.1. $\beta_0(C_6) = con(C_6)$, $\beta_0(C_5) < con(C_5)$, $\beta_0(K_{2,3}) > con(K_{2,3})$. Thus there are graphs for which $\beta_0(G) = con(G)$, $\beta_0(G) < con(G)$, $\beta_0(G) > con(G)$. For prescribed values of β_0 and $con(G) = k$ graphs can be constructed.

2. Realization theorems

Theorem 2.1. *For every pair β_0, k of integers there exists a non-complete connected graph G with $\beta_0 =$ Independence number and $con(G) = k$.*

Proof: For a non-complete connected graph G either $2 \le \beta_0 \le k \le n - 1$ or $2 \le k < \beta_0 \le n - 1$ holds.

(a) Consider $2 \leq \beta_0 \leq k \leq n-1$.
Case (i) $\beta_0 = k$.
When $\beta_0 = k = 2$, C_4 is the required graph. When $\beta_0 = k = n-1$, $K_{1,n-1}$ is the required graph. Therefore $3 \leq \beta_0 = k \leq n-2$, and C_{2k} has $\beta_0 = k$.

Case (ii) $\beta_0 < k$.
When $k = 2$, K_3 has $\beta_0 = 1$. For $\beta_0 = 1$ and any $k > 1$, K_{k+1} is the required graph. When $k = n-1$, K_n has $\beta_0 = 1$. For $\beta_0 = 2$ and any $k > 2$, $K_{1,1,\cdots,1,2}$ (a k - partite graph) is the required graph. Therefore for $3 \leq \beta_0 < k \leq n-2$, consider $F = K_{k-\beta_0+1} + \overline{K_{\beta_0-1}}$. Now order of the graph F is k. Let $\overline{K_{\beta_0-1}} = \{v_1, v_2, \cdots, v_{\beta_0-1}\}$. Let $K_{k-\beta_0+1} = \{u_1, u_2, \cdots, u_{k-\beta_0+1}\}$. For $n = k+1$, construct F_1 from F by joining a pendant u to one of the vertices u_1 of F .

Clearly $\beta_0(F_1) = \{v_1, v_2, \cdots, v_{\beta_0-1}, u\}$ and $con(F_1)$ is the whole graph F_1 with k vertices .

For $n = k+2$, construct F_2 from F such that u_1u, uv and vv_1 are edges.

$\beta_0(F_2) = \{v_1, v_2, \cdots, v_{\beta_0-1}, u\}$ and $con(F_2) = k$ vertices as neither u nor v can be in $con(F_2)$.

For $n = k+3$, construct F_3 from F such that u_1u, uv, vu_2, wv and wv_1 are edges . $\beta_0(F_3) = \{v_1, v_2, \cdots, v_{\beta_0-1}, u$ or $v\}$ and $con(F_3)$ has k vertices.

Now let $n \geq k+4$. Consider $K_{k-\beta_0+m}$ and $\overline{K_{\beta_0-m}}$. Let $V(K_{k-\beta_0+m}) = \{u_1, u_2, \cdots, u_{k-\beta_0+m}\}$. Let $V(\overline{K_{\beta_0-m}}) = \{v_1, v_2, \cdots, v_{\beta_0-m}\}$. Obtain G of order n from $K_{k-\beta_0+m}$ and $\overline{K_{\beta_0-m}}$ by the following construction steps . Let $H = K_{k-\beta_0+m} + \overline{K_{\beta_0-m}}$. Let $v \in V(G)$ be such that $v \notin V(H)$ and $vv_1 \in E(G)$. Now split $n-k-1$ vertices of G among m partitions. Suppose $n-k-1$ is even and equal to $2m$ then m K_2s can be obtained. Let one of the K_2 be uw and let $uv, vv_1, uu_1, wv_1, uv_1, wu_2$ be in $E(G)$ .For rest of the K_2s, one of the vertex must be joined to u, other to u_2 and both to v_1.

For this graph neither v nor the K_2s are included in the convexity number as they all come together. $con(G) = \{v_1, v_2, \cdots, v_{\beta_0-m}, u_1, u_2, \cdots, u_{k-\beta_0+m}\}$ and $\beta_0(G) = \{v_1, v_2, \cdots, v_{\beta_0-m}, m$ vertices joined to$u_2\}$. Suppose $n-k-1 \neq 2m$ and greater than $2m$ then rest of the vertices must be in P_3 , and one vertex of P_3 must be joined to u, rest of the two vertices joined to u_2 and all the three vertices to v_1.

Suppose $n-k-1 < 2m$ then K_2 is transformed to K_1 in some partitions . Here K_1 should be joined to u as well as to v_1. In all the above cases $\beta_0(G) = \beta_0$ vertices and $con(G) = k$ vertices.

(b) Consider $2 \leq k < \beta_0 \leq n-1$. When $k = 2, K_{2,\beta_0}$ where $2 \leq \beta_0 \leq n-1$. When $\beta_0 = n-1, K_{2,n-1}$ is the required graph. Therefore $3 \leq k < \beta_0 \leq n-2$. If $n < 9$ then for $k = 3$ one of the graphs $K_{2,2,4}, K_{1,2,5}, K_{1,1,6}$ holds good. For $k = 4, K_{1,1,2,5}$ is the required graph with $\beta_0 = 5$. If $n \geq 9$ the construction is as follows. Let $V(K_{k+1}) = \{u_1, u_2, \cdots, u_{k+1}\}$, $V(\overline{K_{\beta_0-1}}) = \{v_1, v_2, \cdots, v_{\beta_0-1}\}$ and $V(P_{n-(\beta_0+k)}) = \{w_1, w_2, \cdots, w_{n-(\beta_0+k)}\}$. Let $H = K_{k+1} - u_1u_2$. Now join the vertices of $\overline{K_{\beta_0-1}}$ to u_1 and u_2. Order of H is $k+\beta_0$. Rest of $n-(k+\beta_0)$ vertices are formed as a path. One of the end vertex w_1 (say) of the path is joined to v_1 and v_2. Now w_2 is joined to v_3, w_3 to v_4 and so on untill w_i is joined to v_{β_0-1}. Here w_{i+1} is joined to v_1 and the process repeated till all the vertices of the path exhaust. Let the resulting graph constructed be G. Clearly $con(G) = k$ consisting of vertices $\{u_3, u_4, \cdots, u_{k+1}\}$ with exactly one of u_1 or u_2. Also independence number of G consists of vertices $\{v_1, v_2, \cdots, v_{\beta_0 - 1}\}$ with exactly one vertex of K_{k+1}.

3. Graph characterisation

Lemma 3.1. *Let S be a subset of $V(G)$. S is a maximum convex set if and only if S is not a geodetic set in $\langle S \cup \{u\} \rangle$ for any $u \in V - S$ and $S \cup T$ is a geodetic set in $< G_1 >$ for any T in $V - S$, for some G_1 , $G_1 = G$ for some $T \subset (V - S)$.*

Proof: Let S is a maximum convex set in G. Then there is no $u \in V - S$ such that u lies on a $x - y$ geodesic for any x, y in S that is S is not a geodetic set in $\langle S \cup \{u\} \rangle$ for any $u \in V - S$. Since S is a maximum convex set, S is not contained in any proper convex set of G. Therefore $S \cup T$ for any $T \subset V - S$ is not a convex set. Thus $S \cup T$ for any $T \subset V - S$ is a geodetic set in $< G_1 >$ for some G_1. If $G_1 \neq G$ for any $T \subset V - S$ then G_1 becomes a proper convex set with cardinality greater than S. Thus a contradiction to S a maximum convex set of G.Therefore $S \cup$T is a geodetic set for any $T \subset V - S$ in $\langle G_1 \rangle$ for some $G_1 = G$. Conversely suppose S satisfies the condition given in hypothesis then clearly S is a maximum convex set of G.

Theorem 3.2 *Let G be a non-complete connected graph of order n. Then $\beta_0 = k$ iff G is one of the following graphs.*

(i)$K_{1,n-1}$

(ii)C_n where n is even.

(iii)$K_{n_1,n_2,\cdots,n_r}$ with max.$|V_i| = r$.

(iv) $r - partitegraph$ with $\beta_0(G) = r$ and a set S with r vertices having property in 3.1.

Proof: Let G be a connected graph. Let $k = \beta_0$.

Acyclic: Let $v \in V(G)$. Suppose $\Delta \neq n - 1$. Then there exists a $u \in V(G)$ such that $uv \notin E(G)$. Clearly any two neighbors of v are non-adjacent. Therefore $\beta_0(G) = deg(v)$ and $k = n - 1$. But $deg(v) < n - 1$ which is a contradiction. Hence $\Delta = n - 1$. Thus $G = K_{1,n-1}$.

Cyclic: If G has a single cycle of order $m < n$ then rest of $n - m$ vertices are pendants or paths joined to some vertex of the cycle. Therefore $k = n - 1$ but $\beta_0 \neq n - 1$. Thus $G = C_{n=even}$.

If G has multicycles then as we know any graph can be transformed to an r-partite graph with maimum cardinality of $V_i = \beta_0(G)$, G is either a complete r-partite or non-complete r-partite. Clearly if G is complete r-partite then $k = r$. Let these set of r vertices be a set S (say). $k = r = \beta_0$ if and only if maximum cadinality of V_i is r. If G is transformed to a non-complete r-partite graph then let S be a subset of $V(G)$ with r vertices. Suppose G has an independent set with $\beta_0 = r$ vertices. $k = r$ if and only if S satisfies the hypothesis of 3.1. Conversely if G is one of the following in the hypothesis then $k = \beta_0$.

Theorem 3.3. *Let G be a noncomplete connected graph of order n. Then $\beta_0 > k$ if and only if G is an r - partite graph with $max.|Vi| > r$ and a subset S with r vertices satisfying the hypothesis of 3.1.*

Proof: Let $\beta_0 > k$. Then G has the following properties. G has a cycle. G has no pendant and $deg(u) \geq 2$ for all u. Let $max.|Vi| > r$. By 3.2 we are through.

Theorem 3.4. *For a non-complete connected graph G of order n , $\beta_0 < k$ if and only if G is one of the following graph.*

$G = acyclic$ with $\Delta < n - 1$, $C_{n=odd}$, $K_{n_1,n_2,\cdots,n_r}$ with $max.|V_i| < r$ or non-complete r-partite with $max.|V_i| < r$ and a subset S of cardinality r satisfying the hypothesis in 3.1.

Proof: If G is acyclic then clearly $\Delta < n - 1$. If G is cyclic then the proof is same as in 3.2.

Theorem 3.5. *Let G be a non-complete connected graph. Then clique number $\omega = k$ if and only if G is a complete r-partite graph or G has a maximum clique set S satisfying hypothesis of 3.1 .*

Proof: Clearly $\omega = k \neq 1$. Let $\omega(G) = k$. Then G is not acyclic. If G is unicyclic then $\omega = 2$ but $k > 2$ for $n \geq 5$. If G is a complete graph then $\omega(G) = n$ but $k = n - 1$. Let S be a set containing maximum clique set . Suppose S does not satisfy the hypothesis of 3.1 then there exists a subset T in $(V - S)$ such that $S \cup T$ is a geodetic set in $G_1 \neq G$. Clearly $\omega(G) < k$. Thus a contradiction. Conversely if a maximum clique set S of G satisfies the hypothesis of the theorem then $\omega(G) = k$.

4. Conclusion

In this paper, I have compared two different numbers namely convexity and independence number of a graph. Both the parameters are of hereditary property. I have constructed and characterised graphs using these two parameters. One can similarly compare two or more parameters of same nature. I shall explore the above parameters on product graphs as a part of my future work.

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