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Properties of D_μ-Compact Spaces in Generalized Topological Spaces

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Abstract. The aim of this paper is to introduce a further generalization of compactness in generalized topological spaces. We define and study the concept of D_μ-compact spaces in generalized topological spaces. A space(X, μ) is D_μ-compact if every D_μ-cover of X has a finite D_μ-sub cover. Basic properties and characterizations of D_μ-compact space are established. D_μ-compactness in subspaces, products of generalized topological spaces and in μ-D_2 spaces are also investigated.

Keywords: Generalized topological spaces, D_μ-set, D_μ-compactness, base, (μ, η)-continuous functions, μ-D_2 space

Mathematics Subject Classification (2010): 54A05,54B05,54B10,54C05,54D10,54D30, 54D70

1. Introduction

The concept of generalized topological space (GTS) was introduced by Csaszar [3] is one of the most important developments of general topology in recent years. In the literature, several kinds of compactness in GTS have been introduced; e.g. γ-compact spaces in [2], μ- compact space in [10]. The purpose of the present paper is to show that the concept of a compact space can be generalized by replacing open sets by D_μ-sets. In section 2, all basic definitions and preliminaries useful for subsequent sections are collected. In section 3, we give the definition of D_μ-compact space and establish some of the basic properties and characterizations. In sections 4 and 5, we examine the basic theorems about D_μ-compactness in subspaces, μ-D_2 spaces and inproducts of generalized topological spaces.

2. Preliminaries

We recall some basic definitions and notations of most essential concepts needed in the following. Let X be a non-empty set and denote exp(X) the power set of X. According to [3], a collection $\mu \subset \exp(X)$ of subsets of X is called a generalized topology (GT) on X and (X, μ) is called a generalized topological space (GTS) if μ has the following properties

i. $\emptyset \in \mu$
ii. Any union of elements of μ belongs to μ.

Let μ be a GT on a set $X \neq \emptyset$. Note that $X \in \mu$ must not hold; if $X \in \mu$ then we say that the GT μ is strong [5]. Let M_μ denote the union of all elements of μ; of course, $M_\mu \in \mu$, and $M_\mu = X$ if and only if μ is a strong GT. Throughout this paper a space (X, μ) or simply X will always mean a strong generalized topological space with the strong generalized topology μ. A subset U of X is called μ-open if $U \in \mu$. A subset V of X is called μ-closed if $X \setminus V \in \mu$. A subset U of X is called μ-clopen if U is both μ-open and μ-closed.Let (X, μ) and (Y, η) be two GTS's, then a function $f: (X, \mu) \rightarrow (Y, \eta)$ is said to be (μ, η)-continuous (see [3]) if and only if $U \in \eta \Rightarrow f^{-1}(U) \in \mu$. f is said to be (μ, η)-open [4] if and only if $U \in \mu \Rightarrow f(U) \in \eta$.Throughout this paper, all mappings are assumed to be onto. Let (X, μ) be a GTS and Y be a subset of X, then the collection $\mu_{/Y} = \{U \cap Y: U \in \mu\}$ is a GT on Y called the subspace generalized topology and $(Y, \mu_{/Y})$ is called a sub space of X [11]. Let $\mathfrak{B} \subset \exp(X)$ satisfy $\emptyset \in \mathfrak{B}$. Then all unions of some elements of $\mathfrak{B}$ constitute a GT $\mu(\mathfrak{B})$ called the GT generated by $\mathfrak{B}$, and $\mathfrak{B}$ is said to be a base for $\mu(\mathfrak{B})$ [7]. For a better understanding of developments in GTS one can refer to [1,6, 13, 14]

Definition 2.1. [12] A subset A of a space (X, μ) is called a D_μ-set if there are two sets $U, V \in \mu$ such that $U \neq X$ and $A = U \setminus V$. Letting $A = U$ and $V = \emptyset$ in the above definition, it is easy to see that every proper μ-open set U is a D_μ-set.

Definition 2.2. [12] A space (X, μ) is called μ-D_2 if for any pair of distinct points x and y of X, there exist disjoint D_μ-sets U and V of X containing x and y, respectively.

Definition 2.3. A point $x \in X$ is called a D_μ-cluster point of A if $U \cap (A \setminus \{x\}) \neq \emptyset$ for each $U \in D_\mu$ with $x \in U$.

Definition 2.4. [8] Let $K \neq \emptyset$ be an index set, $X_k \neq \emptyset$ for $k \in K$ and (X_k, μ_k), $k \in K$, a class of GTS's. $X = \prod_{k\in K} X_k$ is the Cartesian product of the sets X_k. Let us consider all sets of the form $\prod_{k\in K} B_k$ where $B_k \in \mu_k$ and, with the exception of a finite number of indices k, $B_k = M_{\mu_k}$. We denote by $\mathfrak{B}$ the collection of all these sets. Clearly $\emptyset \in \mathfrak{B}$ so that we can define a GT $\mu = \mu(\mathfrak{B})$ having $\mathfrak{B}$ for base. We call μ the *product* of the GT's μ_k and denote it by $P_{k\in K}\mu_k$. The GTS (X, μ) is called the product of the GTS's (X_k, μ_k), $k \in K$. Denote $M_k = M_{\mu_k} = \cup \mu_k$ and $M = \cup \mu$. We denote by p_k the projection $p_k : X \rightarrow X_k$ and $x_k = p_k(x)$ for each $x \in X$.

Lemma 2.1. [8,Lemma 2.6**]** $M = \prod_{k\in K} M_k$.

Lemma 2.2. [8,Proposition 2.7**]** If every μ_k is strong, then μ is strong and p_k is (μ, μ_k)-continuous for $k \in K$.

3. D_μ-Compact spaces in GTS's

Definition 3.1. Let (X, μ) be a GTS. A collection $\mathfrak{A}$ of subsets of X is said to be a cover of X if the union of the elements of $\mathfrak{A}$is equal to X.
It is called a D_μ–cover of X if its elements are D_μ-subsets of X

Definition 3.2. Let (X, μ) be a GTS. A sub cover of a cover $\mathfrak{A}$ is a sub collection $\mathfrak{F}$ of $\mathfrak{A}$ which itself is a cover.

Definition 3.3. Let (X, μ) be a GTS. Then (X, μ) is said to be a D_μ-compact space if and only if each D_μ-cover of X has a finite D_μ-sub cover.

Definition 3.4. [10] Let (X, μ) be a GTS. Then (X, μ) is said to be a μ-compact space if and only if each μ-open cover of X has a finite μ-open sub cover.

Theorem 3.1. If (X, μ) is a finite GTS. Then X is D_μ-compact.
Proof: Let $X = \{x_1, x_2, \ldots, x_n\}$. Let $\mathfrak{A}$ be a D_μ-covering of X. Then each element in X belongs to one of the members of $\mathfrak{A}$, say $x_1 \in G_1$, $x_2 \in G_2$, … , $x_n \in G_n$, where $G_i \in \mathfrak{A}$, $G_i = U_i \setminus V_i$, $U_i, V_i \in \mu, U_i \neq X$, i = 1,2, … , n. Since each G_i is a D_μ-set, the collection $\{G_1, G_2, \ldots, G_n\}$ is a finite sub collection of D_μ-sets which cover X. Hence X is D_μ-compact. ■

Theorem 3.2. If (X, μ) is a GT generated by singleton subsets of X. Then any infinite subset of X is not D_μ-compact.
Proof: Suppose U is an infinite subset of the GT (X, μ) generated by singleton subsets of X. Consider the collection $\mathfrak{A} = \{\{x\}: x \in U\}$ of singleton subsets of U. $\mathfrak{A}$ is a D_μ-covering of U, since $\{x\} \in \mu \Rightarrow \{x\} \in D_\mu, \forall x \in X$ and $\bigcup_{x \in U} \{x\} = U$. Also $\mathfrak{A}$ is infinite since U is infinite. Then there is no finite sub collection of $\mathfrak{A}$ which covers U. So U is not D_μ-compact. ■

Theorem 3.3. If (X, μ) is a GTS generated by singleton subsets of X and $U \subset X$. Then U is D_μ-compact if and only if U is finite.

Theorem 3.4. Let (X, μ) be a GTS, where $\mu = \{U \subset X : X \setminus U$ is either finite or is all of $X\}$. Then X is D_μ-compact.
Proof: Let $\mathfrak{A}$ be a D_μ-cover of X. Let A be an arbitrary member of $\mathfrak{A}$. Since $A \in D_\mu \Rightarrow A = (U \setminus V)$ where $U, V \in \mu$, $U \neq X$. Now $U \in \mu \Rightarrow X \setminus U$ is finite. Let $X \setminus U = \{x_1, x_2, \ldots, x_n\}$. Since $\mathfrak{A}$ is a D_μ-cover of X, each x_i belongs to one of the members of $\mathfrak{A}$, say, $x_1 \in A_1$, $x_2 \in A_2$, …, $x_n \in A_n$, where $A_i \in \mathfrak{A}$, $A_i = U_i \setminus V_i$, $U_i, V_i \in \mu, U_i \neq X$, i = 1,2, … ,n. Then the collection $\{A_1, A_2, \ldots, A_n\}$ is a finite sub collection of $\mathfrak{A}$ of D_μ-sets covering $X \setminus U$. Since $U = U \setminus \emptyset$ is a D_μ-set, the collection $\{U, A_1, A_2, \ldots, A_n\}$ is a finite sub collection of $\mathfrak{A}$ of D_μ-sets covering of X. Hence X is D_μ-compact. ■

Theorem 3.5. Let (X, μ) be a GTS. Then finite union of D_μ-compact sets is D_μ-compact.
Proof: Assume that $G \subset X$ and $F \subset X$are any two D_μ-compact subsets of X. Let $\mathfrak{A}$ be a D_μ-cover of $G \cup F$. Then $\mathfrak{A}$ will also be a D_μ-cover of both G and F. So by assumption, there exists a finite sub collections of $\mathfrak{A}$ of D_μ-sets, say, $\{G_1, G_2, \ldots, G_n\}$ and $\{F_1, F_2, \ldots, F_m\}$ covering G and F respectively where $G_i = A_i \setminus B_i$, $A_i \neq X$, $A_i, B_i \in \mu$, i = 1, 2, … , n, $F_j = C_j \setminus D_j$, $C_j \neq X$, $C_j, D_j \in \mu$, j = 1, 2, … , m.Clearly the collection $\{G_1, G_2, \ldots, G_n, F_1, F_2, \ldots, F_m\}$ is a finite sub collection of $\mathfrak{A}$ of D_μ-sets covering $G \cup F$. By induction, every finite union of D_μ-compact sets is D_μ-compact.■

Theorem 3.6. Let (X, μ) be a GTS. Then non-empty D_μ-subsets of a D_μ-compact space (X, μ) is D_μ-compact if μ is the collection of μ-clopen sets.
Proof: Suppose (X, μ) is a D_μ-compact space where μ consists of μ-clopen sets. Let U be a non-empty D_μ-subset of X. Then there exists two sets $P, Q \in \mu$, $P \neq X$ such that $U = P \setminus$

Q. Now $X\backslash U = X\backslash(P\backslash Q) \in \mu$ which implies $X\backslash U \in D_\mu$. Let $\mathfrak{A} = \{A_\alpha\}_{\alpha\in J}$, where $A_\alpha = B_\alpha\backslash C_\alpha$, $B_\alpha \neq X$, $B_\alpha, C_\alpha \in \mu$, be a D_μ-cover of U. Then the collection $\{\{A_\alpha\}_{\alpha\in J}, X \setminus U\}$ is a D_μ-covering of X. Since X is D_μ-compact, there is a finite sub collection of $\mathfrak{A}$ of D_μ-sets covering X which can be either

(i) $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ or

(ii) $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}, X \setminus U\}$

Consider (i). Since $\bigcup_{i=1}^{n} A_{\alpha_i} = X$ and $U \subset X$, $U \subset \bigcup_{i=1}^{n} A_{\alpha_i}$. Then the collection $\{A_{\alpha_i}\}_{i = 1,2, \ldots, n}$ of D_μ-sets is a finite sub collection of $\mathfrak{A}$ covering U. Hence U is D_μ-compact.

Consider (ii). Since $(\bigcup_{i=1}^{n} A_{\alpha_i}) \cup (X \setminus U) = X$, then $U \subset \bigcup_{i=1}^{n} A_{\alpha_i}$ because if $x \in U \Rightarrow x \in X \Rightarrow x \in (\bigcup_{i=1}^{n} A_{\alpha_i}) \cup (X\backslash U) \Rightarrow x \in \bigcup_{i=1}^{n} A_{\alpha_i}$ or $x \in (X \setminus U) \Rightarrow x \in \bigcup_{i=1}^{n} A_{\alpha_i}$ since $x \in U$, $x \notin (X \setminus U)$. So $U \subset \bigcup_{i=1}^{n} A_{\alpha_i}$. Now the collection $\{A_{\alpha_i}\}_{i = 1,2, \ldots, n}$ of D_μ-sets is a finite sub collection of $\mathfrak{A}$ covering U. Hence U is D_μ-compact.■

Theorem 3.7. Let (X, μ) be a GTS. If X is D_μ-compact and the complement of any non-empty set U is a D_μ-set, then U is D_μ-compact.

Theorem 3.8. Let (X, μ) be a GTS. If X is D_μ-compact and U is any non-empty μ-closed subset of X, then U is D_μ-compact.

Theorem 3.9. Every infinite subset A of a D_μ-compact space (X, μ) has at least one D_μ-cluster point in X.

Proof: Suppose X is a D_μ-compact space and let A be an infinite subset of X. If possible assume that A has no D_μ-cluster points in X. Then for each $x \in X$, there exist a D_μ-set U_x such that $U_x \cap A = \{x\}$ or $\emptyset$. Then the collection $\{U_x : x \in X\}$ is a D_μ-covering of X. Since X is D_μ-compact, there exist points $x_1, x_2, \ldots, x_n$ in X such that $\bigcup_{i=1}^{n} U_{x_i} = X$. But $(U_{x_1} \cap A) \cup (U_{x_2} \cap A) \cup \ldots \cup (U_{x_n} \cap A) = \{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_n\}$ or $\emptyset$ which implies $(U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_n}) \cap A = \{x_1, x_2, \ldots, x_n\}$ or $\emptyset$ which implies $X \cap A = \{x_1, x_2, \ldots, x_n\}$ or $\emptyset \Rightarrow A = \{x_1, x_2, \ldots, x_n\}$ or $\emptyset$, contradicts that A is infinite ■

Theorem 3.10. A GTS (X, μ) is D_μ–compact if there exists a base $\mathfrak{B}$ for it such that every cover of X by members of $\mathfrak{B}$ has a finite sub cover.

Proof: Assume that $\mathfrak{B}$ is a base for the GTS (X, μ) with the property that every cover of X by members of $\mathfrak{B}$ has a finite sub cover. Let $\mathfrak{U}$ be any D_μ-cover of X, not necessarily by members of $\mathfrak{B}$. Now if $U \in \mathfrak{U} \Rightarrow U \in D_\mu \Rightarrow$ there exists two μ–open sets P and Q, $P \neq X$ such that $U = P \setminus Q$. Since $P, Q \in \mu \Rightarrow$ there exists two sub families $\mathfrak{B}_P$, $\mathfrak{B}_Q$ of $\mathfrak{B}$ such that $P = \bigcup_{B \in \mathfrak{B}_P} B$ and $Q = \bigcup_{B \in \mathfrak{B}_Q} B$. Let $\mathfrak{B}_U = \mathfrak{B}_P \cup \mathfrak{B}_Q$. Thus for each $U \in \mathfrak{U}$, there exists a sub family $\mathfrak{B}_U$ of $\mathfrak{B}$ such that $U = \left(\bigcup_{B_i \in \mathfrak{B}_U} B_i\right) \setminus \left(\bigcup_{B_j \in \mathfrak{B}_U} B_j\right)$ where $B_i \in \mathfrak{B}_P \subset \mathfrak{B}_U$ and $B_j \in \mathfrak{B}_Q \subset \mathfrak{B}_U$. Let $\mathfrak{D} = \bigcup_{U \in \mathfrak{U}} \mathfrak{B}_U$. Then $\mathfrak{D}$ is a cover of X since $\mathfrak{U}$ is a cover of X and more over $\mathfrak{D} \subset \mathfrak{B}$. So by hypothesis $\mathfrak{D}$ has a finite sub collection, say, $\{V_1, V_2, \ldots, V_n\}$ covering X. For each $i = 1, 2, \ldots, n$ there exists $U_i \in \mathfrak{U}$ such that $V_i \in \mathfrak{B}_{U_i}$. Now $V_i \in \mathfrak{B}_{U_i} \Rightarrow$ there exists two sub families $\mathfrak{B}_{P_i}$ and $\mathfrak{B}_{Q_i}$ of $\mathfrak{B}$, $\mathfrak{B}_{P_i} \cup \mathfrak{B}_{Q_i} = \mathfrak{B}_{U_i}$ and $V_i \in \mathfrak{B}_{U_i} = \mathfrak{B}_{P_i} \cup \mathfrak{B}_{Q_i} \Rightarrow V_i \subset [(\bigcup_{B_k \in \mathfrak{B}_{P_i}} B_k) \setminus (\bigcup_{B_l \in \mathfrak{B}_{Q_i}} B_l)] \Rightarrow V_i \subset (P_i \setminus Q_i)$ where $P_i = (\bigcup_{B_k \in \mathfrak{B}_{P_i}} B_k)$

and $Q_i = (\bigcup_{B_l \in \mathfrak{B}_{Q_i}} B_l)$. Thus each V_i, i= 1, 2, … , n is contained in a D_μ-set. So $\{U_1, U_2, …, U_n\}$ is a finite sub collection of $\mathfrak{U}$ covering X. Hence X is D_μ-compact.■

Theorem 3.11. Let (X, μ) be a GTS. Then the following statements are equivalent.

(i) X is D_μ-compact

(ii) For every collection $\mathfrak{A}$ of complements of D_μ-subsets of X, the intersection of all the elements of $\mathfrak{A}$ is empty then the collection $\mathfrak{A}$ contains a finite sub collection with empty intersection.

Proof: (i) ⇒ (ii) Suppose X is D_μ-compact space. Let $\mathfrak{C} = \{U \setminus V : U, V \in \mu , U \neq X\}$ be the collection of all D_μ-subsets of X and let $\mathfrak{A} = \{X \setminus (U \setminus V) : (U \setminus V) \in \mathfrak{C}\}$ be the collection of all complements of D_μ-subsets of X. Suppose the intersection of all the elements of $\mathfrak{A}$ is empty. i.e, $\bigcap_i[X\setminus (U_i \setminus V_i)]=\emptyset$. Then $X\setminus\bigcap_i[X\setminus (U_i \setminus V_i)]=X\setminus\emptyset$. i.e, $[\bigcap_i[X\setminus (U_i \setminus V_i)]]^c=X$. Therefore by De-Morgan's Law, $\bigcup_i(U_i \setminus V_i) = X$. Then the collection $\{(U_i\setminus V_i)\}_i$ of D_μ-subsets is a covering of X. Since X is D_μ-compact, there is a finite sub collection say, $\{U_1\setminus V_1, U_2\setminus V_2,… , U_n\setminus V_n\}$ of $\{(U_i\setminus V_i)\}_i$ covering X. i.e, $\bigcup_{i=1}^n(U_i \setminus V_i) = X$. Then $X\setminus\bigcup_{i=1}^n(U_i \setminus V_i) = \emptyset$ $\Rightarrow\bigcap_{i=1}^n[(U_i \setminus V_i)]^c = \emptyset\Rightarrow\bigcap_{i=1}^n[X\setminus (U_i \setminus V_i)] = \emptyset$.

(ii) ⇒ (i) Assume that for every collection $\mathfrak{A} = \{X \setminus (U \setminus V) : U, V \in \mu, U \neq X\}$ of complements of D_μ-subsets of X, the intersection of all the elements of $\mathfrak{A}$ is empty implies the collection $\mathfrak{A}$ contains a finite sub collection with empty intersection. Let $\mathfrak{C}=\{U_i\setminus V_i : U_i, V_i\in \mu, U_i \neq X, \forall i\}$ be a D_μ-cover of X. i.e, $\bigcup_i(U_i \setminus V_i) = X$ $\Rightarrow[\bigcup_i(U_i \setminus V_i)]^c = \emptyset$. By De-Morgan's law, $\bigcap_i[(U_i \setminus V_i)]^c = \emptyset$. Then by hypothesis, $\bigcap_{i=1}^n[(U_i \setminus V_i)]^c = \emptyset$. Then $[\bigcap_{i=1}^n[(U_i \setminus V_i)]^c]^c = X$. Again by De-Morgan's law, $\bigcup_{i=1}^n(U_i \setminus V_i) = X$. i.e, the collection $\{U_1\setminus V_1, U_2\setminus V_2,… , U_n\setminus V_n\}$ of D_μ-sets is a finite sub collection of $\mathfrak{C}$ covering X. Hence X is D_μ-compact. ■

Theorem 3.12. A GTS (X, μ) is D_μ-compact iff every collection of complements of D_μ-subsets of X which satisfies the finite intersection property has, itself, a non-empty intersection.

Proof: Suppose X is D_μ-compact. Let $\{X \setminus (U_i\setminus V_i) : U_i,V_i\in\mu, U_i\neq X\}$ be a collection of complements of D_μ-subsets of X which satisfies the finite intersection property. Then $\bigcap_{i=1}^n[X\setminus (U_i \setminus V_i)]\neq \emptyset$. Since X is D_μ-compact, by above theorem $\bigcap_i[X\setminus (U_i \setminus V_i)] = \emptyset\Rightarrow\bigcap_{i=1}^n[X\setminus (U_i \setminus V_i)] = \emptyset$. Therefore $\bigcap_{i=1}^n[X\setminus (U_i \setminus V_i)] \neq \emptyset\Rightarrow\bigcap_i[X\setminus (U_i \setminus V_i)] \neq \emptyset$.

Conversely, suppose $\{X\setminus(U_i\setminus V_i) : U_i,V_i\in \mu, U_i\neq X\}$ is a collection of complements of D_μ-subsets of X which satisfies the finite intersection property has, itself, a non-empty intersection. i.e, $\bigcap_{i=1}^n[X\setminus (U_i \setminus V_i)] \neq \emptyset\Rightarrow \bigcap_i[X\setminus (U_i \setminus V_i)] \neq \emptyset$. Then $\bigcap_i[X\setminus (U_i \setminus V_i)] = \emptyset\Rightarrow\bigcap_{i=1}^n[X\setminus (U_i \setminus V_i)] = \emptyset$. Then by the above theorem X is D_μ-compact. ■

Theorem 3.13. If (X, μ) is D_μ-compact, then for every collection $\mathfrak{A} = \{U_i : U_i\in \mu, U_i \neq X\}$ of μ-open sets covering X ⇒ there exists a finite sub collection of $\mathfrak{A}$ covering X.

Proof: Suppose that X is D_μ-compact. Let $\mathfrak{A} = \{U_i : U_i\in \mu, U_i \neq X\}$ be a collection of μ-open sets covering X. i.e, $\bigcup_{i\in I} U_i = X$ where $U_i\in\mathfrak{A}$. Now the collection $\{(U_i\setminus V_i) : U_i, V_i\in\mathfrak{A}\}$ is a collection of D_μ-sets covering X. Since X is D_μ-compact, there exists a finite sub collection, say, $\{U_1\setminus V_1, U_2\setminus V_2, …, U_n\setminus V_n : U_i, V_i\in\mathfrak{A}, i = 1,2, … , n\}$ ofD_μ-sets covering X. Then the collection $\{U_1, U_2, \dots U_n : U_i \in \mu, U_i \neq X, i = 1,2, \dots, n\}$ is a finite sub collection of $\mathfrak{A}$ covering X. ■

Theorem 3.14. Suppose (X, μ) is μ-compact. If for every collection $\mathfrak{A} = \{U_i \setminus V_i : U_i, V_i \in \mu, U_i \neq X\}$ of D_μ-sets covering X, then there exists a finite sub collection of $\mathfrak{A}$ covering X.

Proof: Suppose (X, μ) is μ-compact. Let $\mathfrak{A} = \{U_i \setminus V_i : U_i, V_i \in \mu, U_i \neq X\}$ be a collection of D_μ-sets covering X. i.e, $\bigcup_{i\in I}(U_i \setminus V_i) = X$ where $U_i, V_i \in \mathfrak{A}$. Now the collection $\{U_i : U_i \in \mu, U_i \neq X, U_i \setminus V_i \in \mathfrak{A}$ for some $V_i \in \mu\}$ is a collection of μ-open sets covering X. Since X is μ-compact, there exists a finite sub collection, say, $\{U_1, U_2, \dots, U_n : U_i \in \mu, U_i \neq X, U_i \setminus V_i \in \mathfrak{A}$ for some $V_i \in \mu, i = 1,2, \dots, n\}$ of μ-open sets covering X. Since each proper μ-open sets are D_μ-sets, the collection $\{U_1, U_2, \dots U_n : U_i \in \mu, U_i \neq X, U_i \setminus V_i \in \mathfrak{A}$ for some $V_i \in \mu, i = 1,2, \dots, n\}$ is a finite sub collection of $\mathfrak{A}$ covering X. ■

4. D_μ-Compact spaces in subspaces of GTS's and in μ-D_2 spaces

Theorem 4.1. Let Y be a subset of a GTS (X, μ). Then the following are equivalent:

(i) Y is D_μ-compact w.r.t. μ

(ii) Y is $D_{\mu/Y}$-compact w.r.t. the subspace GT $\mu_{/Y}$ on Y.

Proof: (i)⇒(ii) Suppose Y is D_μ-compact. Let $\mathfrak{A} = \{H_\alpha\}_{\alpha\in J}$ be a $D_{\mu_{/Y}}$-covering of Y. Then for each α, $H_\alpha \in D_{\mu_{/Y}} \Rightarrow$ there exists $U_\alpha, V_\alpha \in \mu_{/Y}$ such that $H_\alpha = U_\alpha \setminus V_\alpha$. Now $U_\alpha, V_\alpha \in \mu_{/Y} \Rightarrow$ there exists $A_\alpha, B_\alpha \in \mu$ such that $U_\alpha = A_\alpha \cap Y$ and $V_\alpha = B_\alpha \cap Y$. Hence for each α, $H_\alpha = (A_\alpha \cap Y) \setminus (B_\alpha \cap Y) = (A_\alpha \setminus B_\alpha) \cap Y = G_\alpha \cap Y$ where $G_\alpha = (A_\alpha \setminus B_\alpha) \in D_\mu$. Therefore the collection $\{G_\alpha\}_{\alpha\in J}$ of D_μ-sets is a D_μ-covering of Y. Since Y is D_μ-compact w.r.t. μ, by hypothesis, there is a finite sub collection of D_μ-sets, say, $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ covering Y. But then, the collection $\{G_{\alpha_1} \cap Y, G_{\alpha_2} \cap Y, \dots, G_{\alpha_n} \cap Y\} = \{H_{\alpha_1}, H_{\alpha_2}, \dots, H_{\alpha_n}\}$ of $D_{\mu_{/Y}}$-sets is a finite sub collection of $\mathfrak{A}$ covering Y. Hence Y is $D_{\mu_{/Y}}$-compact w.r.t. $\mu_{/Y}$.

(ii)⇒(i) Suppose Y is $D_{\mu_{/Y}}$-compact w.r.t. the subspace GT $\mu_{/Y}$ on Y. Let $\mathfrak{B} = \{G_\alpha\}_{\alpha\in J}$ be a D_μ-covering of Y where $G_\alpha \in D_\mu$, $\forall\, \alpha$. Now $G_\alpha \in D_\mu \Rightarrow$ there exists $A_\alpha, B_\alpha \in \mu$ such that $G_\alpha = (A_\alpha \setminus B_\alpha)$. Set $H_\alpha = G_\alpha \cap Y$. Then $H_\alpha = (A_\alpha \setminus B_\alpha) \cap Y = (A_\alpha \cap Y) \setminus (B_\alpha \cap Y)$ implies $H_\alpha \in D_{\mu_{/Y}}$. But then the collection $\{H_\alpha\}_{\alpha\in J}$ of $D_{\mu_{/Y}}$-sets is a covering of Y w.r.t. $\mu_{/Y}$. Since Y is $D_{\mu_{/Y}}$ -compact, by hypothesis, there is a finite sub collection $\{H_{\alpha_1}, H_{\alpha_2}, \dots, H_{\alpha_n}\}$ of $D_{\mu_{/Y}}$-sets covering Y. i.e, $\{G_{\alpha_1} \cap Y, G_{\alpha_2} \cap Y, \dots, G_{\alpha_n} \cap Y\}$ is a finite sub collection of $D_{\mu_{/Y}}$-sets covering Y. Then the collection $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ of D_μ-sets is a finite sub collection of $\mathfrak{B}$ covering Y. Hence Y is D_μ-compact. ■

Theorem 4.2. Let $(Y, \mu_{/Y})$ be a subspace of the GTS (X, μ) and let $A \subset Y \subset X$. Then A is D_μ-compact if and only if A is $D_{\mu_{/Y}}$-compact.

Proof: Let $\mu_{/A}$ and $(\mu_{/Y})_{/A}$ be the subspace GT's on A. Then by the above theorem, A is D_μ-compact if and only if A is $D_{\mu_{/A}}$-compact and A is $D_{\mu_{/Y}}$-compact if and only if A is $(D_{\mu_{/Y}})_{/A}$-compact. But $D_{\mu_{/A}} = (D_{\mu_{/Y}})_{/A}$. Hence the proof.■

Theorem 4.3. Let (X, μ) be a GTS. If K is D_μ-compact and F is μ-closed then K∩F is D_μ-compact.

Proof: Since F is μ-closed in X, F∩K is $\mu_{/K}$-closed in the subspace GT on K. By theorem 3.8, F∩K is $D_{\mu_{/K}}$-compact. By theorem 4.1, F∩K is D_μ-compact. ■

Theorem 4.4. If E is a D_μ-compact subset of a μ-D_2 space (X, μ) and x ∈ X is not in E, then there is a μ-open set F such that E ⊂ F.

Proof: Suppose E is a D_μ-compact subset of X and x ∈ X is not in E. Since X is μ-D_2, for each p∈ E, there exists D_μ-sets U_x and V_p such that $x \in U_x$, $p \in V_p$, $U_x \cap V_p = \emptyset$, where $U_x=A_x \backslash B_x$, $V_p= G_p \backslash H_p$, A_x, B_x, G_p, $H_p \in \mu$, $A_x \neq X$, $G_p \neq X$. Now the collection $\{V_p : p \in E\}$ is a D_μ-covering of E. Since E is D_μ-compact, there exists a finite sub collection, say, $\{V_{p_1}, V_{p_2}, \dots, V_{p_n}\}$ of D_μ-sets covering E. Then $E \subset \bigcup_{i=1}^{n} V_{p_i} = \bigcup_{i=1}^{n} (G_{p_i} \backslash H_{p_i}) \subset \bigcup_{i=1}^{n} G_{p_i}$. Let $F = \bigcup_{i=1}^{n} G_{p_i}$. Then F is μ-open and E ⊂ F.■

Remark 4.1. If Y is a D_μ-compact subset of a μ-D_2 space (X, μ), then Y need not be μ-closed.

Consider the following example. Let X= {1, 2, 3} and μ = {∅, {1}, {1, 2}, {2, 3}, X}. Then D_μ-sets are {∅, {1}, {1, 2}, {2, 3}, {2}, {3}}. Let Y = {1,3}. Then X is μ-D_2 and Y is D_μ-compact, but Y is not μ-closed.

5. D_μ-Compact spaces in products of GTS's

Theorem 5.1. If (X, μ) and (Y, η) are GTS's and f: (X, μ) → (Y, η) is (μ, η)-continuous and onto. If (X, μ) is D_μ-compact then (Y, η) is D_η-compact.

Proof: Suppose (X, μ) and (Y, η) be two GTS's andf: (X, μ) → (Y, η) be a (μ, η)-continuous function from X onto Y. Assume that X is D_μ-compact. Suppose $\mathfrak{A}$ is any D_η-cover of Y, then the collection $\{f^{-1}(A) : A \in \mathfrak{A}\}$ is a cover of X. Now $A \in \mathfrak{A} \Rightarrow A \in D_\eta \Rightarrow A = B \setminus C$ where B, C ∈η, B≠Y and $f^{-1}(A) = f^{-1}(B \backslash C) = f^{-1}(B) \backslash f^{-1}(C)$.Since f is continuous, $f^{-1}(B)$ and $f^{-1}(C) \in \mu$. Therefore the collection $\{f^{-1}(A) : A \in \mathfrak{A}\} = \{f^{-1}(B) \backslash f^{-1}(C) : f^{-1}(B), f^{-1}(C) \in \mu, f^{-1}(B) \neq X\}$ is a D_μ-cover of X. Since X is D_μ-compact, there is a finite sub collection of D_μ-sets, say, $\{f^{-1}(A_1), f^{-1}(A_2), \dots, f^{-1}(A_n)\}$ covering X, where $A_1, A_2, \dots, A_n \in \mathfrak{A}$. Since the mapping is onto, the collection $\{A_1, A_2, \dots, A_n\}$ of D_η-sets is a finite sub collection of D_η-sets covering Y. Hence Y is D_η-compact.■

Theorem 5.2. Let (X, μ) be the products of the GTS's (X_k, μ_k), k ∈ K. If (X, μ) is D_μ-compact and every μ_k is strong, then every (X_k, μ_k) is $D_{\mu k}$-compact.

Proof: Let $p_k : (X, \mu) \to (X_k, \mu_k)$ be the projection map. By [8, proposition 2.7], p_k is continuous for k ∈ K. By theorem 5.1, since continuous image of a D_μ-compact space is D_μ-compact, every (X_k, μ_k) is D_{μ_k}-compact, k ∈ K.■

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