Annals of Pure and Applied Mathematics
Vol. 9, No. 2, 2015, 215-232
ISSN: 2279-087X (P), 2279-0888(online)
Published on 25 March 2015
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Notes on the Power Sum $1^k+2^k+...+n^k$

Moawwad El-Mikkawy **and** ***Faiz Atlan***

Mathematics Department, Faculty of Science, Mansoura University,
Mansoura-35516, Egypt
E-mails: m_elmikkawy@yahoo.com, faizatlan11@yahoo.com

Received 9 February 2015; accepted 3 March 2015

Abstract. In this paper we present some efficient computational methods to compute the power sum $S_k(n)=\sum_{r=1}^{n} r^k$, for integers $k \geq 0$, and $n \geq 1$. A new method to compute $S_k(n)$ is investigated. Some new identities involving $S_k(n)$ are given.

Keywords: Power sum; Bernoulli numbers; Vandermonde matrix; Stirling numbers; Eulerian numbers; Linear systems; Elementary symmetric polynomials; Exponential generating function

AMS Mathematics Subject Classification (2010): 65F05, 68W30, 15Bxx, 11B73, 05E05

1. Introduction and basic definitions

One of the oldest topics in mathematics is the summation of powers

$$S_k(n)=\sum_{r=1}^{n} r^k = 1^k+2^k+...+n^k, \tag{1}$$

for integers $k \geq 0$, and $n \geq 1$. The interested reader may refer to [1, 2, 4, 5, 7, 8, 12, 13, 14, 15, 16, 19, 22, 23, 24, 29, 31, 32, 33, 35, 38, 40, 42, 43, 44] for the history of this problem. Even nowadays there is a continual stream of publications on the topic. See for instance, [6, 10, 11, 20, 21, 26, 27, 28, 34, 36, 39, 41] and the references therein. Throughout this paper, δ_{ij} is the kronecker delta which is equal to 1 or 0 according as $i=j$ or not. Also $[x^r]P(x)$, denotes the coefficients of x^r in the polynomial $P(x)$, and empty summation is assumed equal to zero.

Definition 1.1. [34] The exponential generating function (EGF) of a sequence $<a_0, a_1, a_2, ...>$ is defined by:

$$f(x)=\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

The EGF satisfies:

$$\left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}, \quad where \quad c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}, \quad n \geq 0. \tag{2}$$

The EGF, $G(n,x)$ of $S_k(n)$ is given by:

$$G(n,x) = \frac{e^{nx}-1}{1-e^{-x}} = \sum_{i=0}^{\infty} S_i(n) \frac{x^i}{i!}. \tag{3}$$

Definition 1.2. [37] For integer numbers n and k with $n \geq k \geq 0$, the Stirling numbers of the first kind, $s(n,k)$ and of the second kind, $S(n,k)$ are defined respectively by:

$$(x)_n = [x-n+1]_n = \sum_{k=0}^{n} s(n,k) x^k, \tag{4}$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \tag{5}$$

where the falling factorial of x, $(x)_n$ and the rising factorial of x, $[x]_n$ are given respectively by:

$$(x)_n = \begin{cases} 1 & if \;\; n = 0, \\ x(x-1)(x-2)...(x-n+1) & if \;\; n \geq 1, \end{cases} \tag{6}$$

and

$$[x]_n = \begin{cases} 1 & if \;\; n = 0, \\ x(x+1)(x+2)...(x+n-1) & if \;\; n \geq 1. \end{cases} \tag{7}$$

It is well known that for integers $n,k \geq 0$, $s(n,k)$ and $S(n,k)$ satisfy the following Pascal-type recurrence relations:

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \tag{8}$$

and

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{9}$$

subject to:

$$s(k,k) = S(k,k) = 1, 1 \leq k \leq n \tag{10}$$

and

$$s(n,k) = S(n,k) = \delta_{nk}, if \;\; k = 0 \; or \; n = 0. \tag{11}$$

The values of the Stirling numbers of the first kind $s(n,k)$ and of the second kind $S(n,k)$ for $0 \leq n,k \leq 10$ are the entries in the following tables:

n/k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	-1	1								
3	0	2	-3	1							
4	0	-6	11	-6	1						
5	0	24	-50	35	-10	1					

6	0	-120	274	-225	85	-15	1				
7	0	720	- 1764	1624	- 735	175	-21	1			
8	0	- 5040	13068	- 13132	6769	- 1960	322	-28	1		
9	0	40320	- 109584	118124	- 67284	22449	- 4536	546	-36	1	
10	0	- 362880	1026576	- 1172700	723680	- 269325	63273	- 9450	870	-45	1

Table 1: The Stirling numbers of the first kind $s(n,k)$ for $0 \le n,k \le 10$.

n/k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	1	1								
3	0	1	3	1							
4	0	1	7	6	1						
5	0	1	15	25	10	1					
6	0	1	31	90	65	15	1				
7	0	1	63	301	350	140	21	1			
8	0	1	127	966	1701	1050	266	28	1		
9	0	1	255	3035	7770	6951	2646	462	36	1	
10	0	1	511	9330	34501	42525	22827	5880	750	45	1

Table 2: The Stirling numbers of the second kind $S(n,k)$ for $0 \le n,k \le 10$.

Definition 1.3. [20] The Bernoulli numbers, $B_i, i \ge 0$ and polynomials, $B_i(x), i \ge 0$ are defined by:

$$B_0 = 1,\, B_i = -\frac{1}{i+1}\sum_{j=0}^{i-1}\binom{i+1}{j}B_j, \quad i \ge 1, \tag{12}$$

and

$$B_i(x) = \sum_{j=0}^{i}\binom{i}{j}B_{i-j}x^j, \quad i \ge 0 \tag{13}$$

respectively.

The exponential generating functions of the Bernoulli numbers, $B_i, i \ge 0$ and polynomials, $B_i(x), i \ge 0$ are given by:

$$H(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{14}$$

and

$$F(x,t) = \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{15}$$

respectively.

Properties of the Bernoulli numbers $B_n, n \ge 0$ and polynomials $B_n(x), n \ge 0$

include [18, 37]:

- $(-1)^{k+1}B_{2k} > 0, \quad k \geq 1.$ (16)
- $B_{2k+1} = 0, \quad k \geq 1.$ (17)
- $\sum_{k=0}^{n-1}\binom{n}{k}B_k = \delta_{n1}, \quad n \geq 0.$ (18)
- $B_i(x+1) - B_i(x) = i\,x^{i-1}, \quad i \geq 1.$ (19)

The first few Bernoulli numbers B_n, and Bernoulli polynomials $B_n(x)$ are given, respectively, in the following tables:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	0	$\frac{7}{6}$

Table 3: The Bernoulli numbers B_n for $0 \leq n \leq 14$.

n	0	1	2	3	4	5
$B_n(x)$	1	$x-\frac{1}{2}$	$x^2-x+\frac{1}{6}$	$x^3-\frac{3}{2}x^2+\frac{1}{2}x$	$x^4-2x^3+x^2-\frac{1}{30}$	$x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x$

Table 4: The Bernoulli polynomials $B_n(x)$ for $0 \leq n \leq 5$.

Definition 1.4. [37] For each $n \geq 1$, the Eulerian numbers $E(n,k)$ are defined by:

$$E(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+1-j)^n, \quad 0 \leq k \leq n-1. \tag{20}$$

The values of the Eulerian numbers $E(n,k)$ for $1 \leq n \leq 10$ and $0 \leq k \leq 9$, are the entries in the following table

n/k	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			

8	1	247	4293	15619	15619	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

Table 5: The Eulerian numbers $E(n,k)$ for $1 \le n \le 10$ and $0 \le k \le 9$.

The Eulerian numbers $E(n,k)$ satisfy:

- $E(n,0) = E(n,n-1) = 1, \quad n \ge 1.$ (21)
- $E(n,k) = E(n,n-1-k), \quad n \ge 1.$ (22)
- $E(n,k) = (k+1)E(n-1,k) + (n-k)E(n-1,k-1), \quad n \ge 2.$ (23)

Definition 1.5. [26] The elementary symmetric polynomial $\sigma_r^{(n)}$ in $x_1, x_2, \ldots, x_n$ is defined by:

$$\sigma_r^{(n)}(x_1, x_2, \ldots, x_n) = \begin{cases} 0 & \text{if } (r > n) \text{ or } (n < 0), \\ 1 & \text{if } r = 0, \\ \sum\limits_{1 \le i_1 < i_2 < \ldots < i_r \le n} x_{i_1} x_{i_2} \ldots x_{i_r} & \text{if } 1 \le r \le n. \end{cases} \tag{24}$$

It should be noted that $\sigma_r^{(n)}$ is a homogenous function and can be written as:

$$\sigma_r^{(n)}(x_1, x_2, \ldots, x_n) = \sum_{\substack{k_1 + k_2 + \ldots + k_n = r \\ k_1, k_2, \ldots, k_n \in \{0,1\}}} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \quad , \; r = 0,1,\ldots,n. \tag{25}$$

The symmetric polynomials $\sigma_r^{(n)}$ satisfies the recurrence relation:

$$\sigma_r^{(n)}(x_1, x_2, \ldots, x_n) = \sigma_r^{(n-1)}(x_1, x_2, \ldots, x_{n-1}) + x_n \sigma_{r-1}^{(n-1)}(x_1, x_2, \ldots, x_{n-1}). \tag{26}$$

A more general form of (26) is given by:

$$\sigma_i^{(n)}(x_1, x_2, \ldots, x_n) = \sigma_i^{(n-1)}(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) + x_j \sigma_{i-1}^{(n-1)}(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), \tag{27}$$

where $1 \le i, j \le n.$

Partial differentiation for both sides of (27) with respect to x_j gives:

$$\partial\partial x_j \sigma_i^{(n)} = \sigma_{i,j}^{(n)} = \sigma_{i-1}^{(n-1)}(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n). \tag{28}$$

Therefore by using (24), we get:

$$\sigma_{ij}^{(n)}(x_1, x_2, \ldots, x_n) = \begin{cases} 0 & \text{if } (i > n) \text{ or } (n < 0), \\ 1 & \text{if } i = 1, \\ \sum\limits_{\substack{1 \le r_1 < r_2 < \ldots < r_{i-1} \le n \\ r_1, r_2, \ldots, r_{i-1} \ne j}} x_{r_1} x_{r_2} \ldots x_{r_{i-1}} & \text{if } 2 \le i \le n. \end{cases} \tag{29}$$

In particular, we have:

$$\sigma_{n,i}^{(n)}(1,2,\ldots,i-1,i+1,\ldots,n)=\frac{n!}{i},\quad 1\le i\le n, \tag{30}$$

and $$\sigma_{2,i}^{(n)}(1,2,\ldots,i-1,i+1,\ldots,n)=\begin{pmatrix} n+1 \\ 2 \\ \end{pmatrix}-i,\quad 1\le i\le n. \tag{31}$$

Definition 1.6. [17] For the $n+1$ real values $m, x_1, x_2, ..., x_n$ we define a generalized Vandermonde matrix $V_n^{(m)}(x_1, x_2, ..., x_n)$ or simply $V_n^{(m)}$ by:

$$V_n^{(m)}=\left(x_j^{m+i-1}\right)_{i,j=1}^{n}. \tag{32}$$

The values $x_1, x_2, ..., x_n$ are called the nodes of the matrix $V_n^{(m)}$. The determinant of $V_n^{(m)}$ is given by:

$$\det(V_n^{(m)})=(\prod_{r=1}^{n}x_r^m)\prod_{1\le j<i\le n}(x_i-x_j). \tag{33}$$

Thus $V_n^{(m)}$ is invertible if and only if the n parameters $x_1, x_2, ..., x_n$ are distinct. Note that for any m, the two matrices $V_n^{(m)}$ and $V_n^{(0)}=V_n$ are related by:

$$V_n^{(m)}=DV_n \tag{34}$$

where $D=diag(x_1^m, x_2^m, ..., x_n^m)$.

The explicit form of the inverse matrix, $(V_n^{(m)})^{-1}=(\alpha_{ij})_{i,j=1}^{n}$ of the Vandermonde matrix $V_n^{(m)}$ is given by [17]:

$$\alpha_{ij}=(-1)^{n-j}\frac{\sigma_{n-j+1,i}^{(n)}(x_1, x_2, ..., x_n)}{x_i^m\prod\limits_{\substack{r=1\\ r\ne i}}^{n}(x_i-x_r)},\quad 1\le i,j\le n. \tag{35}$$

Let $((V_n^{(m)})^T)^{-1}=(\beta_{ij})_{i,j=1}^{n}$, Then we have:

$$\beta_{ij}=(-1)^{n-i}\frac{\sigma_{n-i+1,j}^{(n)}(x_1, x_2, ..., x_n)}{x_j^m\prod\limits_{\substack{r=1\\ r\ne j}}^{n}(x_j-x_r)},\quad 1\le i,j\le n, \tag{36}$$

since the inverse of transpose is the transpose of the inverse.

If $x_i=i$ for $i=1,2,...,k+1$, then (36) takes the form:

$$\beta_{ij}=\frac{(-1)^{i+j}}{(k+1)!}\begin{pmatrix} k+1 \\ j \end{pmatrix}\sigma_{k+2-i,j}^{(k+1)}(1,2,...,k+1),\quad 1\le i,j\le k+1. \tag{37}$$

Definition 1.7. [20] The sequences $<E_0, E_1, E_2, ...>$ of the Euler numbers, $E_n, n\ge 0$ is defined by the exponential generating function

$$\frac{2e^x}{e^{2x}+1}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}. \tag{38}$$

In this paper, we are mainly interested in the power sum $S_k(n)=1^k+2^k+...+n^k$ for integers $k\geq 0$, and $n\geq 1$. We are also interested in the use of the generating function of $S_k(n)$ to obtain new identities. In this framework, some computational methods for computing $S_k(n)$ are given in Section 2. In Section 3, we investigate some new identities for $S_k(n)$. In Section 4, we deal with a method for computing $S_k(n)$ by solving a special linear system of equations.

For the convenience of the reader we are going to list the following facts about $S_k(n)$ [9, 26, 30, 40]:

$S_k(n)$ is a polynomial of degree $k+1$ in n without constant term. Hence $S_k(n)$ takes the form:

$$S_k(n)=\sum_{i=1}^{k+1}c_{k+1,i}\, n^i,\quad k\geq 0, \tag{39}$$

where the coefficients $c_{k+1,i}, i=1,2,...,k+1$ are to be determined.

The sum of the coefficients of any polynomial $S_k(n), k\geq 0$ is always equals 1.

n is a factor of $S_k(n),\quad k\geq 0.$

n and $n+1$ are factors of $S_k(n),\quad k\geq 1.$

$n,\ n+1$ and $2n+1$ are factors of $S_k(n)$, for even $k\geq 2.$

$n^2\ (n+1)^2$ are factors of $S_k(n)$, for odd $k\geq 3.$

$$(g)\quad [n^{k+1}]S_k(n)=c_{k+1,k+1}=\frac{1}{k+1},\qquad k\geq 0. \tag{40}$$

$$(h)\quad [n^k]S_k(n)=c_{k+1,k}=\frac{1}{2},\qquad k\geq 1. \tag{41}$$

$$(i)\quad [n^{k-1}]S_k(n)=c_{k+1,k-1}=\frac{k}{12},\qquad k\geq 2. \tag{42}$$

$$(j)\quad [n^{k-2}]S_k(n)=c_{k+1,k-2}=0,\qquad k\geq 3. \tag{43}$$

$$(k)\quad [n^{k-3}]S_k(n)=c_{k+1,k-3}=-\frac{1}{120}\binom{k}{3},\qquad k\geq 4. \tag{44}$$

$$(l)\quad [n^1]S_k(n)=c_{k+1,1}=\sum_{j=0}^{k}\binom{k}{j}B_j=(-1)^k B_k,\ k\geq 0. \tag{45}$$

2. Some computational methods for computing $S_k(n)$

Perhaps, the most famous formula for computing $S_k(n)$ is the one involving the binomial coefficients $\begin{pmatrix} p \\ q \end{pmatrix}$ and the Bernoulli numbers, B_k which is given by:

$$S_k(n) = \frac{1}{k+1}\sum_{j=1}^{k+1}(-1)^{k+1-j}\begin{pmatrix} k+1 \\ j \end{pmatrix} B_{k+1-j}\, n^j, \quad k \geq 0. \tag{46}$$

From (39) and (46), we see that the coefficients $c_{k+1,j}$ are given by:

$$c_{k+1,j} = \frac{\begin{pmatrix} k+1 \\ j \end{pmatrix}}{k+1}(-1)^{k+1-j} B_{k+1-j}, \quad 1 \leq j \leq k+1. \tag{47}$$

Formula (46) can also be written in the form:

$$S_k(n) = \sum_{j=0}^{k}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix} B_{k-j}\frac{n^{j+1}}{j+1}. \tag{48}$$

From (46)-(48), we see that:

$$S_k(n) = (-1)^k B_k n + k\int_0^n S_{k-1}(x)dx = (-1)^k B_k n + k\sum_{j=1}^{k} c_{k,j}\frac{n^{j+1}}{j+1}. \tag{49}$$

The formula (49) reveals that we can obtain $S_k(n)$ from $S_{k-1}(n)$ recursively for $k \geq 1$ with the initial condition $S_0(n) = n.$ For example, we have:

$$S_1(n) = \frac{1}{2}n + \frac{1}{2}n^2. \quad S_2(n) = \frac{1}{6}n + 2\left(\frac{\frac{1}{2}}{2}n^2 + \frac{\frac{1}{2}}{3}n^3\right) = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3.$$

$$S_3(n) = 0.n + 3\left(\frac{\frac{1}{6}}{2}n^2 + \frac{\frac{1}{2}}{3}n^3 + \frac{\frac{1}{3}}{4}n^4\right) = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4,$$

having used Table 3 and (49).

There are many other ways to express the power sum $S_k(n)$. See for instance [3, 25, 34]

$$S_k(n) = n\delta_{k0} + n(n+1)\sum_{j=1}^{k}\frac{S(k,j)}{j+1}(n-1)_{j-1}, \tag{50}$$

$$S_k(n) = \frac{1}{k+1}\left(B_{k+1}(n+1) - B_{k+1}(1)\right), \tag{51}$$

$$S_k(n) = \sum_{m=0}^{k-1}E(k,m)\begin{pmatrix} n+m+1 \\ k+1 \end{pmatrix} - \delta_{k0}, \quad k \geq 0. \tag{52}$$

$$S_k(n)=\sum_{m=0}^{min(k,n-1)}(n-m)s(n+1,n+1-m)S(n+k-m,n),\quad k\ge 0. \tag{53}$$

and $S_k(n)=XQY,$ (54)

where

$$X=[\binom{n}{1}\binom{n}{2}...\binom{n}{k+1}],\qquad Q=[(-1)^{i-j}\binom{i-1}{j-1}]_{i,j=1}^{k+1}$$ and

$Y=[1^k\ 2^k\ ...(k+1)^k]^T$.

An easy proof for (47) can be found in [30]. Also a very easy proof for the formula (51) can be obtained by using the property:

$$xG(n,x)=F(n+1,x)-F(1,x). \tag{55}$$

By using the property (19) together with the telescoping sum, an additional proof for the formula (51) can be obtained.

3. New identities for $S_k(n)$

The main objective of this section is to introduce new identities involving the power sum, $S_k(n)$.

By noticing that the coefficients $c_{k+1,1},c_{k+1,2},...,c_{k+1,k+1}$ in (39) satisfy the Vandermonde linear system:

$$\begin{bmatrix} 1 & 1^2 & 1^3 & \dots & 1^{k+1} \\ 2 & 2^2 & 2^3 & \dots & 2^{k+1} \\ 3 & 3^2 & 3^3 & \dots & 3^{k+1} \\ \vdots & \vdots & \vdots & \vdots & \\ (k+1) & (k+1)^2 & (k+1)^3 & \dots & (k+1)^{k+1} \end{bmatrix}\begin{bmatrix} c_{k+1,1} \\ c_{k+1,2} \\ c_{k+1,3} \\ \vdots \\ c_{k+1,k+1}\end{bmatrix}=\begin{bmatrix} S_k(1) \\ S_k(2) \\ S_k(3) \\ \vdots \\ S_k(k+1)\end{bmatrix}, \tag{56}$$

then following [17], together with (37) we obtain the coefficients $c_{k+1,1},c_{k+1,2},...,c_{k+1,k+1}$ as follows:

$$c_{k+1,i}=\sum_{j=1}^{k+1}\frac{(-1)^{i+j}}{(k+1)!}\binom{k+1}{j}\sigma_{k+2-i,j}^{(k+1)}S_k(j),\quad 1\le i\le k+1. \tag{57}$$

We are now ready to formulate the following result.

Theorem 3.1. The power sum $S_k(n)$ enjoys the following identities:

$$(i)\quad \sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}S_k(j)=(-1)^k k!,\quad k\ge 0. \tag{58}$$

$(ii)\quad \sum_{j=1}^{k+1}(-1)^{j}[k\binom{k+1}{j}+2\binom{k}{j}]S_k(j)=(-1)^k k!,\quad k\geq 1.$ (59)

$(iii)\quad \sum_{j=1}^{k+1}(-1)^{j+1}\frac{\binom{k+1}{j}}{j}S_k(j)=(-1)^k B_k,\quad k\geq 0.$ (60)

$(iv)\quad \sum_{j=0}^{i-1}(-1)^{i+1-j}\binom{i}{j}S_j(n)=n^i-\delta_{i0},\quad i\geq 0.$ (61)

$(v)\quad \sum_{j=1}^{i}\binom{i}{j}S_j(n-1)=S_i(n)-n,\quad i\geq 0.$ (62)

$(vi)\quad \sum_{j=0}^{i-1}n^{i-j}\binom{i}{j}S_j(n)=S_i(2n)-2S_i(n),\quad i\geq 0.$ (63)

$(vii)\quad \sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}S_j(n)=\delta_{i0}+S_i(n-1),\quad i\geq 0.$ (64)

$(viii)\quad \sum_{j=0}^{i-1}(-1)^{j}n^{i-j}\binom{i}{j}S_j(n-1)=\begin{cases}2S_i(n-1) & \text{if } i \text{ is odd},\\ 0 & \text{if } i \text{ is even.}\end{cases}\quad i\geq 1,$ (65)

$(ix)\quad \sum_{j=0}^{i}(-n)^{i-j}\binom{i}{j}S_j(n)=(-1)^i S_i(n-1)+\delta_{i0},\quad i\geq 0.$ (66)

Proof: (i) Putting $i=k+1$ in (57) together with (40) gives:

$$\frac{1}{k+1}=\sum_{j=1}^{k+1}(-1)^{k+1+j}\frac{\binom{k+1}{j}}{(k+1)!}\sigma_{1,j}^{(k+1)}S_k(j)=\sum_{j=1}^{k+1}(-1)^{k+1+j}\frac{\binom{k+1}{j}}{(k+1)!}S_k(j), \quad (67)$$

having used (29). Therefore

$$\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}S_k(j)=(-1)^k k!,\quad k\geq 0,$$

on simplification.

(ii) Setting $i=k$ in (57) together with (41) yields:

$$\frac{1}{2} = \sum_{j=1}^{k+1}(-1)^{k+j}\frac{\binom{k+1}{j}}{(k+1)!}\sigma_{2,j}^{(k+1)}S_k(j) = \sum_{j=1}^{k+1}(-1)^{k+j}[\binom{k+2}{2}-j]\frac{\binom{k+1}{j}}{(k+1)!}S_k(j), \quad (68)$$

having used (31). Then

$$\begin{aligned}(-1)^k\frac{(k+1)!}{2} &= \sum_{j=1}^{k+1}(-1)^j[\binom{k+2}{2}-j]\binom{k+1}{j}S_k(j) \\ &= \sum_{j=1}^{k+1}(-1)^j\binom{k+1}{j}\binom{k+2}{2}S_k(j)+\sum_{j=1}^{k+1}(-1)^{j+1}(k+1)\binom{k}{j-1}S_k(j), \quad k\ge 1.\end{aligned}$$

having used the identity:

$$\binom{k+1}{j+1} = \frac{k+1}{j+1}\binom{k}{j}.$$

Therefore

$$(-1)^k\frac{k!}{2} = \frac{(k+2)}{2}\sum_{j=1}^{k+1}(-1)^j\binom{k+1}{j}S_k(j)+\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k}{j-1}S_k(j), \quad k\ge 1.$$

Hence

$$\begin{aligned}(-1)^k k! &= (k+2)\sum_{j=1}^{k+1}(-1)^j\binom{k+1}{j}S_k(j)+2\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k}{j-1}S_k(j) \\ &= k\sum_{j=1}^{k+1}(-1)^j\binom{k+1}{j}S_k(j)+2\sum_{j=1}^{k+1}(-1)^j[\binom{k+1}{j}-\binom{k}{j-1}]S_k(j) \quad (69) \\ &= \sum_{j=1}^{k+1}(-1)^j[k\binom{k+1}{j}+2\binom{k}{j}]S_k(j), \quad k\ge 1.\end{aligned}$$

having used the identity:

$$\binom{k+1}{j}-\binom{k}{j-1}=\binom{k}{j}.$$

Consequently, we obtain:

$$\sum_{j=1}^{k+1}(-1)^j[k\binom{k+1}{j}+2\binom{k}{j}]S_k(j) = (-1)^k k!, \quad k\ge 1.$$

(iii) Putting $i=1$ in (57) together with (45) gives:

$$(-1)^k B_k = \sum_{j=1}^{k+1}\frac{(-1)^{j+1}}{(k+1)!}\sigma_{k+1,j}^{(k+1)}\binom{k+1}{j}S_k(j) = \sum_{j=1}^{k+1}\frac{(-1)^{j+1}}{(k+1)!}\frac{(k+1)!}{j}\binom{k+1}{j}S_k(j), \quad k\ge 0.$$

having used (30). Therefore

$$\sum_{j=1}^{k+1}(-1)^{j+1}\frac{\binom{k+1}{j}}{j}S_k(j)=(-1)^k B_k,\quad k\geq 0.$$

(iv) From (3), we have: $(1-e^{-x})G(n,x)=e^{nx}-1$, then

$$[\sum_{i=0}^{\infty}(\delta_{i0}-(-1)^i)\frac{x^i}{i!}][\sum_{i=0}^{\infty}S_i(n)\frac{x^i}{i!}]=\sum_{i=0}^{\infty}(n^i-\delta_{i0})\frac{x^i}{i!},\tag{70}$$

By using (2), we get:

$$n^i-\delta_{i0}\quad=\sum_{j=0}^{i}(\delta_{j0}-(-1)^j)\binom{i}{j}S_{i-j}(n)=\sum_{j=1}^{i}(-1)^{j+1}\binom{i}{j}S_{i-j}(n).\tag{71}$$

Replacing j by $i-j$ in (71), we obtain:

$$\sum_{j=0}^{i-1}(-1)^{i+1-j}\binom{i}{j}S_j(n)=n^i-\delta_{i0},\quad i\geq 0.\tag{72}$$

In particular, we have:

$$\sum_{j=1}^{i}(-1)^{i-j}\binom{i}{j-1}S_{j-1}(n)=n^i,\quad i\geq 1.\tag{73}$$

(v) Using (3), we have: $G(n,x)=e^x(1+G(n-1,x))$, then

$$\sum_{i=0}^{\infty}S_i(n)\frac{x^i}{i!}=[\sum_{i=0}^{\infty}\frac{x^i}{i!}][\sum_{i=0}^{\infty}(\delta_{i0}+S_i(n-1))\frac{x^i}{i!}],\tag{74}$$

Comparing the coefficients of $x^i, i\geq 0$ on both sides in (74) and using (2), we get:

$$\begin{aligned}S_i(n)\quad&=\sum_{j=0}^{i}\binom{i}{j}(\delta_{i-j,0}+S_{i-j}(n-1))=\sum_{j=0}^{i}\binom{i}{j}(\delta_{j0}+S_j(n-1))\\&=1+\sum_{j=0}^{i}\binom{i}{j}S_j(n-1).\end{aligned}\tag{75}$$

Equation (75) can be written in the form:

$$\sum_{j=1}^{i}\binom{i}{j}S_j(n-1)=S_i(n)-n,\quad i\geq 0.\tag{76}$$

(vi) Using (3), we have: $(e^{nx}+1)G(n,x)=G(2n,x)$, then

$$[\sum_{i=0}^{\infty}(n^i+\delta_{i0})\frac{x^i}{i!}][\sum_{i=0}^{\infty}S_i(n)\frac{x^i}{i!}]=\sum_{i=0}^{\infty}S_i(2n)\frac{x^i}{i!},\tag{77}$$

By using (2), we get:

$$\sum_{j=0}^{i}\binom{i}{j}(n^j+\delta_{j0})S_{i-j}(n)=S_i(2n).\tag{78}$$

Equation (78) can be written in the form:

$$\sum_{j=0}^{i-1} n^{i-j}\binom{i}{j} S_j(n) = S_i(2n) - 2S_i(n), \quad i \geq 0.$$

(vii) Using (3), we have: $e^{-x}G(n,x) = 1 + G(n-1,x)$, then

$$[\sum_{i=0}^{\infty}(-1)^i \frac{x^i}{i!}][\sum_{i=0}^{\infty} S_i(n)\frac{x^i}{i!}] = \sum_{i=0}^{\infty}(\delta_{i0} + S_i(n-1))\frac{x^i}{i!}, \tag{79}$$

Using (2), gives:

$$\sum_{j=0}^{i}\binom{i}{j}(-1)^j S_{i-j}(n) = \delta_{i0} + S_i(n-1). \tag{80}$$

Equation (80) can be written in the form:

$$\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j} S_j(n) = \delta_{i0} + S_i(n-1), \quad i \geq 0.$$

(viii) Using (3), we have: $e^{nx}(1 + G(n-1,-x)) = G(n,x)$, then

$$[\sum_{i=0}^{\infty} n^i \frac{x^i}{i!}][\sum_{i=0}^{\infty}(\delta_{i0} + (-1)^i S_i(n-1))\frac{x^i}{i!}] = \sum_{i=0}^{\infty} S_i(n)\frac{x^i}{i!}. \tag{81}$$

Using (2), yields:

$$\sum_{j=0}^{i}\binom{i}{j} n^j(\delta_{i-j,0} + (-1)^{i-j} S_{i-j}(n-1)) = S_i(n). \tag{82}$$

Equation (82) can be written in the form:

$$\sum_{j=0}^{i}(-1)^j n^{i-j}\binom{i}{j} S_j(n-1) = S_i(n-1), \quad i \geq 0,$$

from which the required result follows.

(ix) Using (3), we have: $e^{-nx}G(n,x) = (1 + G(n-1,-x))$, then

$$[\sum_{i=0}^{\infty}(-n)^i \frac{x^i}{i!}][\sum_{i=0}^{\infty} S_i(n)\frac{x^i}{i!}] = \sum_{i=0}^{\infty}(\delta_{i0} + (-1)^i S_i(n-1))\frac{x^i}{i!}, \tag{83}$$

By using (2), we have:

$$\sum_{j=0}^{i}(-n)^j\binom{i}{j} S_{i-j}(n) = \delta_{i0} + (-1)^i S_i(n-1). \tag{84}$$

Equation (84) can be written in the form:

$$\sum_{j=0}^{i}(-n)^{i-j}\binom{i}{j} S_j(n) = (-1)^i S_i(n-1) + \delta_{i0}, \quad i \geq 0.$$

More identities can be obtained. We list some additional identities whose proofs are left to the reader as exercises.

- $$\sum_{j=0}^{i-1}(-1)^{i-j} 2^j\binom{i}{j} S_j(n) = S_i(2n) - 2^{i+1} S_i(n), i \geq 0. \tag{85}$$

- $\sum_{j=0}^{i-1} 2^j \binom{i}{j} S_j(n) = S_i(2n+1) - 2^{i+1} S_i(n) - 1, i \geq 0.$ (86)
- $\sum_{j=0}^{i-1} \binom{i}{j} S_j(n-1) = n^i - 1, i \geq 0.$ (87)
- $\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} [(4n+1)^j - 1] E_{i-j} = 2^{i+1}[2^{i+1} S_i(n) - S_i(2n)], i \geq 0.$ (88)
- $2\sum_{j=0}^{\frac{i}{2}} \binom{i}{2j+1} S_{2j+1}(n) = (n+1)^i + n^i - 1 - \delta_{i0}, \quad i \text{ is even.}$ (89)
- $2\sum_{j=0}^{\frac{i-1}{2}} \binom{i}{2j} S_{2j}(n) = (n+1)^i + n^i - 1 - \delta_{i0}, \quad i \text{ is odd}.$ (90)
- $2\sum_{j=0}^{\frac{i}{2}-1} \binom{i}{2j} S_{2j}(n) = (n+1)^i - n^i - 1 + \delta_{i0}, \quad i \text{ is even.}$ (91)
- $2\sum_{j=0}^{\frac{i-1}{2}} \binom{i}{2j+1} S_{2j+1}(n) = (n+1)^i - n^i - 1 + 2S_i(n) + \delta_{i0}, \quad i \text{ is odd}.$ (92)

For integers $k \geq 0$ and $n > k$, we have:

$$S_i(n) = S_i(k) + \sum_{j=0}^{i} \binom{i}{j} k^{i-j} S_j(n-k), i \geq 0. \quad (93)$$

- $\sum_{j=0}^{i} \binom{i}{j} B_{i-j} S_j(n) = nB_i(n+1) - iS_i(n), i \geq 0.$ (94)
- $\sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j-1} S_j(n) = n^{i+1} - S_i(n-1) - \delta_{i0}, i \geq 0.$ (95)

All the above identities are tested using the Computer Algebra System (CAS), MAPLE. We finish this section by giving the following result.

Theorem 3.2. For $k \geq 0$, define the matrices A, B, S and N as follows:

$$A = [(-1)^{i-j} \frac{\binom{i}{j}}{i} B_{i-j}]_{i,j=1}^{k+1}, \qquad B = [(-1)^{i-j} \binom{i}{j-1}]_{i,j=1}^{k+1}, \quad S = [S_0(n), S_1(n), ..., S_k(n)]^T$$

and $N = [n, n^2, ..., n^{k+1}]^T$.

Then we have $B = A^{-1}$.

Proof. In matrix form the formula (46) can be written as:

$$AN = S. \tag{96}$$

Also, the matrix form of (73) gives:

$$BS = N. \tag{97}$$

From (96) and (97), we get: $B = A^{-1}$.

4. Computing $S_k(n)$ by Solving a Special Linear System

Our main goal in this section is to obtain the power sum $S_k(n)$ by solving a special linear system of equations for $c_{k+1,1}, c_{k+1,2}, ..., c_{k+1,k+1}$ (see (39)):

To do this consider the identity:

$$n^k = f_k(n) - f_k(n-1) = \sum_{i=1}^{k+1} c_{k+1,i}\left[n^i - (n-1)^i\right]$$

$$= \sum_{i=1}^{k+1} c_{k+1,i}\left[n^i - \sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}\, n^j\right] = \sum_{i=1}^{k+1} c_{k+1,i}\sum_{j=0}^{i-1}\binom{i}{j}(-1)^{i-j+1}\, n^j.$$

Reordering the order of summation, we get

$$\sum_{j=0}^{k} c_{k+1,i}\sum_{i=j+1}^{k+1}\binom{i}{j}(-1)^{i-j-1}\, n^j = n^k. \tag{98}$$

Comparing the coefficients of n^j, for $1 \le j \le k+1$, we obtain

$$\sum_{i=j+1}^{k+1}(-1)^{i-j-1}\binom{i}{j} c_{k+1,i} = \delta_{jk}, \quad 1 \le j \le k+1. \tag{99}$$

The system in (99) can also be written as:

$$\sum_{i=j}^{k+1}(-1)^{j-i}\binom{i}{j-1} c_{k+1,i} = \delta_{j-1,k}, \quad 1 \le j \le k+1. \tag{100}$$

Using Theorem 3.2, we get the solution of the system (100) in the form:

$$c_{k+1,j} = \frac{\binom{k+1}{j}}{k+1}(-1)^{k+1-j} B_{k+1-j}, \quad 1 \le j \le k+1. \tag{101}$$

These values are in complete agreement with (47).

Corollary 4.1. For any integer $k \ge 0$, we have:

$$T_k(n) = \sum_{r=1}^{n}(2r-1)^k = S_k(2n) - 2^k S_k(n).$$

Proof. We have

$$T_k(n)=\sum_{r=1}^{n}(2r-1)^k \quad =\sum_{r=1}^{n}[(2r-1)^k+(2r)^k-(2r)^k]$$
$$=\sum_{r=1}^{n}[(2r-1)^k+(2r)^k]-\sum_{r=1}^{n}(2r)^k=\sum_{r=1}^{2n}r^k-2^k\sum_{r=1}^{n}r^k.$$

Thus $$T_k(n)=\sum_{r=1}^{n}(2r-1)^k=S_k(2n)-2^kS_k(n).$$

For example, by applying Corollary 4.1 on $T_5(n)=\sum_{r=1}^{n}(2r-1)^5$, we get:

$$\begin{aligned}T_5(n)&=S_5(2n)-2^5S_5(n)\\&=\frac{1}{3}n^2(2n+1)^2(8n^2+4n-1)-32[\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)]\\&=\frac{1}{3}n^2(16n^4-20n^2+7).\end{aligned}$$

REFERENCES

1. D. Allison, A Note on sums of powers of integers, *Amer. Math. Monthly*, 68 (3) (1961) 272.
2. I.Anderson, Sums of squares and binomial coefficients, *Math. Gazette*, 65 (1981) 87-92.
3. T.Arponen, A matrix approach to polynomials, *Linear Algebra Appl.*, 359 (2003) 181-196.
4. A.F.Beardon, Sums of powers of integers, *Amer. Math. Monthly*, 103 (1996) 201-213.
5. B.L.Burrows and R.F.Talbot, Sums of powers of integers, *Amer. Math. Monthly*, 91 (1984) 394-403.
6. G.-S.Cheon and J.-S.Kim, Factorial Stirling matrix and related combinatorial sequences, *Linear Algebra Appl.*, 357 (2002) 247-258.
7. T.Colledge, *Pascal's Triangle*, Stradbroke: Tarquin, 1992.
8. T.Crilly and S.Millward, Sums of powers of integers: A General method, *Math. Gazette*, 72 (461) (1988) 205-207.
9. W.WU Dane, Bernoulli numbers and sums of powers, *Internat. J. Math. Ed. Sci. Tech.*, 32 (3) (2001) 440-443.
10. J.A.Donaldson, On the Summation of $1^r+2^r+...+n^r$, *Proc. Edinb. Math. Soc.*, (2), 31 (1913) 9-16.
11. J.Doucet and A.Saleh-Jahromi, Sums of powers of integers, Research paper in Proceedings of the Louisiana-Mississippi Section of the Mathematical Association of America, Spring 2002. http://www.mc.edu/campus/users/travis/maa/proceedings/spring2002/doucet.jahroni.pdf
12. S.M.Edmonds, Sums of powers of the natural numbers, *Math. Gazette*, 41 (1957) 187-189.
13. A.W.F.Edwards, Sums of powers of integers: a little of the history, *Math. Gazette*, 66 (1982) 22-28.

14. A.W.F.Edwards, A quick route to sums of powers, *Amer. Math. Monthly,* 93 (1986) 451-455.
15. A.W.F.Edwards, *Pascal's Arithmetical Triangle*, Charles Griffin & Co., 1987.
16. H.M.Edwards, *Fermat's Last Theorem : A Genetic Introduction To Algebraic Number Theory*, Springer-Verlag Inc. New York, 1977.
17. M.E.A.El-Mikkawy, Explicit inverse of a generalized Vandermonde matrix, *Appl. Math. Comput.*, 146 (2003) 643-651.
18. M.El-Mikkawy and F.Atlan, Derivation of identities involving some special polynomials and numbers via generating functions with applications, *Appl. Math. Comput.*, 220 (2013) 518-535.
19. I.Gessel, A formula for power sums, *Amer. Math. Monthly,* 95 (1988) 961-962.
20. I.S.Gradshteyn and I.M.Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, 1980.
21. S.-L.Gue and F.Qi, Recursion formulae for $\sum_{m=1}^{n} m^k$, *Z. Anal. Anweendungen*, 18 (1999) 1123-1130.
22. T.C.T.Kotiah, Sums of powers of integers - A review, *Internat. J. Math. Ed. Sci. Tech.*, 24 (1993) 863-874.
23. D.E.Knuth, Johann Faulhaber and sums of powers, *Math. Comp.*, 61 (1993) 277-294.
24. K.R.Kundert, Calculating sums of powers as sums of products, *Math. Mag.*, 54 (2) (1981) 81-83.
25. W.Lang, A196837: Ordinary generating functions for sums of powers of the first n positive integers, (2011) http://oeis.org/A196837/a196837.pdf.
26. C.Long, Pascal's triangle, difference tables and arithemtic sequences or order N, *College Math. J.*, 15 (1984), 290-298.
27. C.-Y Ma and S.-L.Yang, Pascal type matrices and Bernoulli numbers, *Int. J. Pure Appl. Math.*, 58(3) (2010) 249-254.
28. M.Pal, Intersection graphs: an introduction, *Annals of Pure and Applied Mathematics*, 4(1) (2013) 43-91.
29. N.Marikkannan and V.Ravichandran, Sum of powers of natural numbers using integration, *Reson. J. Sci. Educ.*, 8 (2) (2003) 80-84.
30. J.Nunemacher and R.M.Young, On the sum of consecutive kth powers, *Math. Mag.*, 60 (1987) 237-238.
31. R.W.Owens, Sums of powers of integers, *Math. Mag.*, 65 (1992) 38-40.
32. R.V.Parker, Sums of powers of the integers, *Math. Gazette*, 42 (1958) 91-95.
33. D.E.Penney and C. Pomerance, Multiplicative relations for sums of initial k-th powers, *Amer. Math. Monthly,* 92 (1985) 729-731.
34. T.J.Pfaff, Deriving a formula for sums of powers of integers, *The PME J.*, 12 (2007) 425-430.
35. P.A.Piza, Powers of sums and sums of powers, *Math. Mag.*, 25 (1952) 137-142.
36. R.Pradhan and M.Pal, Intuitionistic fuzzy linear transformations, *Annals of Pure and Applied Mathematics,* 1(1) (2012) 57-68.
37. K.H.Rosen, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, Boca Raton, FL, 2000.
38. H.J.Schultz, The sum of the k-th powers of the first n integers, *Amer. Math. Monthly,*

87 (1980) 478-481.
39. S.H.Scott, Sums of powers of natural numbers by coefficient operation, *Math. Gazette*, 64 (1980) 231-238.
40. S.A.Shirali, On sums of powers of integers, *Resonance*, 12 (7) (2007) 27-43.
41. S.-L.Yang and Z.K.Qiao, Some symmetry identities for the euler polynomials, *J. Math. Res. Exposition*, 30 (3) (2010) 457-464.
42. J.B.Wilker, Sums of powers of natural numbers by efficient cooperation, *Math. Gazette*, 64 (1980) 239-244.
43. E.E.Witmer, The sums of powers of integers, *Amer. Math. Monthly*, 42 (9) (1935) 540-548.
44. D.R.Woodall, Finite sum, matrices and induction, *Math. Gazette*, 65 (1981) 92-103.