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Pairwise Fuzzy Bicontinuous Map in Fuzzy Biclosure Space

***R. Navalakhe*[1] and *U.D.Tapi*[2]**

Department of Applied Mathematics & Computational Science
Shri G.S.Institute of Technology & Science, 23 Park Road, Indore (M.P.) PIN-452003
India. [1]Email: sgsits.rachna@gmail.com, [2]Email: utapi@sgsits.ac.in

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Abstract. The purpose of this paper is to introduce the concept of pairwise fuzzy bicontinuous map in fuzzy biclosure space and study some of their properties.

Keywords: Fuzzy closure operator, fuzzy biclosure space, fuzzy continuous map, Pairwise fuzzy bicontinuous map

AMS Mathematics Subject Classification : 54A40

1. Introduction

Fuzzy closure spaces were first studied by Mashhour and Ghanim [2]. Recently, Chawalit Boonpok [1] introduced the notion of biclosure spaces. Such spaces are equipped with two arbitrary closure operators. He extended some of the standard results of separation axioms in closure space to a biclosure space. Thereafter a large number of papers have been written to generalize the concept of closure space to a biclosure space. Now Tapi and Navalakhe [3] has introduced the notion of fuzzy biclosure spaces and generalized the concept of fuzzy closure space to fuzzy biclosure space.

2. Preliminaries

Definition 2.1. A *fuzzy biclosure space* is a triple (X, u_1, u_2) where X is a non empty set and u_1, u_2 are two fuzzy closure operators on X which satisfy the following properties:

(i) $u_1(0_X) = 0_X$ and $u_2(0_X) = 0_X$

(ii) $\mu \le u_1\mu$ and $\mu \le u_2\mu$ for all $\mu \le I^X$

(iii) $u_1(\mu \vee \upsilon) = u_1\mu \vee u_1\upsilon$ and $u_2(\mu \vee \upsilon) = u_2\mu \vee u_2\upsilon$ for all $\mu, \upsilon \le I^X$.

Definition 2.2. [3] A subset μ of a fuzzy biclosure space (X, u_1, u_2) is called fuzzy closed if $u_1u_2\mu = \mu$. The complement of fuzzy closed set is called fuzzy open.

Definition 2.3, [3] A fuzzy closure space (Y, v_1, v_2) is said to be a subspace of (X, u_1, u_2) if $Y \le X$ and $v_1\mu = u_1\mu \wedge 1_Y$ or $v_2\mu = u_2\mu \wedge 1_Y$ for each fuzzy subset $\mu \le I^Y$. If 1_Y is fuzzy closed in (X, u_1, u_2), then the subspace (Y, v_1, v_2) of (X, u_1, u_2) is also fuzzy closed.

Definition 2.4. [3] Let (X, u_1, u_2) and (Y, v_1, v_2) be fuzzy biclosure spaces. A map $f:(X, u_1, u_2) \to (Y, v_1, v_2)$ is called fuzzy continuous if $f^{-1}(\mu)$ is a fuzzy closed subset of (X, u_1, u_2) for every fuzzy closed subset μ of (Y, v_1, v_2).

Clearly, it is easy to prove that a map $f:(X, u_1, u_2) \to (Y, v_1, v_2)$ is fuzzy continuous if and only if $f^{-1}(\upsilon)$ is a fuzzy open subset of (X, u_1, u_2) for every fuzzy open subset υ of (Y, v_1, v_2).

Definition 2.5. The product of a family $\left\{\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) : \alpha \in J\right\}$ of fuzzy biclosure spaces denoted by $\prod_{\alpha \in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right)$ is the fuzzy biclosure space $\left(\prod_{\alpha \in J} X_\alpha, u^1, u^2\right)$ where $\left(\prod_{\alpha \in J} X_\alpha, u^i\right)$ for $i \in \{1, 2\}$ is the product of the family of fuzzy closure spaces $\left\{X_\alpha, u^i : \alpha \in J\right\}$.

Remark 2.6. Let $\prod_{\alpha \in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) = \left(\prod_{\alpha \in J} X_\alpha, u^1, u^2\right)$. Then for each $\mu \le \prod_{\alpha \in J} X_\alpha$, $u^1u^2\mu = \prod_{\alpha \in J} u_\alpha^1 u_\alpha^2 \pi_\alpha(\mu)$.

Proposition 2.7. [3] *Let* $\left\{\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) : \alpha \in J\right\}$ *be a family of fuzzy biclosure spaces. Then for each* $\beta \in J$, *the projection map* $\pi_\beta : \prod_{\alpha \in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) \to \left(X_\beta, u_\beta^1, u_\beta^2\right)$ *is fuzzy continuous.*

Proposition 2.8.[3] *Let* $\left\{\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) : \alpha \in J\right\}$ *be a family of fuzzy biclosure spaces and let* $\beta \in J$. *Then* $\eta \le X_\beta$ *is a fuzzy closed subset of* $\left(X_\beta, u_\beta^1, u_\beta^2\right)$ *if and only if* $\eta \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ *is a fuzzy closed subset of* $\prod_{\alpha \in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right)$.

Proposition 2.9. [3] *Let* $\left\{\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) : \alpha \in J\right\}$ *be a family of fuzzy biclosure spaces and let* $\beta \in J$. *Then* $\gamma \le X_\beta$ *is a fuzzy open subset of* $\left(X_\beta, u_\beta^1, u_\beta^2\right)$ *if and only if* $\gamma \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ *is a fuzzy open subset of* $\prod_{\alpha \in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right)$.

3. Pairwise fuzzy bicontinuous maps

Definition 3.1. Let $\left(X, u_1, u_2\right)$ and $\left(Y, v_1, v_2\right)$ be fuzzy biclosure spaces. A map $f : \left(X, u_1, u_2\right) \to \left(Y, v_1, v_2\right)$ is called pairwise fuzzy bicontinuous if maps $f : \left(X, u_1\right) \to \left(Y, v_2\right)$ and $f : \left(X, u_2\right) \to \left(Y, v_1\right)$ are fuzzy continuous.

Remark 3.2. Every pairwise fuzzy bicontinuous map is fuzzy bicontinuous map.

Proposition 3.3. *Let* $\left(X, u_1, u_2\right)$ *and* $\left(Y, v_1, v_2\right)$ *be fuzzy biclosure spaces. Then* $f : \left(X, u_1, u_2\right) \to \left(Y, v_1, v_2\right)$ *is pairwise fuzzy bicontinuous if and only if* $u_1 f^{-1}(\upsilon) \le f^{-1}(v_2\upsilon)$ *and* $u_2 f^{-1}(\upsilon) \le f^{-1}(v_1\upsilon)$ *for every* $\upsilon \le Y$.

Proof: Let $\upsilon \le Y$. Then $f^{-1}(\upsilon) \le X$. Since the map f is pairwise fuzzy bicontinuous,

$$f\left(u_1 f^{-1}(\upsilon)\right) \le v_2 f\left(f^{-1}(\upsilon)\right) \le v_2\upsilon \text{ and } f\left(u_2 f^{-1}(\upsilon)\right) \le v_1 f\left(f^{-1}(\upsilon)\right) \le v_1\upsilon$$

Therefore, $u_1 f^{-1}(\upsilon) \le f^{-1}(v_2\upsilon)$ and $f\left(u_2 f^{-1}(\upsilon)\right) \le v_1 f\left(f^{-1}(\upsilon)\right) \le v_1\upsilon$.

Therefore, $u_1 f^{-1}(\upsilon) \le f^{-1}(v_2\upsilon)$ and $u_2 f^{-1}(\upsilon) \le f^{-1}(v_1\upsilon)$.

Conversely, let $\mu \le X$. Then $f(\mu) \le Y$. Thus $u_1 f^{-1}\left(f(\mu)\right) \le f^{-1}\left(v_2 f(\mu)\right)$ and $u_2 f^{-1}\left(f(\mu)\right) \le f^{-1}\left(v_1 f(\mu)\right)$. Consequently

$$f(u_1\mu) \le f\left(u_1 f^{-1}\left(f(\mu)\right)\right) \le f\left(f^{-1}\left(v_2 f(\mu)\right)\right) \le v_2 f(\mu) \text{ and}$$

$$f(u_2\mu) \le f\left(u_2 f^{-1}\left(f(\mu)\right)\right) \le f\left(f^{-1}\left(v_1 f(\mu)\right)\right) \le v_1 f(\mu).$$

Hence, the map f is pairwise fuzzy bicontinuous .

Proposition 3.4. *Let* $\left(X, u_1, u_2\right)$, $\left(Y, v_1, v_2\right)$ *and* $\left(Z, w_1, w_2\right)$ *be fuzzy biclosure spaces. If* $f : \left(X, u_1, u_2\right) \to \left(Y, v_1, v_2\right)$ *is pairwise fuzzy bicontinuous and* $h : \left(Y, v_1, v_2\right) \to \left(Z, w_1, w_2\right)$ *is fuzzy continuous, then* $h \circ f : \left(X, u_1, u_2\right) \to \left(Z, w_1, w_2\right)$ *is pairwise fuzzy bicontinuous.*

Proof: Let $\mu \le X$. Since $h \circ f(u_1\mu) = h\left(f(u_1\mu)\right)$, $h \circ f(u_2\mu) = h\left(f(u_2\mu)\right)$ and the map f is pairwise fuzzy bicontinuous, therefore $h\left(f(u_1\mu)\right) \le h\left(v_2 f(\mu)\right)$ and $h\left(f(u_2\mu)\right) \le h\left(v_1 f(\mu)\right)$. Since the map h is fuzzy continuous, $h\left(v_2 f(\mu)\right) \le w_2 h\left(f(\mu)\right)$ and $h\left(v_1 f(\mu)\right) \le w_1 h\left(f(\mu)\right)$. Thus

$h\circ f(u_1\mu)\le w_2 h\circ f(\mu)$ and $h\circ f(u_2\mu)\le w_1 h\circ f(\mu)$. Consequently, the map $h\circ f$ is pairwise fuzzy bicontinuous.

Proposition 3.5. *Let (X,u_1,u_2) and (Y,v_1,v_2) be fuzzy biclosure spaces and let (Z,w_1,w_2) be a fuzzy closed subspace of (X,u_1,u_2). If the map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is pairwise fuzzy bicontinuous, then the map $f\,|\,z:(Z,w_1,w_2)\to(Y,v_1,v_2)$ is pairwise fuzzy bicontinuous.*

Proof: Let the map f be pairwise fuzzy bicontinuous. If $\upsilon\le Z$, then

$$f\,|\,z(w_1(\upsilon))=f\,|\,z(u_1\upsilon\wedge Z)=f\,|\,z(u_1\upsilon)=f(u_1\upsilon)\le v_2 f(\upsilon)=v_2 f\,|\,z(\upsilon) \text{ and}$$

$$f\,|\,z(w_2(\upsilon))=f\,|\,z(u_2\upsilon\wedge Z)=f\,|\,z(u_2\upsilon)=f(u_2\upsilon)\le v_1 f(\upsilon)=v_1 f\,|\,z(\upsilon).$$

Consequently, the map $f\,|\,z$ is pairwise fuzzy bicontinuous.

Proposition 3.6. *Let (X,u_1,u_2) be a fuzzy biclosure space, $\{(Y_\alpha,v_\alpha^1,v_\alpha^2):\alpha\in J\}$ be a family of fuzzy biclosure spaces and $f:(X,u_1,u_2)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha^1,v_\alpha^2)$ be a map. Then the map $f:(X,u_1,u_2)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha^1,v_\alpha^2)$ is pairwise fuzzy bicontinuous if and only if the map $\pi_\alpha\circ f:(X,u_1,u_2)\to(Y_\alpha,v_\alpha^1,v_\alpha^2)$ is pairwise fuzzy bicontinuous for each $\alpha\in J$.*

Proof: Let f be pairwise fuzzy bicontinuous. Since π_α is fuzzy continuous for each $\alpha\in J$, it follows that $\pi_\alpha\circ f$ is pairwise fuzzy bicontinuous for each $\alpha\in J$.

Conversely, let the map $\pi_\alpha\circ f$ be pairwise fuzzy bicontinuous for each $\alpha\in J$. Suppose that the map f is not pairwise fuzzy bicontinuous. Consequently, $f:(X,u_1)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha^2)$ is not fuzzy bicontinuous or $f:(X,u_2)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha^1)$ is not fuzzy bicontinuous. If the map $f:(X,u_1)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha^2)$ is not fuzzy bicontinuous. Then there exists a fuzzy subset μ of X such that $f(u_1\mu)\not\le\prod_{\alpha\in J}v_\alpha^2\pi_\alpha(f(\mu))$. Therefore, there exists $\beta\in J$ such that $\pi_\beta(f(u_1\mu))\not\le v_\beta^2\pi_\beta(f(\mu))$. This contradicts the continuous for each $\alpha\in J$, it follows that $\pi_\alpha\circ f$ is pairwise fuzzy bicontinuity of $\pi_\beta\circ f$. If $f:(X,u_2)\to\prod_{\alpha\in J}(Y_\alpha,v_\alpha^1)$ is not fuzzy bicontinuous. Then there exists a fuzzy subset μ of X such that $f(u_2\mu)\not\le\prod_{\alpha\in J}v_\alpha^1\pi_\alpha(f(\mu))$. Therefore, there exists

$\beta \in J$ such that $\pi_\beta\left(f(u_2\mu)\right) \nleq v_\beta^1 \pi_\beta\left(f(\mu)\right)$. This contradicts the fuzzy bicontinuity of the map $\pi_\beta \circ f$. Hence, the map f is pairwise fuzzy bicontinuous.

Proposition 3.7. *Let* $\left\{\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) : \alpha \in J\right\}$ *and* $\left\{\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right) : \alpha \in J\right\}$ *be families of fuzzy biclosure spaces. For each* $\alpha \in J$, $f_\alpha : X_\alpha \to Y_\alpha$ *be a map and let* $f : \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in J} Y_\alpha$ *be defined by* $f\left((x_\alpha)_{\alpha\in J}\right) = \left(f_\alpha(x_\alpha)\right)_{\alpha\in J}$. *Then the map* $f : \prod_{\alpha\in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) \to \prod_{\alpha\in J}\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$ *is pairwise fuzzy bicontinuous if and only if the map* $f_\alpha : \left(X_\alpha, u_\alpha^1, u_\alpha^2\right) \to \left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$ *is pairwise fuzzy bicontinuous for each* $\alpha \in J$.

Proof: Let the map f be pairwise fuzzy bicontinuous, let $\beta \in J$ and

$\mu \le X_\beta$.Then $f_\beta\left(u_\beta^1\mu\right) = \pi_\beta\left(f_\beta\left(u_\beta^1\mu\right) \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} f_\alpha\left(u_\alpha^1 X_\alpha\right)\right) = \pi_\beta\left(f\left(u_\beta^1\mu \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} u_\alpha^1 X_\alpha)\right)\right)$

$$= \pi_\beta\left(f\left(\prod_{\alpha\in J} u_\alpha^1 \pi_\alpha\left(\mu \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} X_\alpha\right)\right)\right) \le \pi_\beta\left(\prod_{\alpha\in J} v_\alpha^2 \pi_\alpha\left(f\left(\mu \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} X_\alpha)\right)\right)\right)$$

$$= \pi_\beta\left(\prod_{\alpha\in J} v_\alpha^2 \pi_\alpha\left(f_\beta(\mu) \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} f_\alpha\left(X_\alpha\right)\right)\right) = \pi_\beta\left(v_\beta^2 f_\beta(\mu) \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} v_\alpha^2 f_\alpha\left(X_\alpha\right)\right) = v_\beta^2 f_\beta(\mu)$$

and

$$f_\beta\left(u_\beta^2 A\right) = \pi_\beta\left(f_\beta\left(u_\beta^2\mu\right) \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} f_\alpha\left(u_\alpha^2 X_\alpha\right)\right) = \pi_\beta\left(f\left(u_\beta^2\mu \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} u_\alpha^2 X_\alpha)\right)\right)$$

$$= \pi_\beta\left(f\left(\prod_{\alpha\in J} u_\alpha^2 \pi_\alpha\left(\mu \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} X_\alpha\right)\right)\right) \le \pi_\beta\left(\prod_{\alpha\in J} v_\alpha^1 \pi_\alpha\left(f\left(\mu \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} X_\alpha)\right)\right)\right)$$

$$= \pi_\beta\left(\prod_{\alpha\in J} v_\alpha^1 \pi_\alpha\left(f_\beta(\mu) \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} f_\alpha\left(X_\alpha\right)\right)\right) = \pi_\beta\left(v_\beta^1 f_\beta(\mu) \times \prod_{\substack{\alpha\ne\beta \\ \alpha\in J}} v_\alpha^2 f_\alpha\left(X_\alpha\right)\right) = v_\beta^1 f_\beta(\mu)$$

Hence, the map f_β is pairwise fuzzy bicontinuous.

Conversely, let the map f_α be pairwise fuzzy bicontinuous for each $\alpha \in J$ and let $\mu \le \prod_{\alpha\in J} X_\alpha$. Then

$$f\left(\prod_{\alpha\in J}u_\alpha^1\pi_\alpha(\mu)\right)=\prod_{\alpha\in J}f_\alpha\left(\prod_{\alpha\in J}u_\alpha^1\pi_\alpha(\mu)\right)=\prod_{\alpha\in J}f_\alpha\left(u_\alpha^1\pi_\alpha(\mu)\right)$$

$$\leq\prod_{\alpha\in J}v_\alpha^2 f_\alpha\left(\pi_\alpha(\mu)\right)=\prod_{\alpha\in J}v_\alpha^2\pi_\alpha\left(f(\mu)\right)\text{ and}$$

$$f\left(\prod_{\alpha\in J}u_\alpha^2\pi_\alpha(\mu)\right)=\prod_{\alpha\in J}f_\alpha\left(\prod_{\alpha\in J}u_\alpha^2\pi_\alpha(\mu)\right)=\prod_{\alpha\in J}f_\alpha\left(u_\alpha^2\pi_\alpha(\mu)\right)$$

$$\leq\prod_{\alpha\in J}v_\alpha^1 f_\alpha\left(\pi_\alpha(\mu)\right)=\prod_{\alpha\in J}v_\alpha^1\pi_\alpha\left(f(\mu)\right)$$

Therefore, the map f is pairwise fuzzy bicontinuous.

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