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Mathematical Modeling with Local Volatility Surface by Radial Basis Function Approach

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Abstract. Some obstacles create vulnerable situations in financial market. Overcome this unexpected situation, it is essential to reform the financial market by measuring the risk of share market. This project investigates the sensitivity of radial basis functions to construct different volatility surface by radial basis function approaches to understand the risk of share market. Different types of radial basis functions on the basis of different error measurement such as average error as well as relative average error of Dhaka Stock Exchange (DSE) are measured and multiquadratic function gives the best result with compare to other functions especially Gaussian and Thin plate spline function.

Keywords: Financial market, radial basis function, volatility surface

AMS Mathematics Subject Classification (2010): 34K28, 34K50, 34K60

1. Introduction

Stock market is one of the principal financial institutions of Bangladesh which opens door for companies to raise huge amount of capital from a lot of individual investors inside and outside of the country. To observe the present situation of the market and to find the volatility surface we use radial basis function with some interpolation and also try to introduce different Radial basis functions such as Gaussian function, multi-quadratic function and thin plane spline function to evaluate observations.

The history of radial basis function (RBF) approximations goes back to 1968, when multiquadric radial basis functions were first used by Hardy to represent topographical surfaces given sets of sparse scattered measurements [8]. Today, the literature on different aspects of RBF approximation is extensive. RBFs are used not only for interpolation or approximation of data sets, but also as tools for solving e.g., differential equations [5, 6]. However, their main strength remains the same: The ability to elegantly and accurately approximate scattered data without using any mesh. There have been some concerns about the computational cost and stability of the RBF methods, but many different viable approaches to overcome these difficulties have been proposed, see for example [2, 12, 14] and the references therein. In this project RBFs are mainly focused on the reconstruction of unknown functions from known data.

In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton made a major breakthrough in the pricing of stock options [1]. The famous Black-Scholes model has been intensively studied and used as the foundation for almost any option pricing formula in today's financial markets. The model has a huge influence on the way that trader's price and hedge options [15]. It has also been pivotal to the growth and success of financial engineering in the 1980s and 1990s. Here shows the Black-Scholes model for valuing Bangladeshi call and put options on a non dividend paying stock is derived. In 1987 Hull and White explained the pricing of options on assets with stochastic volatilities [10]. Hon and Mao used radial basis function method for solving options pricing model in 1999 [9]. Coleman and Verma reconstructed the unknown local volatility function for options pricing model also in 1999 [3]. Recently, many researchers working in the field of financial mathematics especially radial basis functions approaches to observe share market situation. In 1999, Schaback improved error bounds for scattered data interpolation by radial basis functions [13] and solved limit problems for interpolation by analytic radial basis functions in 2008. Driscoll, and Fornberg described interpolation problem in the limit of increasingly flat radial basis functions [4]. Kim, et al. reconstructed local volatility function approximation by using radial basis function networks in 2006 [11]. In 2010, Glover used radial basis function approach to reconstructing the local volatility surface of European options [7].
This paper can be explained how volatility can be either estimated from historical data or implied from option prices using the model and show how the Black-Scholes model can be extended to deal with Bangladeshi call and put options on dividend-paying stocks and present some results on the pricing of Bangladeshi call options on dividend paying stocks.

2. Some mathematical tools

The derivation of the Black-Scholes partial differential equation (PDE) is based on the fundamental fact that the option price and the stock price depend on the same underlying source of uncertainty. If S is asset price, σ is volatility, r is risk free rate and $V(S,t)$ the price of a derivative as a function of time and stock price then Black-Scholes partial differential equation [1] is

$$rV = \frac{\partial V}{\partial S} rS + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \qquad (1)$$

The boundary conditions can be easily determined by the option price. For a call option at expiry the option is worth the difference between the current underlying asset price S and the strike price K, if $S > K$ the call boundary condition is

$$V_T = \max(S_T - K,\ 0) \qquad (2)$$

where, S_T is the asset price at maturity T. Again, a similar argument can be used for a put option resulting in the put boundary condition

$$V_T = \max(K - S_T,\ 0) \qquad (3)$$

The Black-Scholes equation shows that inputs required for modeling an option are the underlying asset price S, the strike K, the maturity T, the risk free rate r, and the volatility σ, all these parameters are directly observable except for the volatility. The radial basis function (RBF) is a function of the distance of the point to the origin. That is,

$\phi(x)=\phi(\|x\|)$ is the RBF so that ϕ acts on a vector in R^n space and only through the norm. This means that ϕ can be thought a scalar function. The radial basis functions as the model functions is $f(x)=\alpha_1\phi(\|x-x_1\|)+..........+\alpha_p\phi(\|x-x_p\|)$ (4)

where $\phi: R^+ \to R$ is typically nonlinear and is referred to as the transfer function [13]. Three types of radial basis functions like Gaussian, thin plate, and multiquadric are chosen for the model setup. RBF represents a map from *P*-dimensional input space to the one-dimensional output space i.e. $f: R^p \to R^1$ that consists of a set of weights $\{w^{(i)}\}_{i=1}^m$ and a set of radial basis functions $\{g^{(i)}\}_{i=1}^m$ where $m \le n$. There is a large class of radial basis functions which can be written in a general form

$$g^{(i)}(x)=\phi^{(i)}\left(\|x-c^{(i)}\|\right) \tag{5}$$

where $\|\,.\,\|$ denotes the Euclidean norm and $\{c^{(i)}\}_{i=1}^m$ is a set of the centers that can be chosen from the data points. For function approximation here uses Multiquadric function approximation $\phi^{(i)}(r)=\left(\|x-c^{(i)}\|\right)=\sqrt{r^2+a^{(i)2}}$ for some $a^{(i)}>0$ (6)

Inverse multiquadric approximation and Gaussiann function approximation respectively.

$$\phi^{(i)}(r)=\left(\|x-c^{(i)}\|\right)=\frac{1}{\sqrt{r^2+a^{(i)2}}} \quad \text{for some } a^{(i)}>0 \tag{7}$$

$$\phi^{(i)}(r)=\left(\|x-c^{(i)}\|\right)=\exp\left(-\frac{r^2}{a^{(i)2}}\right) \quad \text{for some } a^{(i)}>0 \tag{8}$$

where $a^{(i)}$ is usually referred to as the width of the i th basis function and

$$r=\|x-c^{(i)}\|=\sqrt{\left(x-c^{(i)}\right).\left(x-c^{(i)}\right)}.$$

Here multivariate interpolation is used for reconstruct volatility surface. Thus the simplest case of reconstruction of a d-variate unknown function u^* from data occurs when only a finite number of data in the form of values $u^*(x_1),........., u^*(x_m)$ at arbitrary locations $x_1,......, x_m$ in R^d forming a set $X=\{x_1,......, x_m\}$ are known. In contrast to the *n* trial points $y_1,......, y_n$ is

$$u(x)=\sum_{k=1}^{n}\alpha_k\phi\left(\|x-y_k\|_2\right) \tag{9}$$

The *m* data locations $x_1,......, x_m$ are called test points or collocation points. To calculate a trial function *u* of the form (9) which reproduces the data $u^*(x_1),........, u^*(x_m)$ well, we have to solve the $m\times n$ linear system $\sum_{k=1}^{n}\alpha_k\phi\left(\|x_i-y_k\|_2\right)\approx u^*(x_i), 1<i<m$ (10)

for *n* coefficients $\alpha_1,......, \alpha_n$ matrices with entries $\phi\left(\|x_i-y_k\|_2\right)$ will occur and they are called kernel matrices in machine learning. If there is no noise in the data, its make then sense to reconstruct u^* by a function u of the form (9) by enforcing the exact interpolation conditions

$$u^*(x_j) = \sum_{k=1}^{n} \alpha_k \phi\left(\left\| x_j - x_k \right\|_2\right), 1 \le j \le m = n \tag{11}$$

This is a system of m linear equations in $m = n$ unknowns $\alpha_1, \ldots\ldots, \alpha_n$ with a symmetric coefficient matrix

$$A_x = \left(\phi\left(\left\| x_j - x_k \right\|_2\right)\right)_{1 \le j,\, k \le m} \tag{12}$$

In general, solvability of such a system is a serious problem, but one of the central features of kernels and radial basis functions is to make this problem.

3. The model

Reconstructing the local volatility surface is using for radial basis functions and the advance taken in the paper follows that of [7] closely, except radial basis functions are used instead of spline to represent the local volatility function. That is, a function

$$\sigma(S,t) = \sum_{j=1}^{m} w_j h_j \tag{13}$$

with *h* a set of *m* radial basis functions and w_j a set of corresponding weights found that satisfies

$$\min_{\sigma(S,t)} \sum_{i=1}^{n} [v_i(\sigma(S,t)) - v_{BS}(f_i)]^2 \tag{14}$$

where $v_{BS}(f_i)$ is the set of n observed Black-Scholes prices and $v_i(\sigma(S,t))$ is the option price at *S* and *t* are given by

$$rV = \frac{\partial V}{\partial S} rS + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma(S,t)^2 S^2 \tag{15}$$

The generated surfaces suffer from over fit and become unstable. To reduce this unstable condition, Tikhonov regularization can be used for this problem

$$\min_{\sigma(S,t)} \sum_{i=1}^{n} [v_i(\sigma(S,t)) - v_{BS}(f_i)]^2 + \lambda \sum_{j=1}^{m} w_j \tag{16}$$

Equations (14) and (16) are nonlinear least squares minimization problems. A number of simplifying assumptions and heuristics are used to reduce the scale of the optimization presented in equation (14). Firstly we assume that the set of *m* radial basis functions, *h* in equation (13) are known and their positioning (centers) are fixed. The range of function sets are used and try to establish the best choice for the local volatility problem. The result of this assumption is to reduce the optimization problem to find the optimal weight vector $\boldsymbol{w}$. Secondly we assume that, if Tikhonov regularization is needed then the regularization parameter λ is chosen by using trial and error methods which are found the optimal λ given the non-linear nature of the problem is out of the scope of this project. The procedure to recover the local volatility function for the problem is presented below for each of the key steps. For the function set *h* with observed market data *f* then

(i) Find an initial weight vector w_0 .

(ii) Evaluate the cost function given in equation (14)

(iii) Using an optimization algorithm update the optimal weight vector.

The whole procedure is very sensitive to initial choice of weight vector. To solve this and generate a surface of realistic volatilities a simple method is used. To find an initial w_0 such that the implied volatilities of observed market data *f* are interpolated using the appropriate radial basis function set *h* and solve the following equation,

$$\min_{w_0} \sum_{i=1}^{n} \left[f_i - \sum_{j=1}^{m} w_0 h_j \left(x_i \right) \right]^2 + \lambda \sum_{j=1}^{m} w_0^2 j \tag{17}$$

This provides an initial weight vector w_0 that gives a volatility surface that is reasonable. Nelder-Mead Simplex Optimization algorithm is used as an optimization algorithm. The basic idea behind the Nelder-Mead Simplex algorithm is the creation and evolution of a simplex of points on the cost function surface to find the minimum. A simplex is a prototype with $n+1$ vertices in *n* dimensions. The vertices of this prototype are evaluated and adjusted using several simple rules depending on the geometry of the function being searched. The first stage of the Nelder-Mead algorithm is creating the simplex.

For the implementation of this project the Preconditioned Conjugate Gradient approach [7] is used and the optimization algorithms for purposes of efficiency and simplicity it is decided that implementations in the MATLAB optimization toolbox would be used. The data which are used in this research obtained from Dhaka Stock Exchange and collect information about different strikes and different maturity rates from Bangladesh Bank. All data which are used in this research are secondary data.

4. Results and discussions

The basic problem of scientific computing that recovers the multivariate functions from discrete data. For this purpose we use radial basis functions and confine to reconstruct from strong data consisting of evaluations of the function itself or its derivatives at discrete points. Using 258 data to recover the functions from data sets are given as integrals against the test functions which are the challenging research problems [12].

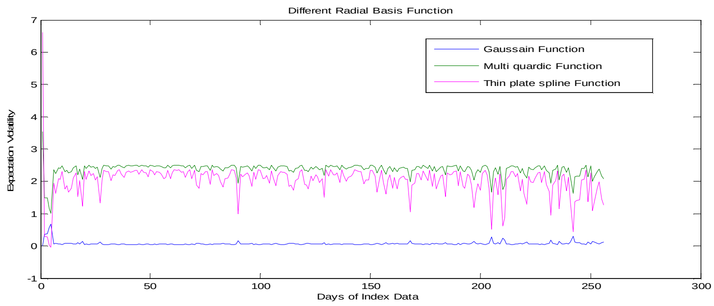

Figure 1: Different radial basis functions

The above figure shows the different radial basis functions like Gaussian, multiquadratic and thin plate spline. It shows that the Gaussian function is better than other radial basis functions. Now investigate the errors for radial basis functions.

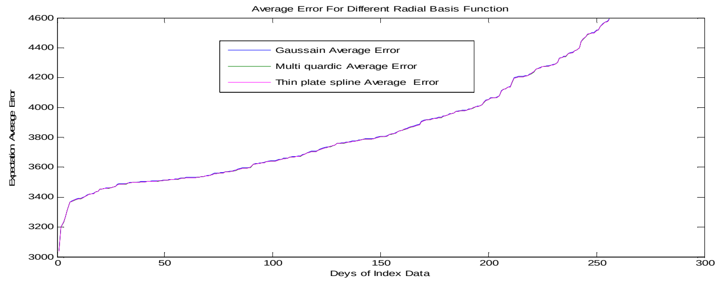

Figure 2: Average error for different radial basis functions

The figure (2) shows average error for the different type of radial basis functions by using 258 data. It shows that all types of radial basis functions are overlapping each other. So it can't identify the best for minimizing the error.

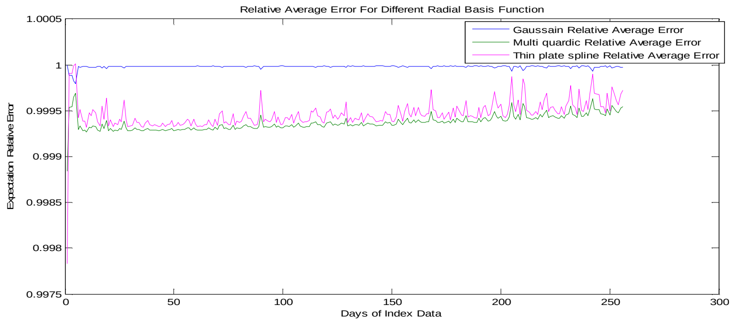

Figure 3: Relative average error for different radial basis functions

The above figure shows the relative average error for the different type of radial basis functions for 258 data and the multiquadratic function is best from other radial basis functions.

For the purposes of comparison, we use the same test problems presented in [7], [12] and several measures of performance are used for consistency. Firstly the average absolute error at each of the *n* known data points in pricing is given by

$$\text{Average error} = \frac{1}{n}\sum_{i=1}^{n}\left|V_i(\sigma) - f_i\right| \tag{18}$$

where $V_i(\sigma)$ is the price at the data point i and f_i is the observed radial basis function. The maximum absolute error is observed at the data points for the equation (8).Then interpolation is also given by

$$\text{Max error} = \text{Max}\left|V_i(\sigma) - f_i\right| \quad \forall\, i = 1,\ldots\ldots, n \tag{19}$$

$$\text{Average relative error} = \frac{1}{n}\sum_{i=1}^{n}\frac{\left|V_i(\sigma) - f_i\right|}{f_i} \tag{20}$$

$$\text{Max relative error} = \text{Max}\left(\frac{\left|V_i(\sigma) - f_i\right|}{f_i}\right) \quad \forall\, i = 1, \ldots\ldots, n \tag{21}$$

This research examines the radial basis interpolating function to reconstruct the surface and to judge the smoothness according to radial basis optimization algorithm. The radial basis function approach is running to use local volatility by the Nelder-Mead optimization, with Gaussian, multiquadratic and thin plate spline function sets are presented in table (1).

Radial Basis Function	Average Absolute Error	Maximum Absolute Error	Average Relative Error	Maximum Relative Error
General	3809.191	4612.344	0.999597	0.999998
Gaussian	3810.291	4612.648	0.999893	1.000000
Multiquadratic	3809.075	4610.683	0.999565	0.999881
Thin plate spline	3809.444	4611.504	0.999651	1.000023

Table 1: Summarized result using Nelder-Mead algorithm

By using the equation (18) and the values from the table (1), we get different figures of Gaussian average error by using MATLAB code whose are shown in the figures bellow.

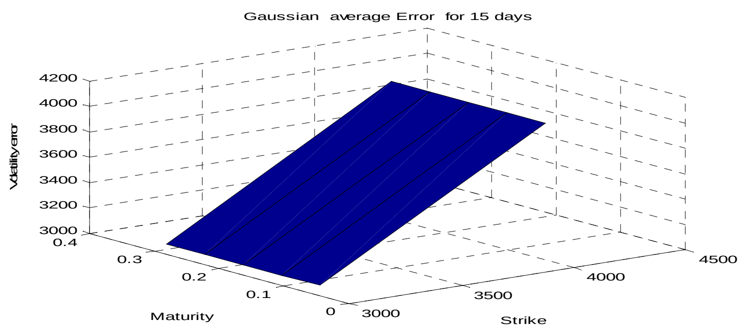

Figure 4: Gaussian average error for 15 days using Nelder-Mead algorithm

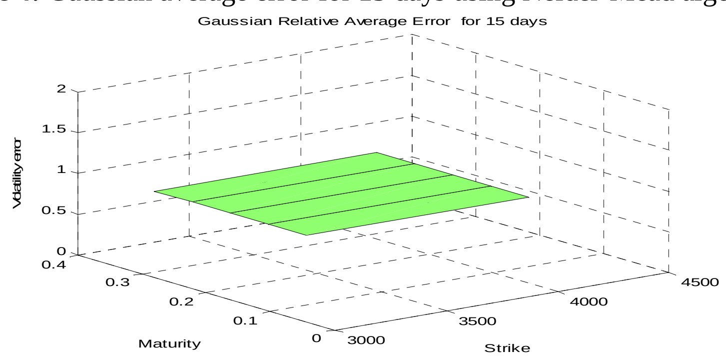

Figure 5: Gaussian relative average error for 15 days using Nelder-Mead algorithm

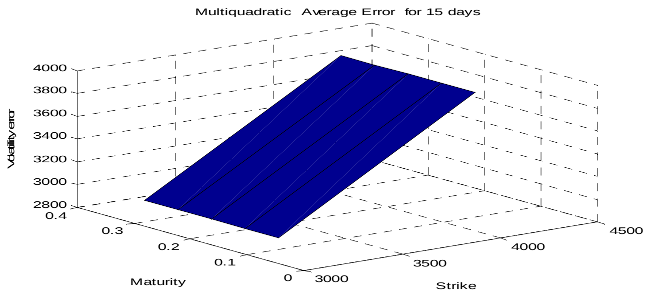

Figure 6: Multiquadratic average error for 15 days using Nelder-Mead algorithm

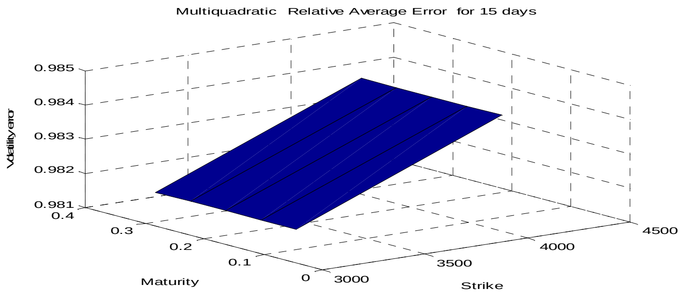

Figure 7: Multiquadratic relative average error for 15 days by Nelder-Mead

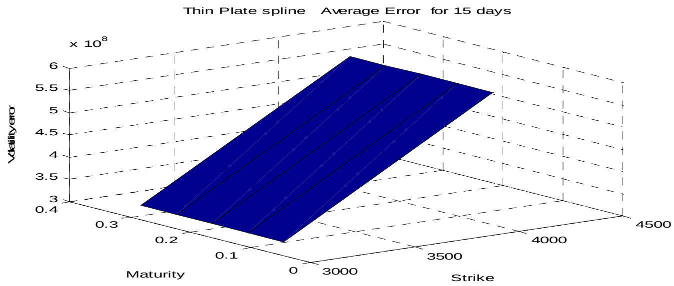

Figure 8: Thin Plate Spline average error for 15 days using Nelder-Mead algorithm

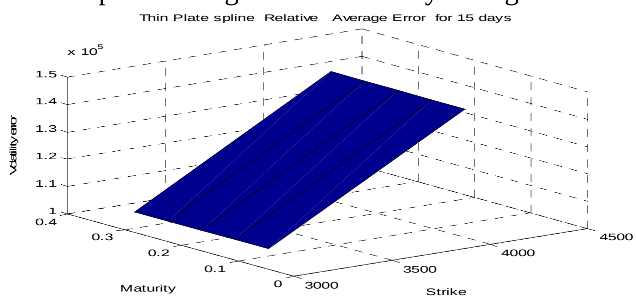

Figure 9: Thin Plate Spline relative average error for 15 days using Nelder-Mead algorithm

It is clear that from the above figures, the thin plate spline performs well closely followed by the multiquadratic while the Gaussian performs poorly show smooth well defined surfaces which closely resemble the known volatility function or both the thin plate spline and the multiquadratic and a very unstable surface for the Gaussian.

In this research we try to find the minimum error among different types of radial basis function. In figure (1) we get three different types of curves of radial basis functions. But it is difficult to measure which is the best function. That's why, we consider radial basis function in term of average error which are shown in figure (2) but face some problem to understand which function is the best. So we use relative average error for different radial basis function and get the best result for minimization the error for multiquadratic function which shown in figure (3).

Again we go for numerical solution to find the absolute error. From table (1) the average absolute error for general radial basis function is 3809.191, the maximum absolute error for general radial basis function is 4612.344, the average relative error for general radial basis function is 0.999597 and the maximum relative error for general radial basis function is 0.999998. The average absolute error for Gaussian radial basis function is 3810.291, the maximum absolute error for Gaussian radial basis function is 4612.648, the average relative error for Gaussian radial basis function is 0.999893 and the maximum relative error for Gaussian radial basis function is 1. The average absolute error for multiquadratic radial basis function is 3809.075, the maximum absolute error for multiquadratic radial basis function is 4610.683, the average relative error for multiquadratic radial basis function is 0.999565and the maximum relative error for multiquadratic radial basis function is 0.999881. The average absolute error for Thin plate spline radial basis function is 3809.444, the maximum absolute error for Thin plate spline radial basis function is 4611.504, the average relative error for Thin plate spilne radial basis function is 0.999651 and the maximum relative error for Thin plate spline radial basis function is 1.000023. Comparing the results both Thin plate spline and multiquadratics perform well, but Gaussian function sets not perform satisfactory level. So, more accurate result has shown by using the multiquadratic radial basis function. Thus multiquadratic radial basis function gives more accurate result than other two radial basis function in terms of error.

5. Conclusion

The main purpose of this research has been carried to investigate the sensitivity of radial basis functions to construct different volatility surface for different kinds of Radial Basis Function approaches to understand market condition and find the best radial basis function approaches both graphically and numerically by different error measurement. For that reason, we use different types of radial basis function to evaluate the absolute average error, relative average error and try to find out the best function to minimize error. At first we try to find the minimum error by graphically but we couldn't identify the best option to minimize the error. Then we go for the numerical solution using an index data (DGEN) and finally get a solution for the multiquadratic function which is the best function to minimize the error from other radial basis function. We have wanted to find out the best function to minimize error with numerical study for my data index (DGEN) but we see a few changes for different types for radial basis function which is unexpected. So in near future we would like to work again with another data index and will be able to measure the best function to minimize the error.

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