Annals of Pure and Applied Mathematics
Vol. 9, No. 2, 2015, 205-213
ISSN: 2279-087X (P), 2279-0888(online)
Published on 23 March 2015
www.researchmathsci.org

Annals of
Pure and Applied
Mathematics

On Invertibility of Matrices over Semirings

***K.R.Chowdhury*[1], *Abeda Sultana*[2], *N.K.Mitra*[3] and *A.F.M.Khodadad Khan*[4]**

[1]Department of Mathematics, Mohammadpur Model School and College
Mohammadpur, Dhaka, Bangladesh, E-mail : krchowdhury1975@yahoo.com
[2]Department of Mathematics, Jahangirnagar University, Savar, Bangladesh
[3]Mathematical and Physical Sciences, Bangladesh University of Business and Technology, Dhaka, Bangladesh
[4]School of Engineering and Computer Science, Independent University, Bangladesh
Bashundhara R/A, Dhaka, Bangladesh

Received 21 February 2015; accepted 10 March 2015

Abstract. In this paper, determinant of matrix and invertibility of matrices over semirings are discussed. Some properties are established on the positive determinant and negative determinant.

Keywords: Positive determinant, negative determinant, permutation matrix, even permutation, odd permutation

AMS Mathematics Subject Classification (2010): 16Y60

1. Introduction

The study of matrices over general semirings has a long history. In 1964, Rutherford [3] gave a proof of Cayley-Hamilton theorem for a commutative semiring avoiding the use of determinants. Since then, a number of works on theory of matrices over semirings were published [1, 12]. In 1999, Golan described semirings and matrices over semirings in his work [5] comprehensively. The techniques of matrices have important applications in optimization theory, models of discrete event network and graph theory. Luce [16] characterized the invertible matrices over a Boolean algebra of at least two elements. Rutherford [2] has introduced that a square matrix over a Boolean algebra of 2 elements is invertible. Additively inverse semirings are studied by Karvellas [14]. Kaplansky [4], Petrich [11], Goodearl [6], Reutenauer [1], Fang [10], Kanak [7,8,9] have studied semiring.

2. Preliminaries

In this section, some fundamental definitions and examples are presented.

Definition 2.1. Let S be non empty set with two binary operations + and **.** . Then the algebraic structure (S; +, .) is called **a *semiring*** iff

$\forall a, b, c \in S$;

(i) (S; +) is a semigroup
(ii) (S; .) is a semigroup
(iii) a. (b+c) = a.b + a.c and (b+c).a = b.a + c.a.

Definition 2.2. Let (S; +, .) be a semiring**.** Then S is called

(i) ***additively commutative*** iff $\forall x, y \in S, x + y = y + x.$

(ii) ***multiplicatively commutative*** iff $\forall x, y \in S,\ x.y = y.x.$

(S; +, .) is called ***a commutative semiring*** iff both (i) and (ii) hold.

Definition 2.3. Let (S; +, .) be a semiring**.** Then an element $0 \in S$ is called ***zero of S*** iff $\forall x \in S$,

$$x + 0 = x = 0 + x \text{ and } x.0 = 0 = 0.x .$$

Definition 2.4. Let (S; +, .) be a semiring**.** Then an element $1 \in S$ is called ***identity of S*** iff

$$\forall x \in S, \quad x.1 = x = 1.x .$$

Definition 2.5. Let (S; +, .) be a commutative semiring with zero (0) and identity (1) . Then (S; +, .) is called ***idempotent semiring*** iff $\forall x \in S$,

$$x + x = x = xx.$$

Example 2.5 (a). (I =[0,1]; +, .) is a ***an idempotent semiring***, where order in [0,1] is usual $\leq$ and + and . are defined as follows:

$$a + b = \max\{a , b\}, a.b = \min\{a , b\}.$$

Proposition 2.6. [8] Let (S; +, .) be an idempotent semiring with zero (0) and identity (1). Then

(a) $\forall x, y \in S$, $x + y = 0 \Rightarrow x = 0 = y$

(b) $\forall x, y \in S$, $xy = 1 \Rightarrow x = 1 = y.$

Definition 2.7. A permutation is said to be ***odd*** iff it is expressible as the product of odd number of transpositions.

Definition 2.8. A permutation is said to be ***even*** iff it is expressible as the product of even number of transpositions.

Definition 2.9. The set of all permutations of n elements is denoted by S_n and is commonly called the ***symmetric group*** of degree n. Clearly S_n contains $n!$ elements.

3. Invertibility of matrices over semirings

In this section, we discuss determinant and invertibility of matrices over semirings. Some properties and examples are presented.

Definition 3.1. Let (S ; +,.) be a semiring and $A \in M_n(S)$. Let $A = (a_{ij})$; where $i, j \in \{1, 2, 3, ..., n\}$. Then the ***transpose of A*** defined as

$$A^t = (a_{ij}^t) = (a_{ji}).$$

Proposition 3.2. For $A, B \in M_n(S)$;

(i) $(A^t)^t = A$

(ii) $(A+B)^t = A^t + B^t$

(iii) $(AB)^t = B^t A^t$.

Definition 3.3. Let (S ; +,.) be a semiring with zero(0) and identity (1) and $A \in M_n(S)$. Then A is called ***a permutation matrix*** if every element (entry) of A is either 0 or 1 and each row and each column contains exactly one 1.

Definition 3.4. Let (S ; +,.) be a semiring with zero(0) and identity (1) and $A \in M_n(S)$. Then A is called ***invertible*** iff $\exists G \in M_n(S)$ such that $AG = GA = I_n$.

Proposition 3.5. Let (S ; +,.) be a semiring with zero (0) and identity (1) and $A \in M_n(S)$ .Then the inverse of A (if exists) is unique.

Proof: Let A be an invertible matrix over S.
If possible suppose B and C are two inverses of A.
Then AB = BA = I_n
and AC = CA = I_n .
Now CAB = C(AB) = C. I_n = C
and CAB = (CA)B = I_n .B = B
Thus B = C.
This shows that inverse of A is unique. Δ

Proposition 3.6. If A is a permutation matrix in $M_n(S)$, then $AA^t = A^t A = I_n$.

Proof: Suppose A is a permutation matrix.
There are two cases:

Case (i) : $i \neq j$

Case (ii) : $i = j$

For case (i):

$$(AA^t)_{ij} = \sum_{k=1}^{n} A_{ik} A^t{}_{kj}$$
$$= \sum_{k=1}^{n} A_{ik} A_{jk}$$

Since there is only one nonzero entry in the *k*th column and $i \neq j$, so A_{ik} and A_{jk} can't both be the nonzero entry.

Therefore $(AA^t)_{ij} = 0$.

For case (ii) :

$$(AA^t)_{ij} = \sum_{k=1}^{n} A_{ik} A^t{}_{ki} = \sum_{k=1}^{n} A_{ik} A_{ik} = \sum_{k=1}^{n} A_{ik}{}^2$$
$$=1 \quad [\because \text{ A is a permutation matrix}]$$

Therefore $AA^t = I_n$.

Similarly, $A^t A = I_n$.

We conclude that $AA^t = A^t A = I_n$. Δ

Proposition 3.7. Let (S ; +,.) be a semiring with zero(0) and identity (1) and $A \in M_n(S)$. If A is invertible matrix over S then $AA^t = A^t A = I_n$.

Proof : Let A be an invertible matrix.

Then $AX = XA = I$ for some $X \in M_n(S)$.

Putting $A = (a_{ij})$ and $X = (x_{ij})$; where $i, j \in \{1, 2, 3, ..., n\}$.

By $AX = I$, we have that if $i \neq j$, then

$$\sum_{k=1}^{n} a_{ik} x_{kj} = 0$$

$$\Rightarrow a_{ik} x_{kj} = 0 \quad \text{for all } k \in \mathbb{N}$$

Again by $XA = I$, we get

$$\sum_{s=1}^{n} x_{is} a_{si} = 1$$

$$\Rightarrow \sum_{s=1}^{n} a_{si} \geq \sum_{s=1}^{n} x_{is} a_{si} = 1 \ ; \ \forall i \in \mathbb{N}$$

Again

$$\sum_{s=1}^{n} x_{is} \geq \sum_{s=1}^{n} x_{is} a_{si} = 1; \quad \forall i \in \mathbb{N}$$

Now

$$a_{ij} = a_{ij}.1 = a_{ij}(\sum_{s=1}^{n} x_{js}) = \sum_{s=1}^{n} a_{ij} x_{js} = a_{ij} x_{ji} + \sum_{s \neq i}^{n} a_{ij} x_{js} = a_{ij} x_{ji} + 0$$

$$= a_{ij} x_{ji} + \sum_{s \neq i}^{n} a_{sj} x_{ji} = \sum_{s=1}^{n} a_{sj} x_{ji} = (\sum_{s=1}^{n} a_{sj}) x_{ji} = 1.x_{ji}$$

$$= x_{ji} \ ; \ \forall i, j \in \mathbb{N}$$

So $A = X^t$

Therefore $X = A^t$.

We conclude that A^t is inverse of A i.e. $AA^t = A^t A = I_n$. Δ

Definition 3.8. Let S_n be the symmetric group of degree $n \geq 2$, A_n be the alternating group of degree n and $B_n = S_n / A_n$, that is **,**

$$A_n = \{\sigma \in S_n : \sigma \text{ is an even permutation}\}$$

$$B_n = \{\sigma \in S_n : \sigma \text{ is an odd permutation}\}.$$

Definition 3.9. Let (S ; +,.) be a commutative semiring with zero (0) and identity (1) and n a positive integer greater than 1, then for $A\in M_n(S)$,

the positive determinant of A is defined by

$$\det{}^+ A=\sum_{\sigma\in A_n}\left(\prod_{i=1}^{n} A_{i\sigma(i)}\right)$$

and ***the negative determinant*** of A is defined by

$$\det{}^- A=\sum_{\sigma\in B_n}\left(\prod_{i=1}^{n} A_{i\sigma(i)}\right)$$

Remark 3.10. (i) $A_n=\{\sigma^{-1}:\sigma\in A_n\}$ and $B_n=\{\sigma^{-1}:\sigma\in B_n\}$.

(ii) $\det^+(I_n)=1$ and $\det^-(I_n)=0$.

Proposition 3.11. For $A,B\in M_n(S)$;

$$(i)\ \det{}^+(A^t)=\det{}^+ A$$
$$(ii)\ \det{}^-(A^t)=\det{}^- A$$

Proof : (i) $\det^+(A^t)=\sum_{\sigma\in A_n}(\prod_{i=1}^{n} A^t{}_{i,\sigma(i)}) = \sum_{\sigma\in A_n}(\prod_{i=1}^{n} A_{\sigma(i),i})$

$$=\sum_{\sigma\in A_n}\left(\prod_{i=1}^{n} A_{\sigma^{-1}(i),i}\right)=\sum_{\sigma\in A_n}\left(\prod_{i=1}^{n} A_{\sigma^{-1}(i),\sigma(\sigma^{-1}(i))}\right)$$

$$=\sum_{\sigma\in A_n}\left(\prod_{i=1}^{n} A_{i,\,\sigma(i)}\right),\ \text{since } \{\sigma^{-1}(1),\sigma^{-1}(2),\ldots\ldots,\sigma^{-1}(n)\}=\{1,2,\ldots\ldots,n\}$$

Therefore $\det^+(A^t)=\det^+ A.$ Δ

(ii) $\det^-(A^t)=\sum_{\sigma\in B_n}(\prod_{i=1}^{n} A^t{}_{i,\sigma(i)}) = \sum_{\sigma\in B_n}(\prod_{i=1}^{n} A_{\sigma(i),i})$

$$=\sum_{\sigma\in B_n}\left(\prod_{i=1}^{n} A_{\sigma^{-1}(i),i}\right)=\sum_{\sigma\in B_n}\left(\prod_{i=1}^{n} A_{\sigma^{-1}(i),\sigma(\sigma^{-1}(i))}\right)$$

$$=\sum_{\sigma\in B_n}\left(\prod_{i=1}^{n} A_{i,\,\sigma(i)}\right),\ \text{since } \{\sigma^{-1}(1),\sigma^{-1}(2),\ldots\ldots,\sigma^{-1}(n)\}=\{1,2,\ldots\ldots,n\}$$

Therefore $\det^-(A^t)=\det^- A.$ Δ

Proposition 3.12. [2] Let (S; +, .) be an idempotent semiring of 2 elements. Then a square matrix A over S is invertible over S iff A is a permutation matrix.

Proposition 3.13. [1] Let (S; +,.) be a commutative semiring with zero(0) and identity (1) and a positive integer $n\geq 2$. If $A,B\in M_n(S)$, then there is an element $r\in S$ such that

(i) $\det^+(AB)=(\det^+ A)(\det^+ B)+(\det^- A)(\det^- B)+r$

(ii) $\det^{-}(AB) = (\det^{+} A)(\det^{-} B) + (\det^{-} A)(\det^{+} B) + r$

Proposition 3.14. [1] Let (S; +,.) be a commutative semiring with zero(0) and identity (1) and n a positive integer. For $A, B \in M_n(S)$ if $AB = I_n$ then $BA = I_n$.

Proposition 3.15. [18] Let (S; +,.) be an idempotent semiring and n a positive integer $n \geq 2$. If $A \in M_n(S)$ is invertible over S, then $\det^{+} A + \det^{-} A = 1$.

Remark 3.16.
(i) Every Boolean ring is commutative semiring.
(ii) Boolean semiring is commutative semiring with zero.
(iii) Every Boolean ring is Boolean semiring.
(iv) Boolean semiring is idempotent.

Proposition 3.17. Let (R ; +,.) be a Boolean ring with identity 1 and n a positive integer $n \geq 2$. If $A \in M_n(R)$ is invertible over R, then $\det^{+} A + \det^{-} A = 1$.

Proof: Let $A \in M_n(R)$ be an invertible matrix.

By definition 3.4, there exists $B \in M_n(R)$ such that $AB = BA = I_n$.

Since Boolean ring is commutative semiring, so by Proposition 3.13 there is an element $r \in R$ such that

$$\det^{+}(AB) = (\det^{+} A)(\det^{+} B) + (\det^{-} A)(\det^{-} B) + r \quad \text{.......(i)}$$

and

$$\det^{-}(AB) = (\det^{+} A)(\det^{-} B) + (\det^{-} A)(\det^{+} B) + r \quad \text{......(ii)}$$

But $\quad \det^{+}(AB) = \det^{+}(I_n) = 1$

$$\det^{-}(AB) = \det^{-}(I_n) = 0$$

From (i) we get

$$1 = (\det^{+} A)(\det^{+} B) + (\det^{-} A)(\det^{-} B) + r \quad \text{...... (iii)}$$

From (ii) we get

$$0 = (\det^{+} A)(\det^{-} B) + (\det^{-} A)(\det^{+} B) + r \quad \text{...... (iv)}$$

By proposition 2.6(a), from (iv) we get

$$(\det^{+} A)(\det^{-} B) = 0 \quad \text{........ (v)}$$

and $\quad (\det^{-} A)(\det^{+} B) + r = 0$

$$\Rightarrow (\det^{-} A)(\det^{+} B) = 0 \quad \text{...... (vi)}$$

and $\quad r = 0 \quad$ ...... (vii)

From (iii) and (vii) we get

$$1 = (\det^{+} A)(\det^{+} B) + (\det^{-} A)(\det^{-} B)$$

$$\Rightarrow (\det^{+} A)(\det^{+} B) + (\det^{-} A)(\det^{-} B) + (\det^{+} A)(\det^{-} B) + (\det^{-} A)(\det^{+} B) = 1$$

[by (v) and (vi)]

$$\Rightarrow (\det^{+} A + \det^{-} A)(\det^{+} B + \det^{-} B) = 1$$

By Proposition 2.6(b) we get

$$\det^{+} A + \det^{-} A = 1. \qquad \Delta$$

Proposition 3.18. Let (S; +,.) be an idempotent and commutative semiring with zero (0) and identity (1), n a positive integer and $A \in M_n(S)$. Then A is invertible over S iff

(i) the product of any two elements in the same row is 0

and (ii) the sum of all elements in each column is 1.

Proof: Suppose that (i) and (ii) hold.

We claim that

$$A^t A = I_n .$$

If $i \in \{1,2,3,......., n\}$, then from (ii)

$$(A^t A)_{ii} = \sum_{k=1}^{n} A^t_{ik} A_{ki} = \sum_{k=1}^{n} A_{ki} A_{ki} = \sum_{k=1}^{n} A^2_{ki}$$

$$= \sum_{k=1}^{n} A_{ki} \qquad [\because \text{ S is idempotent}]$$

$$= 1$$

Also for distinct $i, j \in \{1,2,3,......., n\}$ from (i)

$$(A^t A)_{ij} = \sum_{k=1}^{n} A^t_{ik} A_{kj} = \sum_{k=1}^{n} A_{ki} A_{kj}$$

$$= A_{1i} A_{1j} + A_{2i} A_{2j} + A_{3i} A_{3j} + ... + A_{ni} A_{nj}$$

$$= 0+0+0+.........+0 \qquad [\text{ by (i)}]$$

$$= 0$$

This shows that

$$A^t A = I_n$$

By Proposition 3.14, we get

$$AA^t = I_n$$

Therefore

$$A^t A = AA^t = I_n .$$

Hence A is invertible over S and inverse of A is A^t.

Conversely suppose

A is invertible over S.

By Proposition 2.6(b) , the Proposition is obviously true for n =1.

Let $n > 1$ and assume that A is invertible over S.

Let $B \in M_n(S)$ be such that $AB = BA = I_n$.

Let

$$i, j \in \{1,2,3,......., n\} .$$

Now

$$0=(I_n)_{ij}=(BA)_{ij}=\sum_{l=1}^{n}B_{il}A_{lj}$$

Therefore $B_{il}A_{lj}=0,\ \forall i,j,l\in\{1,2,3,.......,n\}$ such that $i\neq j$ …..(1)

Let

$p,q,k\in\{1,2,3,.......,n\}$ be such that $q\neq k$.

Then

$$\begin{aligned}
A_{pq}A_{pk} &= A_{pq}A_{pk}.1\\
&= A_{pq}A_{pk}(AB)_{pp} = A_{pq}A_{pk}(\sum_{l=1}^{n}A_{pl}B_{lp})\\
&= \sum_{l=1}^{n}A_{pq}A_{pk}A_{pl}B_{lp} = A_{pq}A_{pk}A_{pk}B_{kp}+\sum_{\substack{l=1\\ l\neq k}}^{n}A_{pq}A_{pk}(A_{pl}B_{lp})\\
&= A_{pk}^{2}(B_{kp}A_{pq})+\sum_{\substack{l=1\\ l\neq k}}^{n}A_{pq}A_{pl}(B_{lp}A_{pk})\\
&= 0+0 \qquad \text{from (1)}\\
&= 0
\end{aligned}$$

Hence (i) is proved.

From Proposition 3.15, we have that

$$\det{}^{+}A+\det{}^{-}A=1.$$

By (i) we have

$A_{k_1 1}.A_{k_2 2}.A_{k_3 3}.........A_{k_n n}=0$ if $k_1,k_2,k_3,.........,k_n\in\{1,2,3,......,n\}$ are not all distinct. ….(2)

Then

$$\begin{aligned}
(\sum_{\zeta=1}^{n}A_{\zeta 1})\ (\sum_{\zeta=1}^{n}A_{\zeta 2})\ (\sum_{\zeta=1}^{n}A_{\zeta 3})\ ....(\sum_{\zeta=1}^{n}A_{\zeta n}) &= \sum_{k_1,k_2,k_3,......k_n\in\{1,2,3,.....,n\}}A_{k_1 1}.A_{k_2 2}.A_{k_3 3}.........A_{k_n n}\\
&= \sum_{\sigma\in S_n}A_{\sigma(1)1}A_{\sigma(2)2}A_{\sigma(3)3}........A_{\sigma(n)n} \qquad \text{from (2)}\\
&= \det{}^{+}A+\det{}^{-}A=1.
\end{aligned}$$

By Proposition 2.6(b) we get

$$(\sum_{\zeta=1}^{n}A_{\zeta 1})=(\sum_{\zeta=1}^{n}A_{\zeta 2})=(\sum_{\zeta=1}^{n}A_{\zeta 3})=........=(\sum_{\zeta=1}^{n}A_{\zeta n})=1$$

Hence (ii) is proved. Δ

Acknowledgement. The authors thank to the referees for their suggestions which have made the paper more readable.

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