Annals of Pure and Applied Mathematics
Vol. 10, No.1, 2015, 105-115
ISSN: 2279-087X (P), 2279-0888(online)
Published on 22 June 2015
www.researchmathsci.org

Cubic *G*-subalgebras of *G*-algebras

Chiranjibe Jana[a] **and** ***Tapan Senapati***[b]

[a]Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India
[b]Department of Mathematics, Padima Janakalyan Banipith, Kukurakhupi 721517, India
Email: {math.tapan, jana.chiranjibe7}@gmail.com

Received 1 June 2015; accepted 15 June 2015

Abstract. In this paper, the notion of cubic *G*-subalgebras of *G*-algebras are introduced. Some characterization of cubic *G*-subalgebras of *G*-algebras are given. The homomorphic image and inverse image of cubic *G*-subalgebras are studied and investigated some related properties.

Keywords: *G*-algebra, *G*-subalgebra, cubic set, cubic *G*-subalgebra

AMS Mathematics Subject Classification (2010): 06F35, 03G25, 08A72

1. Introduction

Extending the concept of fuzzy sets [32], many scholars introduced various notions of higher-order fuzzy sets. Among them, interval-valued fuzzy sets [33] provides with a flexible mathematical framework to cope with imperfect and imprecise information. Moreover, Jun et al. [9] introduced the concept of cubic sets, as a generalization of fuzzy set and interval-valued fuzzy set. Jun et al. [10] applied the notion of cubic sets to a group, and introduced the notion of cubic subgroups.

The study of BCK/BCI-algebras [5, 14] was initiated by Imai and Iseki as a generalization of the concept of set-theoretic difference and propositional calculus. Hu and Li [4] introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers et al. [12] introduced Q-algebras and generalized some theorems discussed in BCK/BCI-algebras. Ahn et al. [1] introduced a new notion, called QS-algebras and discussed some properties of the G-part of QS-algebras. Neggers and Kim [13] introduced a new notion, called B-algebras which is related to several classes of algebras of interest such as $BCK/BCI/BCH$-algebras. Kim and Kim [11] introduced the notion of BG-algebras, which is a generalization of B-algebras. Jana et al. [6-8] and Senapati et al. [15-30] has done lot of works on BCK/BCI-algebra and $B/BG/G$-algebras which is related to these algebras. Walendziak [31] introduced a new notion, called a BF-algebra which is a generalization of B-algebra and obtained several results.

Bandru and Rafi [2] introduced a new notion, called G-algebras, which is a generalization of QS-algebras and discussed relationship between these algebras with other related algebras such as Q-algebras, BCI-algebras, BCH-algebras, BF

-algebras and B-algebras. They introduced the concept of 0-commutative, G-part and medial of G-algebras and studied their related properties. Senapati et al. [24] applied the concept of L-fuzzy set to G-subalgebras and introduced the notions of L-fuzzy G-subalgebras of G-algebras.

The objective of this paper is to introduce the concept of cubic set to G-subalgebras of G-algebras. The notion of cubic G-subalgebras of G-algebras are defined and lot of properties are investigated. Section 2 recalls some definitions, viz., G-algebra, G-subalgebra, cubic set and refinement of unit interval. In Section 3, G-subalgebras of cubic sets are defined with some its properties. In Section 4, homomorphism of cubic G-subalgebras and some of its properties are studied. In Section 5, conclusion of the proposed work is given.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.

Definition 2.1. [2] (G-algebra) *A non-empty set X with a constant 0 and a binary peration $*$ is said to be G-algebra if it satisfies the following axioms:*
*G1: x*x=0*
G2: x(x*y)=y, for all x, y ε X.*

A G-algebra is denoted by $(X,*,0)$.

Example 2.2. [2] *Let $X=\{R-\{-n\}, 0\neq n\in Z^{+}\}$ where R is the set of real numbers and Z^{+} be the set of all positive integers. Define a binary operation $*$ on X by $x*y=\dfrac{n(x-y)}{n+y}$. Then $(X,*,0)$ is a G-algebra.*

Any G-algebra X satisfies the following axioms:

(i) $x*0=x$,
(ii) $(x*(x*y))*y=0$,
(iii) $0*(0*x)=x$,
(iv) $x*y=0$ implies $x=y$,
(v) $0*x=0*y$ implies $x=y$,

for all $x,y,z\in X$ [2].

A non-empty subset S of a G-algebra X is called a subalgebra [2] of X if $x*y\in S$, for all $x,y\in S$. A mapping $f:X\to Y$ of G-algebras is called a homomorphism if $f(x*y)=f(x)*f(y)$ for all $x,y\in X$. Note that if $f:X\to Y$ is a homomorphism, then $f(0)=0$.

We now review some fuzzy logic concepts as follows:
Let X be the collection of objects denoted generally by x. Then a fuzzy set [25] A in X is defined as $A=\{<x,\mu_A(x)>: x\in X\}$ where $\mu_A(x)$ is called the membership value of x in A and $0\leq\mu_A(x)\leq 1$.

An interval-valued fuzzy set [26] A over X is an object having the form $A=\{\langle x,\tilde{\mu}_A(x)\rangle : x\in X\}$, where $\tilde{\mu}_A(x): X\to D[0,1]$, where $D[0,1]$ is the set of all subintervals of $[0,1]$. The intervals $\tilde{\mu}_A(x)$ denote the intervals of the degree of membership of the element x to the set A, where $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ for all $x \in X$.

The determination of maximum and minimum between two real numbers is very simple but it is not simple for two intervals. Biswas [3] described a method to find max/sup and min/inf between two intervals or a set of intervals.

Definition 2.3. [3] *Consider two elements* $D_1, D_2 \in D[0,1]$. *If* $D_1 = [a_1^-, a_1^+]$ *and* $D_2 = [a_2^-, a_2^+]$, *then* $rmin(D_1, D_2) = [\min(a_1^-, a_2^-), \min(a_1^+, a_2^+)]$ *which is denoted by* $D_1 \wedge^r D_2$ *. Thus, if* $D_i = [a_i^-, a_i^+] \in D[0,1]$ *for i=1,2,3,4,..., then we define* $rsup_i(D_i) = [\sup_i(a_i^-), \sup_i(a_i^+)]$, *i.e,* $\vee_i^r D_i = [\vee_i a_i^-, \vee_i a_i^+]$. *Now we call* $D_1 \geq D_2$ *iff* $a_1^- \geq a_2^-$ *and* $a_1^+ \geq a_2^+$. *Similarly, the relations* $D_1 \leq D_2$ *and* $D_1 = D_2$ *are defined.*

Based on the (interval-valued) fuzzy sets, Jun et al. [8] introduced the notion of (internal, external) cubic sets, and investigated several properties.

Definition 2.4. [8] *Let* X *be a nonempty set. A cubic set* A *in* X *is a structure* $A = \{\langle x, \tilde{\mu}_A(x), \nu_A(x)\rangle : x \in X\}$ *which is briefly denoted by* $A = (\tilde{\mu}_A, \nu_A)$ *where* $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ *is an interval-valued fuzzy set in* X *and* ν_A *is a fuzzy set in* X .

3. Cubic G-subalgebras of G-algebras

In what follows, let X denote a G-algebra unless otherwise specified. Combined the definitions of G-subalgebras over crisp set and the idea of cubic set, cubic G-subalgebras of G-algebras are defined below.

Definition 3.1. *Let* $A = (\tilde{\mu}_A, \nu_A)$ *be cubic set in* X *, where* X *is a* G*-subalgebra, then the set* A *is cubic* G*-subalgebra over the binary operator* $*$ *if it satisfies the following conditions:*

(F1) $\tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

(F2) $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$

for all $x, y \in X$.

Let us illustrate this definition using the following examples.

Example 3.2. *Let* $X = \{0,1,2,3,4,5,6,7\}$ *be a* G*-algebra with the following Cayley table:*

*	0	1	2	3	4	5	6	7
0	0	2	1	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	2	1	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in X by

$$\tilde{\mu}_A(x)=\begin{cases}[0.6,0.8], & if\ x=0,5\\ [0.5,0.7], & if\ x=3,6\\ [0.4,0.6], & if\ x=1,2,4,7\end{cases} \quad \text{and} \quad \nu_A(x)=\begin{cases}0.3, & if\ x=0,5\\ 0.5, & if\ x=3,6\\ 0.6, & if\ x=1,2,4,7.\end{cases}$$

All the conditions of Definition 3.1 have been satisfy by the set A. Thus $A=(\tilde{\mu}_A,\nu_A)$ is a cubic G-subalgebra of X .

Example 3.3. *Let* $X=\{0,1,2,3,4,5\}$ *be a* G *-algebra with the following Cayley table:*

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	4	0	2	1
4	4	3	5	1	0	2
5	5	1	2	4	3	0

Define a cubic set $A=(\tilde{\mu}_A,\nu_A)$ in X by

$$\tilde{\mu}_A(x)=\begin{cases}[0.4,0.7], & if\ x=0,3\\ [0.2,0.6], & otherwise\end{cases} \text{and} \quad \nu_A(x)=\begin{cases}0.3, & if\ x=0,3\\ 0.5, & otherwise.\end{cases}$$

All the conditions of Definition 3.1 have been satisfied by the set A. Thus $A=(\alpha_A,\beta_A)$ is a cubic G-subalgebra of X .

Proposition 3.4. *If* $A=(\tilde{\mu}_A,\nu_A)$ *is a cubic* G *-subalgebra in* X *, then for all* $x\in X$ *,* $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x)$ *and* $\nu_A(0)\leq\nu_A(x)$*. Thus,* $\tilde{\mu}_A(0)$ *and* $\nu_A(0)$ *are the upper bounds and lower bounds of* $\tilde{\mu}_A(x)$ *and* $\nu_A(x)$ *respectively.*

Proof: For all $x\in X$, we have, $\tilde{\mu}_A(0)=\tilde{\mu}_A(x*x)\geq rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(x)\}=\tilde{\mu}_A(x)$ and $\nu_A(0)=\nu_A(x*x)\leq\max\{\nu_A(x),\nu_A(x)\}=\nu_A(x).$ □

Theorem 3.5. *Let* $A=(\tilde{\mu}_A,\nu_A)$ *be a cubic* G *-subalgebra of* X *. If there exists a sequence* $\{x_n\}$ *in* X *such that* $\lim_{n\to\infty}\tilde{\mu}_A(x_n)=[1,1]$ *and* $\lim_{n\to\infty}\nu_A(x_n)=0$ *. Then* $\tilde{\mu}_A(0)=[1,1]$ *and* $\nu_A(0)=0$.

Proof: By Proposition 3.4, $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x)$ for all $x\in X$, therefore, $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x_n)$ for every positive integer n.

Consider, $[1,1]\geq\tilde{\mu}_A(0)\geq\lim_{n\to\infty}\tilde{\mu}_A(x_n)=[1,1]$. Hence, $\tilde{\mu}_A(0)=[1,1]$.

Again, by Proposition 3.4, $\nu_A(0)\leq\nu_A(x)$ for all $x\in X$, thus $\nu_A(0)\leq\nu_A(x_n)$ for every positive integer n. Now, $0 \le \nu_A(0) \le \lim_{n\to\infty} \nu_A(x_n) = 0$. Hence, $\nu_A(0) = 0$. □

Proposition 3.6. *If a cubic set* $A = (\tilde{\mu}_A, \nu_A)$ *in* X *is a cubic* G*-subalgebra, then for all* $x \in X$, $\tilde{\mu}_A(0 * x) \ge \tilde{\mu}_A(x)$ *and* $\nu_A(0 * x) \le \nu_A(x)$.

Proof: For all $x \in X$,
$\tilde{\mu}_A(0 * x) \ge rmin\{\tilde{\mu}_A(0), \tilde{\mu}_A(x)\} = rmin\{\tilde{\mu}_A(x * x), \tilde{\mu}_A(x)\} \ge rmin$
$\{rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(x)\}, \tilde{\mu}_A(x)\} = \tilde{\mu}_A(x)$ and $\nu_A(0 * x) \le \max\{\nu_A(0), \nu_A(x)\} = \nu_A(x)$.
□

Theorem 3.7. *A cubic set* $A = (\tilde{\mu}_A, \nu_A)$ *in* X *is a cubic* G*-subalgebra of* X *iff* μ_A^-, μ_A^+ *and* ν_A *are fuzzy* G*-subalgebras of* X.

Proof: Let μ_A^-, μ_A^+ and ν_A be fuzzy G-subalgebras of X and $x, y \in X$. Then $\mu_A^-(x * y) \ge \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\nu_A(x * y) \le \max\{\nu_A(x), \nu_A(y)\}$. Now,
$\tilde{\mu}_A(x * y) = [\mu_A^-(x * y), \mu_A^+(x * y)] \ge [min\{\mu_A^-(x), \mu_A^-(y)\}, min\{\mu_A^+(x), \mu_A^+(y)\}]$
$= rmin\{[\mu_A^-(x), \mu_A^+(x)], [\mu_A^-(y), \mu_A^+(y)]\} = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$. Therefore, A is a cubic G-subalgebra of X.

Conversely, assume that, A is a cubic G-subalgebra of X. For any $x, y \in X$,
$[\mu_A^-(x * y), \mu_A^+(x * y)] = \tilde{\mu}_A(x * y) \ge rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$
$= rmin\{[\mu_A^-(x), \mu_A^+(x)], [\mu_A^-(y), \mu_A^+(y)] = [\min\{\mu_A^-(x), \mu_A^-(y)\}, \min\{\mu_A^+(x), \mu_A^+(y)\}]$.
Thus $\mu_A^-(x * y) \ge \min\{\mu_A^-(x), \mu_A^-(y)\}$, $\mu_A^+(x * y) \ge \min\{\mu_A^+(x), \mu_A^+(y)\}$ and $\nu_A(x * y) \le \max\{\nu_A(x), \nu_A(y)\}$. Hence, μ_A^-, μ_A^+ and ν_A are fuzzy G-subalgebras of X. □

Theorem 3.8. *Let* $A = (\tilde{\mu}_A, \nu_A)$ *be a cubic* G*-subalgebra of* X *and let* $n \in \mathbb{N}$ *(the set of natural numbers). Then*

(i) $\tilde{\mu}_A(\prod^{n} x * x) \ge \tilde{\mu}_A(x)$, for any odd number *n*,

(ii) $\nu_A(\prod^{n} x * x) \le \nu_A(x)$, for any odd number *n*,

(iii) $\tilde{\mu}_A(\prod^{n} x * x) = \tilde{\mu}_A(x)$, for any even number *n*,

(iv) $\nu_A(\prod^{n} x * x) = \nu_A(x)$, for any even number *n*.

Proof: Let $x \in X$ and assume that n is odd. Then $n = 2p - 1$ for some positive integer p. We prove the theorem by induction.
Now $\tilde{\mu}_A(x * x) = \tilde{\mu}_A(0) \ge \tilde{\mu}_A(x)$ and $\nu_A(x * x) = \nu_A(0) \le \nu_A(x)$ .Suppose that $\tilde{\mu}_A(\prod^{2p-1} x * x) \ge \tilde{\mu}_A(x)$ and $\nu_A(\prod^{2p-1} x * x) \le \nu_A(x)$. Then by assumption,
$\tilde{\mu}_A(\prod^{2(p+1)-1} x * x) = \tilde{\mu}_A(\prod x * x) = \tilde{\mu}_A(\prod x * (x * (x * x))) = \tilde{\mu}_A(\prod x * x) \ge \tilde{\mu}_A(x)$ and
$\nu_A(\prod^{2(p+1)-1} x * x) = \nu_A(\prod x * x) = \nu_A(\prod x * (x * (x * x))) = \nu_A(\prod x * x) \le \nu_A(x)$, which proves (i) and (ii). Proofs are similar for the cases (iii) and (iv). □

The sets $\{x \in X : \tilde{\mu}_A(x) = \tilde{\mu}_A(0)\}$ and $\{x \in X : \nu_A(x) = \nu_A(0)\}$ are denoted by $I_{\tilde{\mu}_A}$

and I_{ν_A} respectively. These two sets are also G-subalgebra of X.

Theorem 3.9. *Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic G-subalgebra of X, then the sets $I_{\tilde{\mu}_A}$ and I_{ν_A} are G-subalgebras of X.*

Proof: Let $x,y\in I_{\tilde{\mu}_A}$. Then $\tilde{\mu}_A(x)=\tilde{\mu}_A(0)=\tilde{\mu}_A(y)$ and so, $\tilde{\mu}_A(x*y)\geq rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}=\tilde{\mu}_A(0)$. By using Proposition 3.4, we know that $\tilde{\mu}_A(x*y)=\tilde{\mu}_A(0)$ or equivalently $x*y\in I_{\tilde{\mu}_A}$.

Again, let $x,y\in I_{\nu_A}$. Then $\nu_A(x)=\nu_A(0)=\nu_A(y)$ and so, $\nu_A(x*y)\leq \max\{\nu_A(x),\nu_A(y)\}=\nu_A(0)$. Again, by Proposition 3.4, we know that $\nu_A(x*y)=\nu_A(0)$ or equivalently $x*y\in I_{\nu_A}$. Hence, the sets $I_{\tilde{\mu}_A}$ and I_{ν_A} are G-subalgebras of X. □

Theorem 3.10. *Let B be a nonempty subset of X and $A=(\tilde{\mu}_A,\nu_A)$ be cubic set in X defined by $\tilde{\mu}_A(x)=\begin{cases}[\alpha_1,\alpha_2], & \text{if } x\in B\\ [\beta_1,\beta_2], & \text{otherwise}\end{cases}$ and $\nu_A(x)=\begin{cases}\gamma, & \text{if } x\in B\\ \delta, & \text{otherwise}\end{cases}$ for all $[\alpha_1,\alpha_2],[\beta_1,\beta_2]\in D[0,1]$ and γ, $\delta\in[0,1]$ with $[\alpha_1,\alpha_2]\geq[\beta_1,\beta_2]$ and $\gamma\leq\delta$. Then A is a cubic G-subalgebra of X if and only if B is a G-subalgebra of X. Moreover, $I_{\tilde{\mu}_A}=B=I_{\nu_A}$.*

Proof: Let A be a cubic G-subalgebra of X. Let $x,y\in X$ be such that $x,y\in B$. Then $\tilde{\mu}_A(x*y)\geq rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}=rmin\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\}=[\alpha_1,\alpha_2]$ and $\nu_A(x*y)\leq\max\{\nu_A(x),\ \nu_A(y)\}=\max\{\gamma,\gamma\}=\gamma$. So $x*y\in B$. Hence, B is a G-subalgebra of X. Conversely, suppose that B is a G-subalgebra of X. Let $x,y\in X$. Consider two cases

Case (i) If $x,y\in B$ then $x*y\in B$, thus $\tilde{\mu}_A(x*y)=[\alpha_1,\alpha_2]=rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$ and $\nu_A(x*y)=\gamma=\max\{\nu_A(x),\nu_A(y)\}$.

Case (ii) If $x\notin B$ or, $y\notin B$, then $\tilde{\mu}_A(x*y)\geq[\beta_1,\beta_2]=rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$ and $\nu_A(x*y)\leq\delta=\max\{\nu_A(x),\nu_A(y)\}$.

Hence, A is a cubic G-subalgebra of X.

Now, $I_{\tilde{\mu}_A}=\{x\in X,\tilde{\mu}_A(x)=\tilde{\mu}_A(0)\}=\{x\in X,\tilde{\mu}_A(x)=[\alpha_1,\alpha_2]\}=B$ and

$I_{\nu_A}=\{x\in X,\nu_A(x)=\nu_A(0)\}=\{x\in X,\nu_A(x)=\gamma\}=B$. □

Definition 3.11. *Let $A=(\tilde{\mu}_A,\nu_A)$ be a cubic set in X. For $[s_1,s_2]\in D[0,1]$ and $t\in[0,1]$, the set $U(\tilde{\mu}_A:[s_1,s_2])=\{x\in X:\tilde{\mu}_A(x)\geq[s_1,s_2]\}$ is called upper $[s_1,s_2]$-level of A and $L(\nu_A:t)=\{x\in X:\nu_A(x)\leq t\}$ is called lower t-level of A.*

Theorem 3.12. *If $A=(\tilde{\mu}_A,\nu_A)$ is a cubic G-subalgebra of X, then the upper*

$[s_1, s_2]$*-level and lower* t*-level of* A *are* G*-subalgebras of* X .

Proof: Let $x, y \in U(\tilde{\mu}_A : [s_1, s_2])$. Then $\tilde{\mu}_A(x) \geq [s_1, s_2]$ and $\tilde{\mu}_A(y) \geq [s_1, s_2]$. It follows that $\tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \geq [s_1, s_2]$ so that $x * y \in U(\tilde{\mu}_A : [s_1, s_2])$. Hence, $U(\tilde{\mu}_A : [s_1, s_2])$ is a G-subalgebra of X .
Let $x, y \in L(\nu_A : t)$. Then $\nu_A(x) \leq t$ and $\nu_A(y) \leq t$. It follows that $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} \leq t$ so that $x * y \in L(\nu_A : t)$. Hence, $L(\nu_A : t)$ is a G-subalgebra of X . □

Theorem 3.13 . *Let* $A = (\tilde{\mu}_A, \nu_A)$ *be a cubic set in* X *, such that the sets* $U(\tilde{\mu}_A : [s_1, s_2])$ *and* $L(\nu_A : t)$ *are* G*-subalgebras of X for every* $[s_1, s_2] \in D[0,1]$ *and* $t \in [0,1]$. *Then* $A = (\tilde{\mu}_A, \nu_A)$ *is a cubic* G*-subalgebra of* X .

Proof: Let for every $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$, $U(\tilde{\mu}_A : [s_1, s_2])$ and $L(\nu_A : t)$ are G-subalgebras of X. In contrary, let $x_0, y_0 \in X$ be such that $\tilde{\mu}_A(x_0 * y_0) < rmin\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\}$. Let $\tilde{\mu}_A(x_0) = [\theta_1, \theta_2]$, $\tilde{\mu}_A(y_0) = [\theta_3, \theta_4]$ and $\tilde{\mu}_A(x_0 * y_0) = [s_1, s_2]$. Then
$[s_1, s_2] < rmin\{[\theta_1, \theta_2], [\theta_3, \theta_4]\} = [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]$. So, $s_1 < \min\{\theta_1, \theta_3\}$ and $s_2 < \min\{\theta_2, \theta_4\}$. Let us consider,

$$[\rho_1, \rho_2] = \frac{1}{2}[\tilde{\mu}_A(x_0 * y_0) + rmin\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\}] = \frac{1}{2}[[s_1, s_2] + [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]]$$

$$= [\frac{1}{2}(s_1 + \min\{\theta_1, \theta_3\}), \frac{1}{2}(s_2 + \min\{\theta_2, \theta_4\})].$$ Therefore,

$$\min\{\theta_1, \theta_3\} > \rho_1 = \frac{1}{2}(s_1 + \min\{\theta_1, \theta_3\}) > s_1$$ and

$$\min\{\theta_2, \theta_4\} > \rho_2 = \frac{1}{2}(s_2 + \min\{\theta_2, \theta_4\}) > s_2.$$ Hence,

$[\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2] > [s_1, s_2]$, so that $x_0 * y_0 \notin U(\tilde{\mu}_A : [s_1, s_2])$ which is a contradiction, since $\tilde{\mu}_A(x_0) = [\theta_1, \theta_2] \geq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$ and $\tilde{\mu}_A(y_0) = [\theta_3, \theta_4] \geq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$. This implies $x_0 * y_0 \in U(\tilde{\mu}_A : [s_1, s_2])$. Thus $\tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ for all $x, y \in X$.
Again, let $x_0, y_0 \in X$ be such that $\nu_A(x_0 * y_0) > max\{\nu_A(x_0), \nu_A(y_0)\}$. Let $\nu_A(x_0) = \eta_1$, $\nu_A(y_0) = \eta_2$ and $\nu_A(x_0 * y_0) = t$. Then $t > max\{\eta_1, \eta_2\}$. Let us consider, $t_1 = \frac{1}{2}[\nu_A(x_0 * y_0) + max\{\nu_A(x_0), \nu_A(y_0)\}]$. We get that

$t_1 = \frac{1}{2}(t + \max\{\eta_1, \eta_2\})$. Therefore, $\eta_1 < t_1 = \frac{1}{2}(t + \max\{\eta_1, \eta_2\}) < t$ and

$\eta_2 < t_1 = \frac{1}{2}(t + \max\{\eta_1, \eta_2\}) < t$. Hence, $\max\{\eta_1, \eta_2\} < t_1 < t = \nu_A(x_0 * y_0)$, so that

$x_0 * y_0 \notin L(\nu_A : t)$ which is a contradiction, since $\nu_A(x_0) = \eta_1 \leq \max\{\eta_1, \eta_2\} < t_1$ and

$\nu_A(y_0)=\eta_2 \le \max\{\eta_1,\eta_2\} < t_1$. This implies $x_0, y_0 \in L(\nu_A : t)$. Thus $\nu_A(x*y) \le max\{\nu_A(x),\nu_A(y)\}$ for all $x, y \in X$. □

Theorem 3.14. *Any G-subalgebra of X can be realized as both the upper $[s_1,s_2]$-level and lower t-level of some cubic G-subalgebra of X.*

Proof: Let P be a cubic G-subalgebra of X, and A be cubic set on X defined by

$$\tilde{\mu}_A(x)=\begin{cases}[\alpha_1,\alpha_2], & if\ x\in P\\ [0,0], & otherwise\end{cases} \quad \text{and} \quad \nu_A(x)=\begin{cases}\beta, & if\ x\in P\\ 1, & otherwise\end{cases}$$

for all $[\alpha_1,\alpha_2]\in D[0,1]$ and $\beta\in[0,1]$. We consider the following cases:

Case (i) If $x, y\in P$, then $\tilde{\mu}_A(x)=[\alpha_1,\alpha_2]$, $\nu_A(x)=\beta$ and $\tilde{\mu}_A(y)=[\alpha_1,\alpha_2]$, $\nu_A(y)=\beta$. Thus,

$\tilde{\mu}_A(x*y)=[\alpha_1,\alpha_2]=rmin\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\}=rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$ and

$\nu_A(x*y)=\beta=\max\{\beta,\beta\}=\max\{\nu_A(x),\nu_A(y)\}$.

Case (ii) If $x\in P$ and $y\notin P$ then $\tilde{\mu}_A(x)=[\alpha_1,\alpha_2]$, $\nu_A(x)=\beta$ and $\tilde{\mu}_A(y)=[0,0]$, $\nu_A(y)=1$. Thus,

$\tilde{\mu}_A(x*y)\ge[0,0]=rmin\{[\alpha_1,\alpha_2],[0,0]\}=rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$ and

$\nu_A(x*y)\le 1=\max\{\beta,1\}=\max\{\nu_A(x),\nu_A(y)\}$.

Case (iii) If $x\notin P$ and $y\in P$ then $\tilde{\mu}_A(x)=[0,0]$, $\nu_A(x)=1$ and $\tilde{\mu}_A(y)=[\alpha_1,\alpha_2]$, $\nu_A(y)=\beta$. Thus, $\tilde{\mu}_A(x*y)\ge[0,0]=rmin\{[0,0],[\alpha_1,\alpha_2]\}=rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$ and $\nu_A(x*y)\le 1=\max\{1,\beta\}=\max\{\nu_A(x),\nu_A(y)\}$.

Case (iv) If $x\notin P$ and $y\notin P$ then $\tilde{\mu}_A(x)=[0,0]$, $\nu_A(x)=1$ and $\tilde{\mu}_A(y)=[0,0]$, $\nu_A(y)=1$. Now $\tilde{\mu}_A(x*y)\ge[0,0]=rmin\{[0,0],[0,0]\}=rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$ and $\nu_A(x*y)\le 1=\max\{1,1\}=\max\{\nu_A(x),\nu_A(y)\}$.

Therefore, A is a cubic G-subalgebra of X. □

Theorem 3.15. *Let P be a subset of X and A be cubic set on X which is given in the proof of Theorem 3.14. If A be realized as lower level G-subalgebra and upper level G-subalgebra of some cubic G-subalgebra of X, then P is a cubic G-subalgebra of X.*

Proof: Let A be a cubic G-subalgebra of X, and $x, y\in P$. Then

$\tilde{\mu}_A(x)=[\alpha_1,\alpha_2]=\tilde{\mu}_A(y)$ and $\nu_A(x)=\beta=\nu_A(y)$. Thus

$\tilde{\mu}_A(x*y)\ge rmin\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}=rmin\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\}=[\alpha_1,\alpha_2]$ and

$\nu_A(x*y)\le\max\{\nu_A(x),\nu_A(y)\}=\max\{\beta,\beta\}=\beta$, which imply that $x*y\in P$. Hence, the theorem. □

4. Homomorphism of cubic G-subalgebras

In this section, homomorphism of cubic G-subalgebra is defined and some results are studied. Let f be a mapping from a set X into a set Y. Let $B=(\tilde{\mu}_B,\nu_B)$ be cubic set in Y. Then the inverse image of B, is defined as

$f^{-1}(B)=\{\langle x, f^{-1}(\tilde{\mu}_B), f^{-1}(\nu_B)\rangle : x\in X\}$ with the membership function and non-membership function respectively are given by $f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x))$ and $f^{-1}(\nu_B)(x) = \nu_B(f(x))$. It can be shown that $f^{-1}(B)$ is cubic set.

Theorem 4.1. *Let $f : X \to Y$ be a homomorphism of G-algebras. If $B = (\tilde{\mu}_B, \nu_B)$ is a cubic G-subalgebra of Y, then the preimage $f^{-1}(B) = \{\langle x, f^{-1}(\tilde{\mu}_B), f^{-1}(\nu_B)\rangle : x \in X\}$ of B under f is a cubic G-subalgebra of X.*

Proof: Assume that $B = (\tilde{\mu}_B, \nu_B)$ is a cubic G-subalgebra of Y and let $x, y \in X$. Then

$f^{-1}(\tilde{\mu}_B)(x * y) = \tilde{\mu}_B(f(x * y)) = \tilde{\mu}_B(f(x) * f(y)) \geq rmin\{\tilde{\mu}_B(f(x), \tilde{\mu}_B(f(y))\} = rmin\{f^{-1}(\tilde{\mu}_B)(x), f^{-1}(\tilde{\mu}_B)(y)\}$ an

$f^{-1}(\nu_B)(x * y) = \nu_B(f(x * y)) = \nu_B(f(x) * f(y)) \leq \max\{\nu_B(f(x), \nu_B(f(y))\}$
$= \max\{f^{-1}(\nu_B)(x), f^{-1}(\nu_B)(y)\}$.

Therefore, $f^{-1}(B) = \{\langle x, f^{-1}(\tilde{\mu}_B), f^{-1}(\nu_B)\rangle : x \in X\}$ is a cubic G-subalgebra of X.
□

Definition 4.2. *A cubic set A in the G-algebra X is said to have the rsup-property and inf-property if for any subset T of X there exist $t_0 \in T$ such that $\tilde{\mu}_A(t_0) = rsup_{t_0 \in T} \tilde{\mu}_A(t)$ and $\nu_A(t_0) = \inf_{t_0 \in T} \nu_A(t)$ respectively.*

Definition 4.3. *Let f be a mapping from the set X to the set Y. If $A = (\tilde{\mu}_A, \nu_A)$ is cubic set in X, then the image of A under f, denoted by $f(A)$, and is defined as $f(A) = \{\langle x, f_{rsup}(\tilde{\mu}_A), f_{\inf}(\nu_A)\rangle : x \in Y\}$, where*

$f_{rsup}(\tilde{\mu}_A)(y) = \{rsup_{x \in f^{-1}(y)} \tilde{\mu}_A(x),$ f^-1(y)[0,0], o*therwise*.

and $f_{\inf}(\nu_A)(y) = \{\inf_{x \in f^{-1}(y)} \nu_A(x),$ f^-1(y)1, o*therwise*..

Theorem 4.4 . *Let $f : X \to Y$ be a homomorphism from a G-algebra X onto a G-algebra Y. If $A = (\tilde{\mu}_A, \nu_A)$ is a cubic G-subalgebra of X, then the image $f(A) = \{\langle x, f_{rsup}(\tilde{\mu}_A), f_{\inf}(\nu_A)\rangle : x \in Y\}$ of A under f is a cubic G-subalgebra of Y.*

Proof: Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic G-subalgebra of X and let $y_1, y_2 \in Y$. We know that, $\{x_1 * x_2 : x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X : x \in f^{-1}(y_1 * y_2)\}$. Now,

$$
\begin{aligned}
f_{rsup}(\tilde{\mu}_A)(y_1 * y_2) &= rsup\{\tilde{\mu}_A(x) : x \in f^{-1}(y_1 * y_2)\} \\
&\geq rsup\{\tilde{\mu}_A(x_1 * x_2) : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\
&\geq rsup\{rmin\{\tilde{\mu}_A(x_1), \tilde{\mu}_A(x_2)\} : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\
&= rmin\{rsup\{\tilde{\mu}_A(x_1) : x_1 \in f^{-1}(y_1)\}, rsup\{\tilde{\mu}_A(x_2) : x_2 \in f^{-1}(y_2)\}\} \\
&= rmin\{f_{rsup}(\tilde{\mu}_A)(y_1), f_{rsup}(\tilde{\mu}_A)(y_2)\}
\end{aligned}
$$

and

$$\begin{aligned} f_{\inf}(\nu_A)(y_1 * y_2) &= \inf\{\nu_A(x) : x \in f^{-1}(y_1 * y_2)\} \\ &\le \inf\{\nu_A(x_1 * x_2) : x_1 \in f^{-1}(y_1) \ \ and \ \ x_2 \in f^{-1}(y_2)\} \\ &\le \inf\{\max\{\nu_A(x_1), \nu_A(x_2)\} : x_1 \in f^{-1}(y_1) \ \ and \ \ x_2 \in f^{-1}(y_2)\} \\ &= \max\{\inf\{\nu_A(x_1) : x_1 \in f^{-1}(y_1)\}, \inf\{\nu_A(x_2) : x_2 \in f^{-1}(y_2)\}\} \\ &= \max\{f_{\inf}(\nu_A)(y_1), f_{\inf}(\nu_A)(y_2)\}. \end{aligned}$$

Hence, $f(A) = \{\langle x, f_{rsup}(\tilde{\mu}_A), f_{\inf}(\nu_A)\rangle : x \in Y\}$ is a cubic G-subalgebra of Y. □

5. Conclusion

To investigate the structure of an algebraic system, it is clear that G-subalgebras with special properties play an important role. In the present paper, we considered the notions of cubic G-subalgebras of G-algebras and investigated some of their useful properties. The homomorphism of G-subalgebras has been introduced and some important properties are of it are also studied. It is our hope that this work would other foundations for further study of the theory of G-algebras.

REFERENCES

1. S.S.Ahn and H.S.Kim, On QS-algebras, *J.Chungcheong Math. Soc.*, 12 (1999) 33-41.
2. R.K.Bandru and N.Rafi, On G-algebras, *Scientia Magna*, 8(3) (2012) 1-7.
3. R.Biswas, Rosenfeld's fuzzy subgroups with interval valued membership function, *Fuzzy Sets and Systems*, 63(1) (1994) 87-90.
4. Q.P.Hu and X.Li, On BCH-algebras, *Mathematics Seminar Notes*, 11(2) (1983) 313-320.
5. Y.Huang, BCI-algebra, Science Press, Beijing, 2006.
6. C.Jana, T.Senapati, M.Bhowmik and M.Pal, On intuitionistic fuzzy G-subalgebras of G-algebras, *Fuzzy Inf.Eng.*, 7(2) (2015) 195-209.
7. C.Jana, M.Pal, T.Senapati and M.Bhowmik, Atanassov's intutionistic L-fuzzy G-subalgebras of G-algebras, *J.Fuzzy Math.*, 23(2) (2015) 195-209.
8. C.Jana, T.Senapati and M.Pal, Derivation, f-Derivation and Generalized Derivation of KUS-algebras, *Cogent Mathematics* (appear).
9. Y.B.Jun, C.S.Kim and K.O.Yang, Cubic sets, *Ann. Fuzzy Math. Inform.*, 4(1) (2012) 83-98.
10. Y.B.Jun, S.T.Jung, and M.S.Kim, Cubic subgroups, *Ann. Fuzzy Math. Inform.*, 2(1) (2011) 9-15.
11. C.B.Kim and H.S.Kim, On BG-algebras, *Demonstratio Mathematica*, 41 (2008) 497-505.
12. J.Neggers, S.S.Ahn and H.S.Kim, On Q-algebras, *Int.J. Math. Math. Sci.*, 27(12) (2001) 749-757.
13. J.Neggers and H.S.Kim, On B-algebras, *Math. Vensik*, 54 (2002) 21-29.
14. T.Senapati, M.Bhowmik and M.Pal, Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy H-ideals in BCK/BCI-algebras, *Notes on Intuitionistic Fuzz Sets*, 19(1) (2013) 32-47.
15. M.Bhowmik, T.Senapati and M.Pal, Intuitionistic L-fuzzy ideals of BG-algebras, *Afr. Mat.*, available as online first article, DOI 10.1007/s13370-013-0139-5

16. T.Senapati, M.Bhowmik and M.Pal, Interval-valued intuitionistic fuzzy BG-subalgebras, *J. Fuzzy Math.*, 20(3) (2012) 707-720.
17. T.Senapati, M.Bhowmik and M.Pal, Fuzzy B-subalgebras of B-algebra with respect to t-norm, *Journal of Fuzzy Set Valued Analysis*, 2012 (2012) 11 pages.
18. T.Senapati, M.Bhowmik and M.Pal, Fuzzy dot subalgebras and fuzzy dot ideals of B-algebras, *Journal of Uncertain Systems*, 8(1) (2014) 22-30.
19. T.Senapati, M.Bhowmik and M.Pal, Intuitionistic fuzzifications of ideals in BG-algebras, *Mathematica Aeterna*, 2(9) (2012) 761-778.
20. T.Senapati, M.Bhowmik and M.Pal, Interval-valued intuitionistic fuzzy closed ideals of BG-algebra and their products, *Int. J. of Fuzzy Logic Systems*, 2(2) (2012) 27-44.
21. T.Senapati, M.Bhowmik and M.Pal, Fuzzy closed ideals of B-algebras, *Int. J. of Computer Science, Engineering and Technology*, 1(10) (2011) 669-673.
22. T.Senapati, M.Bhowmik and M.Pal, Fuzzy closed ideals of B-algebras with interval-valued membership function, *Int. J. of Fuzzy Mathematical Archive*, 1 (2013) 79-91.
23. T.Senapati, M.Bhowmik and M.Pal, Fuzzy dot structure of BG-algebras, to appear in *Fuzzy Information and Engineering*.
24. T.Senapati, C.Jana, M.Bhowmik and M.Pal, L-fuzzy G-subalgebras of G-algebras, *Journal of the Egyptian Mathematical Society*, (23) 2 (2015) 219-223.
25. T.Senapati, T- fuzzy KU-subalgebras of KU-algebras, *Ann. Fuzzy Math. Inform.* Available as online first article.
26. T.Senapati, Bipolar fuzzy structure of BG-subalgebras, *J. Fuzzy Math.*, 23(1) (2015) 209-220.
27. T.Senapati, C.S.Kim, M.Bhowmik and M.Pal, Cubic subalgebras and cubic closed Ideals B-algebras, to appear in *Fuzzy Inf. Eng.*, 7(2) (2015).
28. T.Senapati, M.Bhowmik and M.Pal, Fuzzy dot structure of BG-algebras, *Fuzzy Inf. Eng.*, 6(3) (2015) 315-329.
29. T.Senapati, M.Bhowmik, M.Pal and B.Davvaz, Fuzzy translation of fuzzy H-ideals in BCK/BCI-algebras, *J. Indones. Math. Soc.*, 21(1) (2015) 45-58.
30. T.Senapati, M.Bhowmik and M.Pal, Triangular norm based fuzzy BG-algebras, *Afrika Mathematika*, (2015) DOI 10.1007/s13370-015-0330-y.
31. A.Walendziak, On BF-algebras, *Math. Slovaca*, 57(2) (2007) 119-128.
32. L.A.Zadeh, Fuzzy sets, *Inform. and Control*, 8 (1965) 338-353.
33. L.A.Zadeh, The concept of a linguistic variable and its application to approximate reasoning. I, *Inform. Sci.*, 8 (1975) 199-249.