q-Continuous Functions in Quad Topological Spaces

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Abstract. The purpose of this paper is to study the properties of q-open sets and q-closed sets and introduce q-continuous function in quad topological spaces (q-topological spaces).

Keywords: quad topological spaces, q-open sets, q-interior, q-closure, q-continuous function.

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1. Introduction

The concept of bitopological spaces was introduced by Kelly [2] as an extension of topological spaces in 1963. A nonempty set X with two topologies is called bitopological spaces. The study of tri-topological spaces was first initiated by Kovar [3] in 2000, where a non empty set X with three topologies is called tri-topological spaces. Biswas [1] defined some mapping in topological spaces. tri α Continuous Functions and tri β continuous functions introduced by Palaniammal [5] in 2011. Mukundan [4] introduced the concept on topological structures with four topologies, quad topology (4-tuple topology) and defined new types of open (closed) sets. In year 2011, Sweedy and Hassan [6] defined δ**-continuous function in tritopological space. In this paper, we study the properties of q-open sets and q-closed sets and q-continuous function in quad topological space (q-topological spaces).

2. Preliminaries

Definition 2.1.[4] Let X be a nonempty set and T₁, T₂, T₃ and T₄ are general topologies on X. Then a subset A of space X is said to be quad-open (q-open) set if A ⊆ T₁ ∪ T₂ ∪ T₃ ∪ T₄ and its complement is said to be q-closed and set X with four topologies called q-topological spaces (X, T₁, T₂, T₃, T₄). q-open sets satisfy all the axioms of topology.

Note 2.2.[4] We will denote the q-interior (resp. q-closure) of any subset, say of A by q-intA (q-clA), where q-intA is the union of all q-open sets contained in A, and q-clA is the intersection of all q-closed sets containing A.

3. Properties of q-open and q-closed sets

Theorem 3.1. Arbitrary union of q-open sets is q-open.
Proof: Let {Aₐ / α ∈ I} be a family of q-open sets in X.
For each $\alpha \in I$, $A_\alpha \subset T_1 \cup T_2 \cup T_3 \cup T_4$. Therefore, $\bigcup A_\alpha \subset T_1 \cup T_2 \cup T_3 \cup T_4$. (by definition of q-open sets). Therefore $\bigcup A_\alpha$ is q-open.

**Theorem 3.2.** Arbitrary intersection of q-closed sets is q-closed.

**Proof:** Let $\{B_\alpha / \alpha \in I\}$ be a family of q-closed sets in $X$. Let $A_\alpha = \overline{B_\alpha \cap \{A_\alpha / \alpha \in I\}}$ be a family of q-open sets in $X$. Arbitrary union of q-open sets is q-open. Hence $\bigcup A_\alpha$ is q-open and hence $(\bigcup A_\alpha)^c$ is q-closed i.e $\bigcap A_\alpha$ is q-closed. Hence arbitrary intersection of q-closed sets is q-closed.

**Definition 3.3.** [4] Let $(X, T, T, T)$ be a q-topological space. Let $A \subset X$, an element $x \in A$ is called q-interior point of $A$, if $\exists$ a q-open set $\overline{A}$ such that $x \in \overline{A}$.

**Definition 3.4** [4] The set of all q-interior points of $A \subset A$ is called q-interior of $A$ and is denoted as $q-int A$.

**Note 3.5.** (1) $q-int A \subset A$.
(2) $q-int A$ is q-open.
(3) $q-int A$ is the largest q-open set contained in $A$.

**Theorem 3.6.** Let $(X, T, T, T, T)$ be a q-topological space. Let $A \subset X$ then $A$ is q-open iff $A = q-int A$.

**Proof:** $A$ is q-open and $A \subset A$. Therefore, $A \in \{B / B \subset A, B$ is q-open\} $A$ is in the collection and every other member in the collection is a subset of $A$ and hence the union of this collection is $A$. Hence $\bigcup \{B / B \subset A, B$ is q-open\} = $A$ and hence $q-int A = A$.

Conversely, since $q-int A$ is q-open, $A = q-int A$ implies that $A$ is q-open.

**Theorem 3.7.** $q-int (A \cup B) \supset q-int A \cup q-int B$

**Proof:** $q-int A \subset A$ and q-int $A$ is q-open.

$q-int B \subset B$ and q-int B is q-open.

Union of two q-open sets is q-open and hence $q-int A \cup q-int B$ is a q-open set. Also $q-int A \cup q-int B \subset A \cup B$.

$q-int A \cup q-int B$ is one q-open subset of $A \cup B$ and $q-int (A \cup B)$ is the largest q-open subset of $A \cup B$.

Hence, $q-int (A \cup B) \supset q-int A \cup q-int B$.

**Definition 3.8.** [4] Let $(X, T, T, T, T)$ be a quad topological space and let $A \subset X$. The intersection of all q-closed sets containing $A$ is called the q-closure of $A$ & denoted by $q-cl A$. $q-cl A = \bigcap \{B / B \supset A, B$ is tri a closed\}.

**Note 3.9.** Since intersection of q-closed sets is q-closed, q-cl $A$ is a q-closed set.

**Note 3.10.** q-cl $A$ is the smallest q-closed set containing $A$. 
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**Theorem 3.11.** A is q-closed iff \( A = q - cl A \).

**Proof:** \( q - cl A = \cap \{ B / B \owns A, B \text{ is q-closed} \} \).

If A is a q-closed then A is a member of the above collection and each member contains A. Hence their intersection is A. Hence \( q - cl A = A \). Conversely if \( A = q - cl A \), then A is q-closed because q-cl A is a q-closed set.

**Theorem 3.12.** Let \((X, T_1, T_2, T_3, T_4)\) be a quad topological space for any \( A \subseteq X \).

\[(q - int A)^C = q - cl A^C.\]

**Proof:**
\[
\begin{align*}
(q - int A)^C &= \cup \{ G / G \subseteq A \text{ and } G \text{ is } q - \text{open} \}^C \\
&= \cap \{ G^C / G^C \owns A^C \text{ and } G^C \text{ is } q - \text{closed} \} \\
&= \cap \{ F = G^C / F \owns A^C \text{ and } F \text{ is } q - \text{closed} \} \text{ where } F = G^C \\
&= q - cl A^C.
\end{align*}
\]

**Definition 3.13.** Let \( A \subseteq X \), be a quad topological space. \( x \in X \) is called a q-limit point of A, if every q-open set U containing x, intersects \( A - \{x\} \). (i.e.) every q-open set containing x, contains a point of A other than x.

**Example 3.14.** Let \( X = \{a, b, c\} \), \( T_1 = \{\emptyset, \{a\}, \{a, b\}, X\} \), \( T_2 = \{\emptyset, \{a\}, X\} \), \( T_3 = \{\emptyset, \{a\}, \{a, c\}, X\} \), \( T_4 = \{\emptyset, \{a, b\}, \{a, c\}\} \).

q-open sets are \( \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\} \).

Consider A = \{a, c\}. Then b is a q-limit point of A.

**Definition 3.15.** Let \( A \subseteq X \). The set of all q-limit points of A is called the q-derived set of A and is denoted as \( q - D(A) \).

**Theorem 3.16.** \( q - cl A = A \cup q - D(A) \).

**Proof:** Let \( x \in q - cl A \). If \( x \in cl A \), then \( x \in A \cup q - D(A) \). If \( x \not\in A \), then we claim that \( x \) is a q-limit point of \( A \). Let \( U \) be a q-open set containing \( x \). Suppose \( U \cap A = \emptyset \).

Then \( A \subseteq U^c \) and \( U^c \) is q-closed and hence \( q - cl A \subseteq U^c \). This implies \( x \in U^c \). Therefore every q-open set \( U \) containing \( x \) intersects \( A - \{x\} \).

Hence \( x \in q - D(A) \) and \( x \in A \cup q - D(A) \). Therefore \( q - cl A \subseteq A \cup q - D(A) \). Conversely, it is clear that \( A \subseteq q - cl A \). It is enough to prove \( q - D(A) \subseteq q - cl A \).

Let \( x \in q - D(A) \). If \( x \in A \) then it is true. So let us take \( x \not\in A \). Now we have to prove that \( x \) is an q-closed set containing A. Suppose not, \( x \not\in B \) where B is a q-closed set containing A. \( B \owns A \). Now \( x \in B^c \). \( B^c \) is q-open and \( B^c \cap A = \emptyset \). Contradiction to the fact that \( x \) is a q-limit point of A. Hence \( x \in q - cl A \). Therefore \( x \in q - cl A \).

Hence \( q - cl A = A \cup q - D(A) \).

4. q-continuous function

**Definition 4.1.** Let \((X, T_1, T_2, T_3, T_4)\) and \((Y, T_1', T_2', T_3', T_4')\) be two quad topological spaces. A function \( f : X \to Y \) is called q-continuous function if \( f^{-1}(V) \) is q-open in \( X \) for every q-open set \( V \) in \( Y \).
Example 4.2. Let $X = \{1, 2, 3, 4\}$, $T_1 = \emptyset, \{1\}, X$, $T_2 = \emptyset, \{1, 3\}, X$.

$T_2 = \emptyset, \{1\}, \{1, 2\}, X$, $T_4 = \emptyset, \{4\}, \{1, 4\}, X$

Let $Y = \{a, b, c, d\}$, $T_1' = \emptyset, \{a\}, Y$, $T_2' = \emptyset, \{a\}, \{a, b\}, Y$, $T_3' = \emptyset, \{a, b\}, \{a, d\}, Y$

Let $f : X \to Y$ be a function defined as $f(1) = a; f(2) = b; f(3) = c; f(4) = d$.

$q$-open sets in $(X, T_1, T_2, T_3, T_4)$ are $\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{4\}, \{1, 4\}, X$.

$q$-open sets in $(Y, T_1', T_2', T_3', T_4')$ are $\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{d\}, \{a, d\}, Y$.

Since $f^{-1}(V)$ is $q$-open in $X$ for every $q$-open set $V$ in $Y$, $f$ is $q$-continuous.

Definition 4.3. Let $X$ and $Y$ be two $q$-topological spaces. A function $f : X \to Y$ is said to be $q$-continuous at a point $a \in X$ if for every $q$-open set $V$ containing $f(a)$, $\exists$ a $q$-open set $U$ containing $a$, such that $f(U) \subset V$.

Theorem 4.4. $f : X \to Y$ is $q$-continuous iff $f$ is $q$-continuous at each point of $X$.

Proof: Let $f : X \to Y$ be $q$-continuous.

Take any $a \in X$. Let $V$ be a $q$-open set containing $f(a)$.

$f : X \to Y$ is $q$-continuous, since $f^{-1}(V)$ is $q$-open set containing $a$.

Let $U = f^{-1}(V)$. Then $f(U) \subset V \Rightarrow \exists$ a $q$-open set $U$ containing $a$ and $f(U) \subset V$.

Hence $f$ is $q$-continuous at $a$.

Conversely, suppose $f$ is $q$-continuous at each point of $X$.

Let $V$ be a $q$-open set of $Y$. If $f^{-1}(V) = \emptyset$ then it is $q$-open.

Take any $a \in f^{-1}(V)$ $f$ is $q$-continuous at $a$.

Hence $\exists U_a$, $q$-open set containing $a$ and $f(U_a) \subset V$.

Let $U = U \cup \{U_a / a \in f^{-1}(V)\}$.

Claim: $U = f^{-1}(V)$.

$a \in f^{-1}(V) \Rightarrow U_a \subset U \Rightarrow a \in U$.

$x \in U \Rightarrow x \in U_a$ for some $a \Rightarrow f(x) \in V \Rightarrow x \in f^{-1}(V)$. Hence $U = f^{-1}(V)$.

Each $U_a$ is $q$-open. Hence $U$ is $q$-open. $f^{-1}(V)$ is $q$-open in $X$.

Hence $f$ is $q$-continuous.

Theorem 4.5. Let $(X, T_1, T_2, T_3, T_4)$ and $(Y, T_1', T_2', T_3', T_4')$ be two $q$-topological spaces. Then $f : X \to Y$ is $q$-continuous function iff $f^{-1}(V)$ is $q$-closed in $X$ whenever $V$ is $q$-closed in $Y$.

Proof: Let $f : X \to Y$ be $q$-continuous function.

Let $V$ be any $q$-closed in $Y$.

$V^c$ is tri $a$ open in $Y$ $\Rightarrow f^{-1}(V^c)$ is $q$-open in $X$.

$[f^{-1}(V)]^c$ is $q$-open in $X$.

$f^{-1}(V)$ is $q$-closed in $X$.

Hence $f^{-1}(V)$ is $q$-closed in $X$ whenever $V$ is $q$-closed in $Y$.

Conversely, suppose $f^{-1}(V)$ is $q$-closed in $X$ whenever $V$ is $q$-closed in $Y$.

$V$ is a $q$-open set in $Y$.

$V^c$ is $q$-closed in $Y$.  

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⇒ \( f^{-1}(V^c) \) is tri α closed in X.
⇒ \([f^{-1}(V)]^c\) is q-closed in X.
⇒ \( f^{-1}(V) \) is q-open in X.

Hence \( f \) is q-continuous.

**Theorem 4.6.** Let \((X,T_1,T_2,T_3,T_4)\) and \((Y,T'_1,T'_2,T'_3,T'_4)\) be two q-topological spaces. Then, \( f : X \rightarrow Y \) is q-continuous iff \( f[q - cl(A)] \subseteq q - cl(f(A)) \forall A \subseteq X \).

**Proof:** Suppose \( f : X \rightarrow Y \) is q-continuous. Since \( q - cl(f(A)) \) is q-closed in Y. Then by theorem (4.5) \( f^{-1}(q - cl(f(A))) \) is q-closed in X.

\[
q - cl[f^{-1}(q - cl(f(A))] = f^{-1}(q - cl(f(A)).
\]

Now : \( f(A) \subseteq q - cl(f(A)) \subseteq f^{-1}(q - cl(f(A)). 
Then \( q - cl(A) \subseteq q - cl[f^{-1}(q - cl(f(A))] = f^{-1}(q - cl(f(A))) \) by (1)

Then \( f(q - cl(f(A))) \subseteq q - cl(f(A)) \).
Conversely, let \( f(q - cl(f(A))) \subseteq q - cl(f(A)) \forall A \subseteq X \).

Let \( F \) be q-closed set in Y., so that \( q - cl(F) = F \).

Now \( f^{-1}(F) \subseteq X \).

By hypothesis, \( f(q - cl(F)) \subseteq q - cl(f^{-1}(F)) \)

Therefore \( q - cl(f^{-1}(F)) \subseteq f^{-1}(F). \)

But \( f^{-1}(F) \subseteq q - cl(f^{-1}(F)) \) always.

Hence \( q - cl(f^{-1}(F)) = f^{-1}(F) \) and so \( f^{-1}(F) \) is q-closed in X.

Hence by theorem (4.5) \( f \) is q-continuous.

5. q-Homomorphism

**Definition 5.1.** Let \((X,T_1,T_2,T_3,T_4)\) and \((Y,T'_1,T'_2,T'_3,T'_4)\) be two q-topological spaces. A function \( f : X \rightarrow Y \) is called q-open map if \( f(V) \) q-open in Y for every q-open set \( V \) in X.

**Example 5.2.** In example 4.2 \( f \) is q- open map also.

**Definition 5.3.** Let \((X,T_1,T_2,T_3,T_4)\) and \((Y,T'_1,T'_2,T'_3,T'_4)\) be two q-topological spaces .Let \( f : X \rightarrow Y \) be a mapping . \( f \) is called q-closed map if \( f(F) \) is q-closed in Y for every q-closed set \( F \) in X.

**Example 5.4.** The function \( f \) defined in the example 4.2 is q-closed map.

**Result 5.5.** Let \( X \) & \( Y \) be two q-topological spaces. Let \( f : X \rightarrow Y \) be a mapping. \( f \) is q-continuous iff \( f^{-1} \) is q-open map.

**Definition 5.6.** Let \((X,T_1,T_2,T_3,T_4)\) and \((Y,T'_1,T'_2,T'_3,T'_4)\) be two q-topological spaces. Let \( f : X \rightarrow Y \) be a mapping . \( f \) is called a q-homeomorphism.

If (i) \( f \) is a bijection.
(ii) \( f \) is q-continuous.
(iii) \( f^{-1} \) is q-continuous.

**Example 5.7.** The function \( f \) defined in the example 4.2 is
(i) a bijection. (ii) \( f \) is q-continuous. (iii) \( f^{-1} \) is q-continuous.

Therefore \( f \) is a q-homeomorphism.
6. Conclusion
In this paper, the idea of q-continuous function in quad topological spaces were introduced and studied. Also properties of q-open and q-closed sets were studied.

REFERENCES