On Pre Generalized Pre Regular Weakly Open Sets and Pre Generalized Pre Regular Weakly Neighbourhoods in Topological Spaces

R.S. Wali and Vivekananda Dembre

1Department of Mathematics
Bhandari and Rathi College, Guledagudd-587203, Karnataka State, India
Corresponding author. Email: rswali@rediffmail.com

2Department of Mathematics
Rani Channamma University, Belagavi-591156, Karnataka State, India
Email: vbdembre@gmail.com

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Abstract. In this paper a new class of sets called pre generalized pre regular weakly open sets (briefly pgprw open sets) and pgprw neighbourhoods are introduced and studied in topological spaces and some properties of new concepts have been studied.

Keywords: pgprw closed sets, pgprw open sets, pgprw -neighbourhoods

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1. Introduction

Stone [1] introduced and studied Regular open sets then Regular semi open sets, Pre-open sets, gspr closed sets, gpr closed sets, gp closed sets, Rg closed sets, rga-closed sets, π-g-closed sets, pgprw closed sets are introduced and studied by Cameron [2], Mashhour, Abd El-Monsef and El-Deeb [3], Govindappa Navalagi, Chandrashakarappa and Gurushanthanavar [4], Gnanambal [5], Maki, Umehara and Noiri [6], Palaniappan and Rao [7], Vadivel and Vairamamanickam [8], Dontchev and Noiri [9], Wali and Vivekananda Dembre [10] respectively.

2. Preliminaries

Throughout this paper space (X, τ) and (Y, σ) (or simply X and Y) always denote topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, Cl(A), Int(A), A', P-Cl(A) and P-int(A) denote the Closure of A, Interior of A, Compliment of A, pre-closure of A and pre-interior of (A) in X respectively.

Definition 2.1. A subset A of a topological space (X, τ ) is called
(i) Regular open set [1] if A = int(cl(A)) and a regular closed set if A = cl(int(A)).
(ii) Regular semi open set [2] if there is a regular open set U such that U ⊆ A ⊆ cl(U).
(iii) Pre-open set [3] if A ⊆ int(cl(A)) and pre-closed set if cl(int(A)) ⊆ A.
(iv) Generalized semi pre regular closed (briefly, gspr-closed) set \([4]\) if \(spcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\).

(v) Generalized pre regular closed set (briefly, gpr-closed) \([5]\) if \(pcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\).

(vi) Generalized pre closed (briefly, gp-closed) set \([6]\) if \(pcl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(vii) Regular generalized closed set (briefly, rg-closed) \([7]\) if \(cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\).

(viii) Regular-generalized-\(\alpha\) closed set \([8]\) if \(\alpha-cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular \(\alpha\)-open in \((X, \tau)\).

(ix) \(\pi\)-generalized closed set (briefly, \(\pi\)g-closed) \([9]\) if \(cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\pi\)-open in \(X\).

(x) pre generalized pre regular \(\omega\)-weakly closed set (briefly pgpr\(\omega\)-closed) \([10]\) if \(pCl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is rg\(\alpha\)-open in \((X, \tau)\).

3. Pre generalized pre regular weakly closed sets in topological spaces

**Definition 3.1.** \([10]\) A subset \(A\) of topological space \((X, \tau)\) is called a pre generalized pre regular weakly closed set (briefly pgpr\(\omega\)-closed set) if \(pCl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular \(\alpha\)-open in \((X, \tau)\).

**Results 3.2.** From \([10]\)

(i) Every closed set is pgpr\(\omega\)-closed set in \(X\).

(ii) Every regular closed set is pgpr\(\omega\)-closed set in \(X\).

(iii) Every pgpr\(\omega\)-closed set is gspr, gpr, gp, rg, \(\pi\)g closed set.

(iv) The Union of two pgpr\(\omega\)-closed subsets of \(X\) is pgpr\(\omega\) – closed set.

(v) If \(A\) is pre generalized pre regular weakly closed set in \(X\) and \(A \subseteq B \subseteq pCl(A)\) then \(B\) is also pre generalized pre regular weakly closed set in \(X\).

(vi) If a subset \(A\) of topological space \(X\) is a pre generalized pre regular weakly closed set in \(X\); then \(pCl(A) - A\) does not contain any non empty rg\(\alpha\)-closed set in \(X\).

4. Pre generalized pre regular weakly open sets

**Definition 4.1.** A subset \(A\) of a topological space \((X, \tau)\) is called pre generalized pre regular weakly open (briefly pgpr\(\omega\)-open) set in \(X\) if \(A^c\) is pgpr\(\omega\)-closed in \(X\).

The following theorem is the analogue of results 3.2 (i) to (iv).

**Theorem 4.2.** For any topological spaces \((X, \tau)\) we have the following .

(i) Every open set is pgpr\(\omega\)-open.

(ii) Every regular open set is pgpr\(\omega\) closed set.

(iii) Every pgpr\(\omega\)-open set is gspr, gpr, gp, rg, \(\pi\)g-open set.

**Theorem 4.3.** If \(A\) and \(B\) are pgpr\(\omega\)-open sets in space \(X\), then \(A \cap B\) is also an pgpr\(\omega\)-open in \(X\).

**Proof:** Let \(A\) and \(B\) be two pgpr\(\omega\)-open sets in \(X\). Then \(A^c\) and \(B^c\) are pgpr\(\omega\)-closed sets in \(X\) by Results 3.2 (iv), \(A^c \cup B^c\) is also pgpr\(\omega\)-closed set in \(X\), that is \(A^c \cup B^c = (A \cap B)^c\) is pgpr\(\omega\)-closed set in \(X\). Therefore \(A \cap B\) is an pgpr\(\omega\)-open set in \(X\).
Remark 4.4. The union of pgpr\textsuperscript{ω}-open set in X is generally not an pgpr\textsuperscript{ω}-open set in X.

Example 4.5. Let X={a, b, c, d}, \( \tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\} \). If A={c} B={a} Then A & B are pgpr\textsuperscript{ω}-open set in X but A \bigcup B={a,c} is not an pgpr\textsuperscript{ω}-open set in X.

Theorem 4.6. A subset A of a topological space X is pgpr\textsuperscript{ω}-open iff \( U \subseteq \text{p-int}(A) \) whenever U is rg\textsuperscript{α}-closed and U \subseteq A.

Proof: Assume that A is pgpr\textsuperscript{ω}-open set in X and U is rg\textsuperscript{α}-closed set of (X, \( \tau \)) s.t U \subseteq A. Then X-A is a pgpr\textsuperscript{ω}-closed set in (X, \( \tau \)). Also X-A \subseteq X-U and X-U is rg\textsuperscript{α}-open set of (X, \( \tau \)). This implies that pcl(X-A) \subseteq X-U. But pcl(X-A)=X-p-int(A).

Thus, X-p.int(A) \subseteq X-U, so U \subseteq p-int(A). Conversely: Suppose U \subseteq p-int (A) whenever U is rg\textsuperscript{α}-closed and U \subseteq A. To prove that A is pgpr\textsuperscript{ω}-open set. Let F be rg\textsuperscript{α}-open set of (X, \( \tau \)) s.t X-A \subseteq F. Then X-F \subseteq A. Now X-F is rg\textsuperscript{α}-closed set containing A. So X-F \subseteq p-int(A). Thus X-p.int(A) \subseteq F but pcl(X-A)=X-p-int(A) \subseteq F. Thus pcl(X-A) \subseteq F i.e X-A is pgpr\textsuperscript{ω}-closed set & hence A is pgpr\textsuperscript{ω}-open set.

Theorem 4.7. If p-int(A) \subseteq B \subseteq A and A is pgpr\textsuperscript{ω}-open set, then B is pgpr\textsuperscript{ω}-open set.

Proof: Let p-int(A)\subseteq B\subseteq A, Thus X-A \subseteq X-B \subseteq X-p-int(A), i.e. X-A \subseteq X-B \subseteq cl(X-A). Since X-A is pgpr\textsuperscript{ω}-closed set, then from result 3.2 (v) [10] X-B is pgpr\textsuperscript{ω}-open set. Therefore B is pgpr\textsuperscript{ω}-open set.

Theorem 4.8. If A \subseteq X is pgpr\textsuperscript{ω}-closed then pcl(A) \rightarrow A is pgpr\textsuperscript{ω}-open set.

Proof: Let A be pgpr\textsuperscript{ω}-closed. Let F \subseteq pcl(A) \rightarrow A, where F is rg\textsuperscript{α}-closed; then from result 3.2 (vi) [10] we have F=\phi . Therefore F \subseteq p-int (pcl(A) \rightarrow A) and Theorem 4.6 pcl(A) \rightarrow A is pgpr\textsuperscript{ω}-open set.

The reverse implication does not hold good.

Example 4.9. Let X={a, b, c, d}, \( \tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\} \) Let A={a,d}, pcl(A)\{a,c,d\} then pcl(A) \rightarrow A={c} which is pgpr\textsuperscript{ω}-open set in X, but A is not pgpr\textsuperscript{ω}-closed.

Theorem 4.10. A set A is a pgpr\textsuperscript{ω}-open set in (X, \( \tau \)) if and only if U=X whenever U is rg\textsuperscript{α}-open and \( U \subseteq \text{p-int}(A) \) \( U \subseteq (X-A) \subseteq U \).

Proof: Suppose that A is pgpr\textsuperscript{ω}-open set in X. Then U be rg\textsuperscript{α}-open and p-int(A) \( U \subseteq (X-A) \subseteq U \). Then \( U \subseteq (p-int(A)) \cap A \) i.e. \( U \subseteq (p-int(A)) \cap A \) (because A \( \rightarrow B = A \cap B \)). Thus \( U \subseteq pcl(A) \rightarrow A \) (because \( p-int(A) = pcl(A) \)). Now \( A \rightarrow \) is also pgpr\textsuperscript{ω}-closed and \( U \) is rg\textsuperscript{α}-closed then from result 3.2 (vi) [10] it follows U=\phi then U=X. Conversely: Suppose F is pgpr\textsuperscript{ω}-closed and F \subseteq A. Then p-int(A) \( U \subseteq p-int(A) \subseteq U \). It follows that p-int(A) \( U \subseteq (X \rightarrow) = X \).

Theorem 4.11. If A and B be subsets of space (X, \( \tau \)). If B is pgpr\textsuperscript{ω}-open and p-int(B) \subseteq A then A \cap B is pgpr\textsuperscript{ω}-open.
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Proof: Let $B$ be pgprω-open in $X$. Then $\text{p-int}(B) \subseteq A$ and $\text{p-int}(B) \subseteq B$ is always true. If $\text{p-int}(B) \subseteq A \cap B$ and also $\text{p-int}(B) \subseteq B$, then $B$ is pgprω-open set by Theorem 4.7, and $A \cap B$ is also pgprω-open set in $X$.

5. pgprw-neighbourhood

Definition 5.1. (i) Let $(X, \tau)$ be a topological space and let $x \in X$. A subset $N$ of $X$ is said to be pgprω-neighbourhood of $x$ if there exists an pgprω-open set $G$ such that $x \in G \subseteq N$.

(ii) The collection of all pgprω-neighbourhood of $x \in X$ is called pgprω-neighbourhood system at $x$ and shall be denoted by $\text{pgprw-N}(x)$.

Theorem 5.2. Every neighbourhood $N$ of $x \in X$ is a pgprω-neighbourhood of $X$.

Proof: Let $N$ be neighbourhood of point $x \in X$. To prove that $N$ is a pgprω-neighbourhood of $x$ by definition of neighbourhood, there exists an open set $G$ such that $x \in G \subseteq N$. Hence $N$ is pgprω-neighbourhood of $x$.

Remark 5.3. In general, a pgprω-neighbourhood $N$ of $x \in X$ need not be a nbhd of $x$ in $X$, as seen from the following example.

Example 5.4. Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $\text{pgprω}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. The set $\{c, d\}$ is pgprω-neighbourhood of the point $c$, since the pgprω-open set $\{c\}$ is such that $c \in \{c\} \subseteq \{c, d\}$. However, the set $\{c, d\}$ is not a nbhd of the point $c$, since no pgprw open set $G$ exists such that $c \in G \subseteq \{c, d\}$.

Theorem 5.5. If a subset $N$ of a space $X$ is pgprω-open, then $N$ is a pgprω-neighbourhood of each of its points.

Proof: Suppose $N$ is pgprω-open. Let $x \in N$. We claim that $N$ is pgprω-neighbourhood of $x$. For $N$ is a pgprω-open set such that $x \in N \subseteq N$. Since $x$ is an arbitrary point of $N$, it follows that $N$ is a pgprω-neighbourhood of each of its points.

Remark 5.6. The converse of the above theorem is not true in general as seen from the following example.

Example 5.7. Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $\text{pgprω}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. The set $\{b, c\}$ is pgprω-neighbourhood of the point $b$, since the pgprω-open set $\{b\}$ is such that $b \in \{b\} \subseteq \{b, c\}$. Also the set $\{b, c\}$ is pgprω-neighbourhood of the point $c$. Since the pgprω-open set $\{c\}$ is such that $c \in \{c\} \subseteq \{b, c\}$. That is $\{b, c\}$ is a pgprω-neighbourhood of each of its points. However the set $\{b, c\}$ is not a pgprω-open set in $X$.

Theorem 5.8. Let $X$ be a topological space. If $F$ is a pgprω-closed subset of $X$, and $x \in F^c$. Prove that there exists a pgprω-nbhd $N$ of $x$ such that $N \cap F = \emptyset$.

Proof: Let $F$ be pgprω-closed subset of $X$ and $x \in F^c$. Then $F^c$ is pgprω-open set of $X$. So by theorem 5.5 $F^c$ contains a pgprω-neighbourhood of each of its points. Hence there exists a pgprω-neighbourhood $N$ of $x$ such that $N \subseteq F^c$. That is $N \cap F = \emptyset$. 

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Theorem 5.9. Let $X$ be a topological space and for each $x \in X$, let $pgpr\text{-}N (x)$ be the collection of all $pgpr\text{-}nbhds$ of $x$. Then we have the following results.

(i) $\forall x \in X$, $pgpr\text{-}N (x) \neq \emptyset$

(ii) $N \in pgpr\text{-}N (x) \Rightarrow x \in N$

(iii) $N \in pgpr\text{-}N (x), M \supseteq N \Rightarrow M \in pgpr\text{-}N (x)$

(iv) $N \in pgpr\text{-}N (x) \Rightarrow \text{there exists } M \in pgpr\text{-}N (x) \text{ such that } M \subseteq N$ and $M \in pgpr\text{-}N (y)$ for every $y \in M$

Proof: (i) Since $X$ is a $pgpr\text{-}open$ set, it is a $pgpr\text{-}nbhd$ of every $x \in X$. Hence there exists at least one $pgpr\text{-}nbhd$ (namely $X$) for each $x \in X$. Hence $pgpr\text{-}N (x) \neq \emptyset$ for every $x \in X$.

(ii) If $N \in pgpr\text{-}N (x)$, then $N$ is a $pgpr\text{-}nbhd$ of $x$. So by definition of $pgpr\text{-}nbhd$, $x \in N$.

(iii) Let $N \in pgpr\text{-}N (x)$ and $M \supseteq N$. Then there is a $pgpr\text{-}open$ set $G$ such that $x \in G \subset N$. Since $N \subseteq M$, $x \in G \subseteq M$ and so $M$ is $pgpr\text{-}nbhd$ of $x$. Hence $M \in pgpr\text{-}N (x)$.

(iv) Let $N \in pgpr\text{-}N (x)$ and $M \in pgpr\text{-}N (x)$. Then by definition of $pgpr\text{-}nbhd$ there exists $pgpr\text{-}open$ sets $G_1$ and $G_2$ such that $x \in G_1 \subseteq N$ and $x \in G_2 \subseteq M$. Hence $x \in G_1 \cap G_2 \subseteq N \cap M$ -- (1). Since $G_1 \cap G_2$ is a $pgpr\text{-}open$ set, (being the intersection of two $pgpr\text{-}open$ sets), it follows from (1) that $N \cap M$ is a $pgpr\text{-}nbhd$ of $x$. Hence $N \cap M \in pgpr\text{-}N (x)$.

(v) If $N \in pgpr\text{-}N (x)$, then there exists a $pgpr\text{-}open$ set $M$ such that $x \in M \subseteq N$. Since $M$ is a $pgpr\text{-}open$ set, it is $pgpr\text{-}nbhd$ of each of its points. Therefore $M \in pgpr\text{-}N (y)$ for every $y \in M$.

Theorem 5.10. Let $X$ be a nonempty set, and for each $x \in X$, let $pgpr\text{-}N (x)$ be a nonempty collection of subsets of $X$ satisfying following conditions.

(i) $N \in pgpr\text{-}N (x) \Rightarrow x \in N$

(ii) $N \in pgpr\text{-}N (x), M \in pgpr\text{-}N (x) \Rightarrow N \cap M \in pgpr\text{-}N (x)$

Let $\tau$ consists of the empty set and all those non-empty subsets of $G$ of $X$ having the property that $x \in G$ implies that there exists an $N \in pgpr\text{-}N (x)$ such that $x \in N \subseteq G$. Then $\tau$ is a topology for $X$.

Proof: (i) $\emptyset \in \tau$ by definition. We now show that $x \in \tau$. Let $x$ be any arbitrary element of $X$. Since $pgpr\text{-}N (x)$ is nonempty, there is an $N \in pgpr\text{-}N (x)$ and so $x \in N$ by (i). Since $N$ is a subset of $X$, we have $x \in N \subseteq X$. Hence $x \in \tau$.

(ii) Let $G_1 \in \tau$ and $G_2 \in \tau$. If $x \in G_1 \cap G_2$ then $x \in G_1$ and $x \in G_2$. Since $G_1 \in \tau$ and $G_2 \in \tau$, there exists $N \in pgpr\text{-}N (x)$ and $M \in pgpr\text{-}N (x)$, such that $x \in N \subseteq G_1$ and $x \in M \subseteq G_2$. Then $x \in N \cap M \subseteq G_1 \cap G_2$. But $N \cap M \in pgpr\text{-}N (x)$ by (2). Hence $G_1 \cap G_2 \in \tau$. Let $G_3 \in \tau$ for every $x \in \Lambda$. If $x \in \bigcup \{G_3 : \lambda \in \Lambda\}$, then $x \in G_{3x}$ for some $\lambda x \in \Lambda$. 

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Since $G_{\lambda x} \in \tau$, there exists an $N \in gpro-N(x)$ such that $x \in N \subseteq G_{\lambda x}$ and consequently $x \in N \subseteq \bigcup\{G_{\lambda} : \lambda \in \Lambda\}$. Hence $\bigcup\{G_{\lambda} : \lambda \in \Lambda\} \in \tau$. It follows that $\tau$ is topology for $X$.

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