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An Algorithm to Solve the Games under Incomplete Information

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Abstract. Classical game theory assumes that the structure of the game is common knowledge among the players but in most of the real life game theoretic problems incomplete information available about payoffs of agents. But it is possible to determine lower and upper bounds on payoffs. In this paper an algorithm to determine the value of game in fuzzy environment, using octagonal fuzzy numbers is proposed. Octagonal fuzzy numbers are converted into interval numbers using α-cut method. The effectiveness of proposed algorithm is illustrated by means of a numerical example.

Keywords: Octagonal fuzzy numbers, interval numbers, saddle point, fuzzy payoffs

AMS Mathematics Subject Classification (2010): 91A80, 90C70

1. Introduction

Fuzzy numbers are widely used in the research on fuzzy set theory. In order to identify the preference ranking of the fuzzy numbers, one such numbers need to evaluated and compared with the other fuzzy numbers. This is not easy in practical situations. Game theory had its beginnings in the 1920s, and significantly advanced at Princeton University through the work of Nash [3,4]. The method of finding the fuzzy game value using intervals as elements of matrix is explained in [5, 7]. By defining a ranking of octagonal fuzzy numbers, transportation problems solved in [1]. Also using the ranking of these fuzzy numbers in [2] a two person zero sum game is solved. Definitions of intervals and interval arithmetic are studied in [6]. By comparing the intervals and using dominance principle in [8] a fuzzy environment we find the fuzzy game value of the interval valued matrix.

2. Octagonal fuzzy numbers

An octagonal fuzzy number denoted by $\tilde{A}_w$ is defined to be the ordered quadruple,

$\tilde{A}_w = (l_1(r), s_1(t), s_2(t), l_2(r))$, where $r \in [0,k]$ and $t \in [k,w]$, where

(1) $l_1(r)$ Is a bounded left continuous non decreasing function over $[0, w_1], [0 \le w_1 \le k]$

(2) $s_1(t)$ Is a bounded left continuous non decreasing function over

$[k, w_2], [k \le w_2 \le w]$

(3) $s_2(t)$ Is a bounded left continuous non increasing function over $[k, w_2],[k \le w_2 \le w]$
(4) $l_2(r)$ Is a bounded left continuous non increasing function over $[0, w_1],[0 \le w_1 \le k]$
If w=1, then the above defined number is called a normal octagon number. A fuzzy number $\tilde{A}$ is a normal octagon fuzzy number denoted by $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ where $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7 \le a_8$ are real numbers and its membership function $\mu_{\tilde{A}}(x)$ is given as following

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, x < a_1 \\ k\left(\dfrac{x-a_1}{a_2-a_1}\right), a_1 \le x \le a_2 \\ k, a_2 \le x \le a_3 \\ k+(1-k)\left(\dfrac{x-a_3}{a_4-a_3}\right), a_3 \le x \le a_4 \\ 1, a_4 \le x \le a_5 \\ k+(1-k)\left(\dfrac{a_6-x}{a_6-a_5}\right), a_5 \le x \le a_6 \\ k, a_6 \le x \le a_7 \\ k\left(\dfrac{a_8-x}{a_8-a_7}\right), a_7 \le x \le a_8 \\ 0, x \ge a_8 \end{cases}$$

where $0 < k < 1$.

3. The interval number system

Closed interval denoted by $[X, Y]$ is the set of real numbers given by,

$[X,Y] = \{x \in \mathbb{R} : X \le x \le Y\}$

Although various other types of intervals (open, half-open) appear throughout mathematics, our work will center primarily on closed intervals. In this paper, the term interval will mean closed interval. We will adopt the convention of denoting intervals and their endpoints by capital letters. The left and right endpoints of an interval X will be denoted by $\underline{X}$ and $\overline{X}$, respectively. Thus, $X = [\underline{X}, \overline{X}]$

Two intervals X and Y are said to be equal if they are the same sets. Operationally, this happens if their corresponding endpoints are equal:

$X = Y \iff \underline{X} = \underline{Y}$ And $\overline{X} = \overline{Y}$

We say that X is degenerate if $\underline{X} = \overline{X}$. Such an interval contains a single real number x. By convention, we agree to identify a degenerate interval $[X, X]$ with the real number x.

Operations on intervals:

Intersection of the intervals : $X \cap Y = [\max\{\underline{X}, \underline{Y}\}, \min\{\overline{X}, \overline{Y}\}]$

Union of intervals : $X \cup Y = [\min\{\underline{X}, \underline{Y}\}, \max\{\overline{X}, \overline{Y}\}]$

Interval hull of intervals : $X \underline{\cup} Y = [\min\{\underline{X}, \underline{Y}\}, \max\{\overline{X}, \overline{Y}\}]$

Intersection plays a key role in interval analysis. If we have two intervals containing a result of interest, regardless of how they were obtained, then the intersection, which may be narrower, also contains the result.

In general, the union of two intervals is not an interval. However, the interval hull of two intervals is always an interval and can be used in interval computations. We have: $X \cup Y \subseteq X \underline{\cup} Y$

The width of an interval X is defined and denoted by: $\omega(X) = \overline{X} - \underline{X}$

The absolute value of X, denoted by $|X|$, and is the maximum of the absolute values of its endpoints: $|X| = \max\{|\underline{X}|, |\overline{X}|\}$ note that $|x| \le |X|, \forall x \in X$

The midpoint of X is given by: $m(X) = \frac{1}{2}(\underline{X} + \overline{X})$

Order Relations for Intervals: $X < Y$ means that $\overline{X} < \underline{Y}$

Transitive order relation for intervals is set inclusion: $X \subseteq Y \Leftrightarrow \underline{Y} \le \underline{X}$ and $\overline{X} \le \overline{Y}$

The sum of two intervals : $X + Y = \{x + y : x \in X, y \in Y\}$

Subtraction of two intervals : $X - Y = \{x - y : x \in X, y \in Y\}$

Product of two intervals : $X \cdot Y = \{xy : x \in X, y \in Y\}$

Quotient of two intervals : $X / Y = \{x / y : x \in X, y \in Y\}$.

α-cut of an octagonal fuzzy number

The α-cut of an normal octagonal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ for $\alpha \in [0,1]$ is given by

$$[\tilde{A}]_\alpha = \begin{cases} [a_1 + \left(\frac{\alpha}{k}\right)(a_2 - a_1), [a_8 - \left(\frac{\alpha}{k}\right)(a_8 - a_7)], \alpha \in [0, k] \\ [a_3 + \left(\frac{\alpha - k}{1-k}\right)(a_4 - a_3), a_6 - \left(\frac{\alpha - k}{1-k}\right)(a_6 - a_5)], \alpha \in (k, 1] \end{cases}$$

4. Algorithm

Step 1: formulate the given problem in the form of octagonal fuzzy number.

Step 2: convert the octagonal fuzzy numbers into interval numbers using α-cut method to get the interval matrix.

Step 3: For each row $\{1, 2, \ldots, m\}$ find the entry g_{ij} that is less than or equal to all other entries in the i^{th} row.

Step 4: For each column $\{1,2,......,n\}$ find the entry g_{ij} that is greater than or equal to all other entries in the j^{th} column.

Step 5: Determine if there is an entry g_{ij} that is simultaneously the minimum of the i^{th} row and the maximum of the j^{th} column which is known as saddle point and value of saddle interval is known as value of game, if any of the above values cannot be found then saddle point does not exist. Then we go for dominance principle.

Step 6: if all the elements of the i^{th} row are less than or equal to the corresponding elements of j^{th} row then j^{th} row will be dominated row.

Step 7: if all the elements of k^{th} column are greater than or equal to the corresponding elements of r^{th} column then k^{th} column will be dominated column.

Step 8: dominated row or column can be deleted to reduce the size of payoff matrix as optimal strategies will remain unaffected. A given strategy can also said to be dominated if it is inferior to an average of two or more other pure strategies. More generally if some convex linear combination of some rows dominates the i^{th} row then i^{th} row will be deleted. Similar arguments follow for columns.

Step 9: using the above arguments matrix can be reduced to a simple matrix for game value can be evaluated easily.

Step 10: if there is no saddle point and dominance principle also failed then we define fuzzy membership of an interval being a minimum and a maximum of an interval vector and then we define the notion of a least and greatest interval in□ as defined in next step.

Step 11: The binary fuzzy operator $\circ$ of two intervals X and Y returns a real number between 0 and 1 as follows:

$$X \circ Y = \begin{cases} 1, X = Y; \overline{X} \le \underline{Y}, X \ne Y; \underline{X} < \underline{Y} < \overline{X} < \overline{Y} \\ 0, \overline{Y} \le \underline{X}, X \ne Y; \underline{Y} < \underline{X} < \overline{Y} < \overline{X} \\ \dfrac{\overline{Y} - \overline{X}}{\omega(Y) - \omega(X)}, \underline{Y} \le \underline{X} \le \overline{X} \le \overline{Y}, \omega(Y) > 0, X \ne Y \\ \dfrac{\underline{Y} - \underline{X}}{\omega(X) - \omega(Y)}, \underline{X} \le \underline{Y} \le \overline{Y} \le \overline{X}, \omega(X) > 0, X \ne Y \end{cases}$$

The binary fuzzy operator $\pm$ of two intervals X and Y is defined as,
$X \pm Y = 1$ If $X = Y$ and $X \pm Y = 1 - (X \circ Y)$ otherwise, i.e.

$$X \pm Y = \begin{cases} 1, X = Y; \overline{Y} \le \underline{X}, X \ne Y \\ 0, \overline{X} \le \underline{Y}, X \ne Y; \underline{X} < \underline{Y} < \overline{X} < \overline{Y}; \underline{Y} < \underline{X} < \overline{Y} < \overline{X} \\ \dfrac{\underline{X} - \underline{Y}}{\omega(Y) - \omega(X)}, \underline{Y} \le \underline{X} \le \overline{X} \le \overline{Y}, \omega(Y) > 0, X \ne Y \\ \dfrac{\overline{X} - \overline{Y}}{\omega(X) - \omega(Y)}, \underline{X} \le \underline{Y} \le \overline{Y} \le \overline{X}, \omega(X) > 0, X \ne Y \end{cases}$$

5. Numerical example

If pay off matrix of a game is given as following

$$\begin{pmatrix} (0,1,2,3,4,5,6,7) & (8,9,10,11,12,13,14,15) & (4,5,6,7,8,9,10,11) \\ (2,3,4,5,6,7,8,9) & (11,12,14,15,16,17,18,21) & (5,6,8,9,10,11,12,15) \\ (3,6,7,8,9,10,12,13) & (5,6,8,10,12,13,15,17) & (1,3,5,6,7,8,10,12) \end{pmatrix}$$

Now we will use α-cut method to convert given octagonal fuzzy numbers into intervals. Using,

$$[\tilde{A}]_\alpha = \begin{cases} [a_1 + \left(\dfrac{\alpha}{k}\right)(a_2 - a_1), [a_8 - \left(\dfrac{\alpha}{k}\right)(a_8 - a_7)], \alpha \in [0, k] \\ [a_3 + \left(\dfrac{\alpha - k}{1-k}\right)(a_4 - a_3), a_6 - \left(\dfrac{\alpha - k}{1-k}\right)(a_6 - a_5)], \alpha \in (k, 1] \end{cases}$$

For this problem taking k=0.5 and α=1, we get following,

Octagonal fuzzy number	Corresponding interval number
(0, 1, 2, 3, 4, 5, 6, 7)	[3, 4]
(8, 9, 10, 11, 12, 13, 14, 15)	[11, 12]
(4, 5, 6, 7, 8, 9, 10, 11)	[7, 8]
(2, 3, 4, 5, 6, 7, 8, 9)	[5,6]
(11, 12, 14, 15, 16, 17, 18, 21)	[15, 16]
(5, 6, 8, 9, 10, 11, 12, 15)	[9, 10]
(3, 6, 7, 8, 9, 10, 12, 13)	[8, 9]
(5, 6, 8, 10, 12, 13, 15, 17)	[10, 12]
(1, 3, 5, 6, 7, 8, 10, 12)	[6,7]

Hence we get the following interval matrix,

$$\begin{pmatrix} [3,4] & [11,12] & [7,8] \\ [5,6] & [15,16] & [9,10] \\ [8,9] & [10,12] & [6,7] \end{pmatrix}$$

First of all we check weather saddle interval exist or not for this,

$$\min\{[3,4],[11,12],7,8]\}=[3,4], \min\{[5,6],[15,16],[9,10]\}=[5,6], \min\{[8,9],[10,12],[6,7]\}=[6,7]$$

$$\max\{[3,4],[5,6],[8,9]\}=[8,9], \max\{[11,12],[15,16],[10,12]\}=[15,16], \max\{[7,8],[9,10],[6,7]\}=[9,10]$$

From above discussion saddle interval does not exist.

Since all the interval numbers of first row are less than or equal to the interval numbers of second row hence first row is dominated by second row. So we can delete first row.

$$\begin{pmatrix} [5,6] & [15,16] & [9,10] \\ [8,9] & [10,12] & [6,7] \end{pmatrix}$$

Now all the interval numbers of second column are greater than or equal to the interval numbers of first column hence second column is dominated by first column. So we can delete second column. Remaining matrix will be,

$$\begin{pmatrix} [5,6] & [9,10] \\ [8,9] & [6,7] \end{pmatrix}$$

Now firstly we are looking for least intervals,

$$\min\{[5,6]\prec[9,10],[5,6]\prec[8,9],[5,6]\prec[6,7]\}=\min\{1,1,1\}=1$$

$$\min\{[9,10]\prec[5,6],[9,10]\prec[8,9],[9,10]\prec[6,7]\}=\min\{0,0,0\}=0$$

$$\min\{[8,9]\prec[5,6],[8,9]\prec[9,10],[8,9]\prec[6,7]\}=\min\{0,1,0\}=0$$

$$\min\{[6,7]\prec[5,6],[6,7]\prec[9,10],[6,7]\prec[8,9]\}=\min\{0,1,1\}=0$$

Hence $\max\{1,0,0,0\}=1$, is corresponding to interval [5, 6].

For greatest intervals,

$$\max\{[5,6]\succ[9,10],[5,6]\succ[8,9],[5,6]\succ[6,7]\}=\max\{0,0,0\}=0$$

$$\max\{[9,10]\succ[5,6],[9,10]\succ[8,9],[9,10]\succ[6,7]\}=\max\{1,1,1\}=1$$

$$\max\{[8,9]\succ[5,6],[8,9]\succ[9,10],[8,9]\succ[6,7]\}=\max\{1,0,1\}=1$$

$$\max\{[6,7]\succ[5,6],[6,7]\succ[9,10],[6,7]\succ[8,9]\}=\max\{1,0,0\}=1$$

Hence $\min\{0,1,1,1\}=0$, is corresponding to interval [5, 6]. So from minimax theorem [5, 6] is saddle interval.

6. Conclusion

In the above discussion an algorithm for solving the games having payoffs in form of octagonal fuzzy numbers is suggested. A numerical example is given to clarify the proposed algorithm. Many problems in the present scenarios having competitive situations require best strategy for the agents. These competitive situations may be seen in business, military battles, sports, elections, advertising, marketing and different cases of conflict.

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