Common Fixed Point Theorems for Weakly Compatible Mapping Satisfying Generalized Contraction Principle in Complete G-Metric Spaces

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Abstract. In this paper, we study some common fixed point results for weakly compatible mapping satisfying Generalized Contraction Principle in G-metric space by using a control function.

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1. Introduction

Some generalizations of the notion of a metric space have been proposed by some authors. Gahler [1,2] coined the term of 2-metric spaces. This is extended to D-metric space by Dhage (1992) [3, 4]. Dhage proved many fixed point theorems in D-metric space. In 2006, Mustafa in collaboration with Sims introduced a new notion of generalized metric space called G-metric space [5]. In fact, Mustafa et al. studied many fixed point results for a self mapping in G-metric spaces under certain conditions; see [5, 6, 7, 8, 9].

2. Definitions and preliminaries

Definition 2.1. (Altering Distance Function [see 10]) A mapping $f: [0, \infty) \to [0, \infty)$ is called an Altering Distance Function if the following properties are satisfied.

(a) $f$ is continuous and non-decreasing.

(b) $f(t) = 0$ if and only if $t = 0$.

Definition 2.2. (Control Function [see 10]) A Control Function $\phi$ is defined as $\phi: R^+ \to R^+$ which is continuous at zero, monotonically increasing and $\phi(t) = 0$ if and only if $t = 0$.

Definition 2.3. [5] Let $X$ be a non empty set, and let $G: X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms

$(G1) \ G(x, y, z) = 0$ if $x = y = z$,
(G2) $G(x, x, y) > 0$ for all $x, y \in X$, with $x \neq y$.
(63) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$.
(64) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables)
(65) $G(x, y, z) \leq G(x, a, a) + G(a, y, x)$, for all $x, y, z, a \in X$ (rectangular inequality)

Then the function $G$ is called a generalized metric, or more specially a $G$-metric on $X$, and the pair $(X, G)$ is called a $G$-metric space.

**Example 1.1.** Let $(X, d)$ be a usual metric space. Then $(X, G_{\delta})$ and $(X, G_{m})$ are $G$-metric spaces, where

$$G_{\delta}(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in X$ and

$$G_{m}(x, y, z) = \max \{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in X$.

**Definition 2.4.** [5] Let $(X, G)$ and $(X', G')$ be $G$-metric spaces and let $f: (X, G) \to (X', G')$ be a function, then $f$ is said to be $G$-continuous at a point $a \in X$ if given $\varepsilon > 0$ there exist $\delta > 0$ such that $x, y \in X, G(a, x, y) < \delta$ implies that $G'(fa, fx, fy) < \varepsilon$. A function $f$ is $G$-continuous on $X$ if and only if it is $G$-continuous at all $a \in X$.

**Definition 2.5.** [5] Let $(X, G)$ be a $G$-metric space, and let $\{x_n\}$ be a sequence of points of $X$, then we say that $\{x_n\}$ is $G$-convergent to $x$ if $\lim_{n,m \to \infty} G(x, x_n, x_m) = 0$: that is, for any $\varepsilon > 0$, there exist $N \in N$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq N$. We call $x$ the limit of the sequence $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = x$.

**Proposition 2.6.** [5] Let $(X, G)$ and $(X', G')$ be metric spaces, then a function $f: X \to X'$ is said to be $G$-continuous at a point $x \in X$ if and only if it is $G$-sequentially continuous, that is, whenever $\{x_n\}$ is $G$-convergent to $x$, $\{fx_n\}$ is $G'$-convergent to $f(x)$.

**Proposition 2.7.** [5] Let $(X, G)$ be a $G$-metric space. Then the following statements are equivalent

(a) $\{x_n\}$ is $G$-convergent to $x$.
(b) $G(x_n, x, x) \to 0$ as $n \to \infty$.
(c) $G(x_n, x, x) \to 0$ as $n \to \infty$.
(d) $G(x_n, x_m, x) \to 0$ as $n \to \infty$.

**Proposition 2.8.** [5] Let $(X, G)$ be a $G$-metric space. A sequence $\{x_n\}$ is called $G$-cauchy sequence if given $\varepsilon > 0$, there is $N \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$; that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

**Proposition 2.9.** [5] In a $G$-metric space $(X, G)$, the following two statements are equivalent.

(1) The sequence $\{x_n\}$ is $G$-cauchy.
(2) For every $\varepsilon > 0$, there exist $N \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m \geq N$.
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Definition 2.10. [5] A $G$-metric space $(X, G)$ is said to be $G$-complete (or a complete $G$-metric space) if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent in $(X, G)$.

Proposition 2.11. [5] Let $(X, G)$ be a $G$-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 2.12. [5] A $G$-metric space $(X, G)$ is called a symmetric $G$-metric space if $G(x, y, z) = G(y, x, z)$ for all $x, y, z \in X$.

Proposition 2.13. [5] Every $G$-metric space $(X, G)$ defines a metric space $(X, d_G)$ by $d_G(x, y, z) = G(x, y, z)$ for all $x, y, z \in X$.

Note that, if $(X, G)$ is a symmetric space $G$-metric space, then

$\frac{2}{3} G(x, y, z) \leq d_G(x, y, z) \leq 3 G(x, y, z)$

for all $x, y \in X$.

In general, these inequalities cannot be improved.

Proposition 2.14. [5] A $G$-metric space $(X, G)$ is $G$-complete if and only if $(X, d_G)$ is a complete metric space.

Definition 2.16. Two self maps $T$ and $f$ of a G-Metric Space $(X, G)$ are said to be weakly compatible if $Tf(x) = fTx$ whenever $fx = Tx$ for all $x \in X$.

Definition 2.17. Let $T$ and $f$ be two self maps of a non empty subset $M$ of a metric space $X$. The mapping $T$ is called $f$-contraction mapping, if there exist a real number $0 \leq k < 1$ such that $G(Tx, Ty, Tz) \leq k G(fx, fy, fz)$ for all $x, y, z \in M$.

Definition 2.18. A mapping $T : X \rightarrow X$, where $(X, G)$ is a $G$-metric space, is said to be a weak contraction if

$G(Tx, Ty, Tz) \leq G(x, y, z) - \Phi(G(x, y, z))$

where $x, y, z \in X$ and $\Phi: [0, \infty) \rightarrow [0, \infty)$ is continuous and non-decreasing function such that $\Phi(t) = 0$ if and only if $t = 0$.

Theorem 2.19. [11] Let $(X, G)$ be a complete G-metric space and $T : X \rightarrow X$ be a mapping satisfying

$G(Tx, Ty, Tz) \leq G(x, y, z) - \Phi(G(x, y, z))$, 

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for all \( x, y, z \in X \). If \( \emptyset \colon [0, \infty) \to [0, \infty) \) is a continuous and non decreasing function with 
\( \emptyset(t) = 0 \) if and only if \( t = 0 \), then \( T \) has a unique fixed point in \( X \).

**Definition 2.20.** A self mapping \( T \) of a metric space \((X, G)\) is said to be \textbf{Weakly Contractive with respect to a self mapping} \( f \colon X \to X \) if for all \( x, y, z \in X \)
\[
G(Tx, Ty, Tz) \leq G(fx, fy, fz) - \emptyset(G(fx, fy, fz)).
\]
where \( \emptyset \colon [0, \infty) \to [0, \infty) \) is a continuous and non-decreasing function such that \( \emptyset \) is positive on \((0, \infty) \), \( \emptyset(0) = 0 \), \( \lim_{t \to \infty} \emptyset(t) = \infty \).

**Note 2.1.** If \( I \), the identity mapping, then the above definition is as follows. A self mapping \( T \) of a metric space \((X, G)\) is said to be Weakly Contractive with respect to a self mapping \( f \colon X \to X \) if for all \( x, y, z \in X \)
\[
G(Tx, Ty, Tz) \leq G(x, y, z) - \emptyset(G(x, y, z)).
\]
This is a \textbf{Weakly Contractive Mapping}.

**Note 2.2.** Combining the generalization of Contraction Principle and Weakly Contractive Mapping with respect to a self map in G-Metric Space we can obtain the following result.

**Theorem 2.21.** Let \((X, G)\) be a complete G-Metric Space and a self map \( T \colon X \to X \) be weakly contractive mapping with respect to a self mapping \( f \colon X \to X \) if for all \( x, y, z \in X \) and \( T \colon X \to X \) is satisfying
\[
\varphi(G(Tx, Ty, Tz)) \leq \varphi(G(fx, fy, fz)) - \emptyset(G(fx, fy, fz))
\]
where \( \emptyset \colon [0, \infty) \to [0, \infty) \), \( \varphi \colon [0, \infty) \to [0, \infty) \) are continuous and monotone non-decreasing functions with \( \varphi(t) = 0 = \emptyset(t) \) if and only if \( t = 0 \), then \( T \) has a unique fixed point.

**Theorem 2.22.** [see 12] Let \( T \) and \( f \) be self maps of a G - metric space \((X, G)\) satisfying
\[
\varphi(d(Tx, Ty)) \leq \varphi(M(x, y)) - \emptyset(M(x, y))
\]
for all \( x, y \in X \) where
\[
M(x, y) = \max \{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{\xi}(d(fy, Tx) + d(fx, Ty))\}
\]
and \( \emptyset, \varphi \colon [0, \infty) \to [0, \infty) \) are both continuous monotone non-decreasing functions with
\[
\varphi(t) = 0 = \emptyset(t) \text{ if and only if } t = 0.
\]
If \( TX \) is complete metric space and \( TX \subset fx \), then \( T \) and \( f \) have coincidence point in \( X \). Further, if \( T \) and \( f \) are weakly compatible, then they have a unique common fixed point in \( X \).

Motivated by the above result, we address the same question on G -metric space for weakly compatible mappings satisfying a Generalized Contraction Principle condition given by (1), we establish a fixed point results in the third part of the paper. Our results are the following.

3. \textbf{Main results}

**Theorem 3.1:** let \( T \) and \( f \) be self maps of a complete G -metric space \((X, G)\) satisfying
\[
\varphi(G(Tx, Ty, Tz)) \leq \varphi(M(x, y, z)) - \emptyset(M(x, y, z))
\]
for all \( x, y, z \in X \) where
\[
M(x, y, z) = \max \{G(fx, fy, fz), G(fx, Tx, Tx), G(fy, Ty, Ty), G(fz, Tz, Tz),
\]
\[
\frac{1}{\xi} (G(fy, Tx, Tx) + G(fx, Ty, Ty)) + \frac{1}{\xi} (G(fz, Ty, Ty) + G(fy, Tz, Tz))
\]

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\[ (G(fx, Tx, Tz) + G(fz, Tx, Tx)) \]

(3)

and \( \emptyset, \phi: [0, \infty) \to [0, \infty) \) are both continuous monotone non-decreasing functions with \( \phi(t) = 0 = \emptyset(t) \) if and only if \( t = 0 \). If \( TX \) is complete metric space and \( TX \subset fX \), then \( T \) and \( f \) have coincidence point in \( X \). Further, if \( T \) and \( f \) are weakly compatible, then they have a unique common fixed point in \( X \).

**Proof:** let \( x_0 \) be an arbitrary point. Construct the sequence \( \{x_n\} \) such that

\[ f x_n = T x_{n-1} \text{ for } n = 1, 2, 3, \ldots \ldots \ldots \]

(4)

this is possible since \( TX \subset fX \).

Now

\[ \phi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \phi(M(x_n, x_{n+1}, x_{n+1})) - \emptyset(M(x_n, x_{n+1}, x_{n+1})) \]  

(5)

where

\[ M(x_n, x_{n+1}, x_{n+1}) = \max \{ G(fx_n, x_{n+1}, f x_{n+1}), G(fx_n, Tx_n, Tx_n), G(f x_{n+1}, Tx_{n+1}, Tx_{n+1}) \} \]

\[ \leq \frac{1}{3} G(fx_n, x_{n+1}, f x_{n+1}) + \frac{1}{3} G(fx_n, Tx_n, Tx_n) + \frac{1}{3} G(f x_{n+1}, Tx_{n+1}, Tx_{n+1}) \]

(6)

From (5) and (6), we have

\[ \phi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \phi(M(x_n, x_{n+1}, x_{n+1})) \leq \phi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \]

(7)

This implies

\[ \phi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \phi(G(Tx_n, Tx_n, Tx_n)) - \emptyset(G(Tx_n, Tx_n, Tx_n)) \]

\[ \phi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \phi(G(Tx_n, Tx_n, Tx_n)) \]

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By monotone property of the function $\varphi$, we have
$$G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq G(Tx_{n-1}, Tx_{n}, Tx_{n}) \quad \text{for } n = 1, 2, 3 \ldots$$
Therefore the sequence $\{G(Tx_n, Tx_{n+1}, Tx_{n+1})\}$ is monotonic decreasing and continuous.

Therefore there exist a real number $r \geq 0$ such that
$$\lim_{n \to \infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = r \quad (8)$$
Taking $n \to \infty$ in equation (7), we get
$$\varphi(r) \leq \varphi(r) - \varphi(r)$$
This is possible only when $r = 0$.

Therefore $\lim_{n \to \infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0 \quad (9)$
Next, we claim that $\{Tx_n\}$ is a Cauchy sequence.

Assume that $\{Tx_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and subsequences $\{n(i)\}, \{m(i)\}$ such that $m(i) < n(i) < m(i + 1)$ along with
$$G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) \geq \varepsilon \quad \text{and} \quad G(Tx_{m(i)}, Tx_{n(i)-1}, Tx_{n(i)-1}) < \varepsilon \quad (10)$$
Then it follows that
$$\varepsilon \leq G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) \leq G(Tx_{m(i)}, Tx_{n(i)-1}, Tx_{n(i)-1}) + G(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)})$$
$$\varepsilon \leq G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) \leq \varepsilon + G(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}) \quad (11)$$
Let $i \to \infty$ and using (9) in (11)
$$\varepsilon \leq \lim_{i \to \infty} G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) \leq \varepsilon + \lim_{i \to \infty} G(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)})$$
$$\varepsilon \leq \lim_{i \to \infty} G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) \leq \varepsilon$$
Therefore $\lim_{i \to \infty} G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) = \varepsilon \quad (12)$

Now
$$G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) \leq G(Tx_{m(i)}, Tx_{m(i)-1}, Tx_{m(i)-1}) + G(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)})$$
$$\quad + G(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)})$$
$$G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) \leq 2 \quad (13)$$

(Lessing $i \to \infty$ in (13)
$$\varepsilon \leq 2(0) + \lim_{i \to \infty} G(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1})$$
$$\varepsilon \leq \lim_{i \to \infty} G(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}) \quad (14)$$
Again
$$G(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}) \leq G(Tx_{m(i)-1}, Tx_{m(i)}, Tx_{m(i)})$$
$$\quad + G(Tx_{m(i)}, Tx_{n(i)-1}, Tx_{n(i)-1}) + G(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)})$$
$$G(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}) \leq G(Tx_{m(i)-1}, Tx_{m(i)}, Tx_{m(i)})$$
$$\quad + G(Tx_{m(i)}, Tx_{n(i)-1}, Tx_{n(i)-1}) + 2 G(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}) \quad (15)$$
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(Since \( G(x, y, z) \leq 2G(y, x, x) \))

Letting \( i \to \infty \) in (15)

\[
\lim_{i \to \infty} G(Tx_m(i-1), Tx_n(i-1), Tx_n(i-1)) \leq 0 + \varepsilon + 2(0)
\]

\[
\lim_{i \to \infty} G(Tx_m(i-1), Tx_n(i-1), Tx_n(i-1)) \leq \varepsilon
\]

(16)

From (14) & (16)

\[
\varepsilon \leq \lim_{i \to \infty} G(Tx_m(i-1), Tx_n(i-1), Tx_n(i-1)) \leq \varepsilon
\]

\[
\lim_{i \to \infty} G(Tx_m(i-1), Tx_n(i-1), Tx_n(i-1)) = \varepsilon
\]

(17)

Now using inequalities (2) and (10)

\[
\phi(\varepsilon) \leq \phi \left( G(Tx_m(i), Tx_n(i), Tx_n(i)) \right) \leq 
\]

\[
\phi(M(x_m(i), x_n(i), x_n(i))) - \phi(M(x_m(i), x_n(i), x_n(i)))
\]

(18)

where

\[
M(x_m(i), x_n(i), x_n(i)) = \max \{ G(f x_m(i), f x_n(i), f x_n(i)), G(f x_m(i), Tx_m(i), Tx_m(i)), G(f x_n(i), Tx_n(i), Tx_n(i)), \}
\]

\[
\frac{1}{3} \{ G(Tx_n(i-1), Tx_m(i), Tx_m(i)) + G(Tx_n(i-1), Tx_n(i), Tx_n(i)), \}
\]

\[
\frac{1}{3} \{ G(Tx_n(i-1), Tx_m(i), Tx_m(i)) + G(Tx_n(i-1), Tx_n(i), Tx_n(i)), \}
\]

\[
\frac{1}{3} \{ G(Tx_n(i-1), Tx_m(i), Tx_m(i)) + G(Tx_n(i-1), Tx_m(i), Tx_m(i)) \}
\]

Taking \( i \to \infty \) on both sides in above equation, we obtain

\[
\lim_{i \to \infty} M(x_m(i), x_n(i), x_n(i)) = \max \{ \varepsilon, 0, 0, \leq \varepsilon \}
\]

Therefore

\[
\lim_{i \to \infty} M(x_m(i), x_n(i), x_n(i)) = \varepsilon
\]

(19)
Letting \( i \to \infty \) in (18) and using (19) in that, then we obtain
\[
\varphi(x) \leq \varphi(x) - \emptyset(x)
\]
which is a contradiction, has \( \varepsilon > 0 \). Thus \( \{Tx_n\} \) is a Cauchy Sequence in \( TX \) which in turn implies that \( \{fx_n\} \) is also Cauchy Sequence in \( X \). Since \( TX \) is complete, \( \{Tx_n\} \) converges to some \( v \in TX \).

Since \( TX \subset X \) and \( v = fu \) for some \( u \in X \), thus \( \{fx_n\} \) converges to \( fu \).

Now
\[
\lim_{n \to \infty} \varphi(G(Tx_n, Tu, Tu)) \leq \lim_{n \to \infty} [\varphi(M(x_n, u, u) - \emptyset(M(x_n, u, u))]
\]
where
\[
\lim_{n \to \infty} M(x_n, u, u) = \lim_{n \to \infty} \max \{G(x_n, fu, fu), G(fu, Tu, Tu) G(fu, Tu, Tu), \frac{1}{3} \{G(fu, Tu, Tu) + G(fu, Tu, Tu) + G(fu, Tu, Tu)\},
\]
\[
\frac{1}{3} \{G(v, Tu, Tu) + G(v, Tu, Tu) + G(v, Tu, Tu)\}, \frac{1}{3} \{G(v, Tu, Tu) + G(v, Tu, Tu) + G(v, Tu, Tu)\},
\]
\[
M(x_n, u, u) = \max \{0,0, G(v, Tu, Tu), G(v, Tu, Tu), \frac{1}{3} \{G(v, Tu, Tu) + G(v, Tu, Tu) + G(v, Tu, Tu)\},
\]
\[
\frac{1}{3} \{G(v, Tu, Tu) + G(v, Tu, Tu) + G(v, Tu, Tu)\}, \frac{1}{3} \{G(v, Tu, Tu) + G(v, Tu, Tu) + G(v, Tu, Tu)\}.
\]

Therefore \( M(x_n, u, u) = G(v, Tu, Tu) \).

By monotone increasing property of \( \varphi \) & \( \emptyset \), we have
\[
\varphi(G(v, Tu, Tu)) \leq \varphi(G(v, Tu, Tu)) - \emptyset(G(v, Tu, Tu))
\]
which is possible only when\( G(v, Tu, Tu) = 0 \).

Thus \( v = Tu = fu \) and \( u \) is the coincidence point of \( T \) and \( f \).

Since \( T \) and \( f \) are weekly compatible, they commute at their coincidence point.

Hence \( Tf u = fT u \) which implies \( Tv = f v \).

\[
(20)
\]

Now
\[
\varphi(G(Tu, Tv, Tv)) \leq \varphi(M(u, v, v)) - \emptyset(M(u, v, v)),
\]
where
\[
M(u, v, v) = \max \{G(v, v, f), G(fu, Tu, Tu), G(fu, Tu, Tu), G(v, Tu, Tu), G(v, Tu, Tu), \frac{1}{3} \{G(fu, Tu, Tu) + G(fu, Tu, Tu) + G(fu, Tu, Tu)\},
\]
\[
\frac{1}{3} \{G(fu, Tu, Tu) + G(fu, Tu, Tu) + G(fu, Tu, Tu)\}, \frac{1}{3} \{G(fu, Tu, Tu) + G(fu, Tu, Tu) + G(fu, Tu, Tu)\}.
\]

\[
M(u, v, v) = \max \{G(v, v, f), G(Tv, Tv, Tv), G(Tv, Tv, Tv), G(Tv, Tv, Tv), G(Tv, Tv, Tv), \frac{1}{3} \{G(Tv, v, v) + G(v, Tv, Tv)\}, \frac{1}{3} \{G(Tv, v, v) + G(v, Tv, Tv)\},
\]
\[
\frac{1}{3} \{G(Tv, v, v) + G(v, Tv, Tv)\}.
\]

\[
M(u, v, v) = \max \{G(v, v, f), 0,0,0, \frac{1}{3} \{G(Tv, v, v) + G(v, Tv, Tv)\}, 0,
\]
\[
\frac{1}{3} \{G(v, v, f) + G(Tv, v, v)\}.
\]

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\[ M(u,v,v) = \max\{G(v,Tv,v), \frac{1}{3}(G(Tv,v,v) + G(v,Tv,v))\} \]

\[ M(u,v,v) = G(v,Tv,Tv). \]  

(22)

Since \( \frac{1}{3}\{G(v,Tv,Tv) + G(Tv,v,v)\} \leq \frac{1}{3}\{2G(v,Tv,Tv) + G(v,Tv,Tv)\} \)

\[ \frac{1}{3}\{G(v,Tv,Tv) + G(Tv,v,v)\} \leq G(v,Tv,Tv). \]

Hence by using (22) in (23), we get

\[ \varphi(G(v,Tv,Tv) = \varphi(G(Tv,Tv,Tv) \leq \varphi(G(v,Tv,Tv)) - \varphi(G(Tv,Tv,Tv)) \]

\[ \varphi(G(v,Tv,Tv) \leq \varphi(G(v,Tv,Tv)) - \varphi(G(v,Tv,Tv)) \]

This implies \( \varphi(G(v,Tv,Tv) \leq 0 \)

which is possible only when \( G(v,Tv,Tv) = 0 \).

Therefore \( v = Tv \).

Thus \( v = Tv = f v. \) (from (20))

Hence \( v \) is the common fixed point of \( T \) and \( f \).

**Uniqueness:**

Let \( v \) and \( w \) be two fixed points of \( T \) and \( f \).

That is \( v = Tv = f v \) and \( w = Tw = f w \).

By using inequality (4), we have

\[ \varphi(G(Tv,Tw,Tw) \leq \varphi(M(v,w,w)) - \varphi(M(v,w,w)) \]  

(23)

where

\[ M(v,w,w) = \max\{G(v,fw,fw), G(fw,Tv,Tw), G(fw,Tw,Tw), G(fw,Tw,Tw), \]

\[ \frac{1}{3}\{G(fw,Tv,Tw) + G(fw,Tw,Tw)\}, \frac{1}{3}\{G(fw,Tw,Tw) + G(fw,Tw,Tw)\} \]

\[ \frac{1}{3}\{G(fw,Tw,Tw) + G(fw,Tw,Tw)\} \]

\[ M(v,w,w) = \max\{G(v,w,w), 0, 0, 0, \frac{1}{3}\{G(w,v,w) + G(v,w,w)\}, 0\} \]

(24)

\[ M(v,w,w) = \max\{G(v,w,w), 0, 0, 0, 0, G(w,v,v) + G(v,w,w)\} \]

\[ M(v,w,w) = G(v,w,w). \]

Since \( \frac{1}{3}\{G(w,v,v) + G(v,w,w)\} \leq \frac{1}{3}\{2G(v,w,w) + G(v,w,w)\} \)

\[ \frac{1}{3}\{G(w,v,v) + G(v,w,w)\} \leq G(v,w,w). \]

Hence by using (24) in (23), we get

\[ \varphi(v,w,w) = \varphi(G(Tv,Tw,Tw) \leq \varphi(G(v,w,w)) - \varphi(G(v,w,w)) \]

\[ \varphi(v,w,w) \leq \varphi(G(v,w,w)) - \varphi(G(v,w,w)) \]

\[ \varphi(G(v,w,w) \leq 0 \]

This is possible only when \( G(v,w,w) = 0 \).

Therefore \( v = w \)

This proves the uniqueness of the common fixed point of \( T \) and \( f \).

**Example 3.1.** Let \( X = [0,1] \) and \( d(x,y) = |x - y| \). Define

\( G(x,y,z) = |x - y| + |y - z| + |z - x| \), then \( (X,G) \) is a complete \( G \)-metric space.

Consider two self mappings \( T \) and \( f \) of \( X \) by \( Tx = \frac{x}{2} \) and \( fx = x \) for all \( x \in X \).
Let \( \varphi: [0, \infty) \rightarrow [0, \infty) \) be defined by
\[
\varphi(t) = \begin{cases} 
  t + \frac{t^2}{2} & \text{if } 0 \leq t \leq 1 \\
  0 & \text{if } t > 1
\end{cases}
\] (25)
and \( \emptyset: [0, \infty) \rightarrow [0, \infty) \) defined by
\[
\emptyset(t) = \begin{cases} 
  \frac{3t^2}{8} & \text{if } 0 \leq t \leq 1 \\
  0 & \text{if } t > 1
\end{cases}
\] (26)

Now to verify inequality (2), LHS of (2)
\[
\varphi(G(Tx, Ty, Tz)) = \varphi(|Tx - Ty| + |Ty - Tz| + |Tz - Tx|)
\]
\[
\varphi(G(Tx, Ty, Tz)) = \varphi \left( \frac{|x-y|}{2} + \frac{|y-z|}{2} + \frac{|z-x|}{2} \right),
\]
\[
\varphi(G(Tx, Ty, Tz)) = \varphi \left( \frac{|x-y|}{2} + \frac{|y-z|}{2} + \frac{|z-x|}{2} \right),
\]
\[
\varphi(G(Tx, Ty, Tz)) = \varphi \left( \frac{G(x, y, z)}{2} \right),
\]
\[
\varphi(G(Tx, Ty, Tz)) = \frac{G(x, y, z)}{2} + \frac{(G(x, y, z))^2}{8}.
\] (27)

Now to verify inequality (2), RHS of (2) is
\[
\varphi(M(x, y, z)) - \emptyset(M(x, y, z)),
\] (28)

where
\[
M(x, y, z) = \max \{ G(fx, fy, fz), G(fx, Tx, Tx), G(fy, Ty, Ty), G(fz, Tz, Tz),
\]
\[
\frac{1}{3} \left( G(fy, Tx, Tx) + G(fx, Ty, Ty) \right), \frac{1}{3} \left( G(fz, Ty, Ty) + G(fy, Tz, Tz) \right), \frac{1}{3} \left( G(fx, Tz, Tz) + G(fz, Tz, Tx) \right) \}
\]
\[
M(x, y, z) = \max \{ G(x, y, z), G \left( \frac{x}{2}, \frac{y}{2}, \frac{z}{2} \right), G \left( \frac{y}{2}, \frac{x}{2}, \frac{z}{2} \right), G \left( \frac{z}{2}, \frac{y}{2}, \frac{x}{2} \right) \} +
\]
\[
G \left( \frac{x}{2}, \frac{y}{2}, \frac{z}{2} \right)
\]
\[
\frac{1}{3} \left( G \left( z, \frac{y}{2}, \frac{z}{2} \right) + G \left( \frac{z}{2}, \frac{z}{2}, \frac{z}{2} \right) \right), \frac{1}{3} \left( G \left( \frac{z}{2}, \frac{y}{2}, \frac{z}{2} \right) + G \left( \frac{z}{2}, \frac{z}{2}, \frac{z}{2} \right) \right) \}.
\]

M(x, y, z) = \max \{ |x - y| + |y - z| + |z - x|, |x|, |y|, |z|, \frac{2}{3} \left( |x - y| + |y - z| + |z - x| \right) \}
\]
\[
M(x, y, z) = \max \{ |x - y| + |y - z| + |z - x| \} \text{ for all } x, y, z \in X,
\]
\[
M(x, y, z) = G(x, y, z) \text{ for all } x, y, z \in X.
\] (29)

Substitute (29) in (28), we obtain RHS of (2) is
\[
\varphi(G(x, y, z)) - \emptyset(G(x, y, z)),
\]

From (24) and (25), we obtain RHS of (2) is
\[
G(x, y, z) + \frac{(G(x, y, z))^2}{2} - \frac{3(G(x, y, z))^2}{8},
\]
\[
\text{RHS of (2) is } = G(x, y, z) + \frac{(G(x, y, z))^2}{8}.
\] (30)

From (26) and (29), we obtain
\[
\frac{G(x, y, z)}{2} + \frac{(G(x, y, z))^2}{8} \leq G(x, y, z) + \frac{(G(x, y, z))^2}{8}.
\]
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This implies LHS \( \leq \) RHS and inequality (2) is verified. Now, it is easy to see that
\[
TX = \left[0, \frac{1}{2}\right] \subset fX = [0,1].
\]
Moreover, \( T \) and \( f \) are weakly compatible in \( X \). Hence all the conditions of theorem 3.1 are satisfied. It may be noted that \( 0 \) is unique common fixed point of \( T \) and \( f \).

**Theorem 3.2.** let \( T \) and \( f \) be self maps of a \( G \)-metric space \((X, G)\) satisfying
\[
\varphi(G(Tx,Ty,Tz)) \leq k \varphi(M(x,y,z)) \text{ for all } x, y, z \in X
\]
where
\[
M(x, y, z) = \max \{G(fx, fy, fz), G(fx, Tx, Tx), G(fy, Ty, Ty), G(fz, Tz, Tz),
\]
\[
\frac{1}{3}(G(fy, Tx, Tx) + G(fx, Ty, Ty)) + \frac{1}{3}(G(fz, Ty, Ty) + G(fy, Tz, Tz)),
\]
\[
G(fy, Tz, Tz)\}
\]
and \( \varphi: [0, \infty) \to [0, \infty) \) is continuous monotone non-decreasing function with \( \varphi(t) = 0 \) if and only if \( t = 0 \). If \( TX \) is complete metric space and \( TX \subset fX \), then \( T \) and \( f \) have coincidence point in \( X \). Further, if \( T \) and \( f \) are weakly compatible, then they have a unique common fixed point in \( X \).

**Proof:** By taking \( \varphi(t) = (1 - k) \varphi(t) \) in theorem 3.1 then condition (2) reduced to the condition (32), and the proof follows the theorem (3.1).

**Theorem 3.3.** let \( T \) and \( f \) be self maps of a \( G \)-metric space \((X, G)\) satisfying
\[
G(Tx,Ty,Tz) \leq G(fx, fy, fz) - \varnothing(G(fx, fy, fz)) \text{ for all } x, y, z \in X
\]
and \( \varnothing: [0, \infty) \to [0, \infty) \) is continuous monotone non-decreasing function with \( \varnothing(t) = 0 \) if and only if \( t = 0 \). If \( TX \) is complete metric space and \( TX \subset fX \), then \( T \) and \( f \) have coincidence point in \( X \). Further, if \( T \) and \( f \) are weakly compatible, then they have a unique common fixed point in \( X \).

**Proof:** By taking \( \varphi(t) = t \) and \( M(x, y, z) = G(fx, fy, fz) \) in theorem 3.1, then condition (2) reduced to the condition (33), and the proof follows the theorem (3.1).

**Theorem 3.4.** Let \((X, G)\) be a complete \( G \)- metric space and \( T: X \to X \) be a mapping satisfying
\[
G(Tx,Ty,Tz) \leq G(x, y, z) - \varnothing(G(x, y, z)),
\]
for all \( x, y, z \in X \). If \( \varnothing: [0, \infty) \to [0, \infty) \) is a continuous and non decreasing function with \( \varnothing(t) = 0 \) if and only if \( t = 0 \), then \( T \) has a unique fixed point in \( X \).

**Proof:** By taking \( \varphi(t) = t \), \( M(x, y, z) = G(fx, fy, fz) \) and \( f = I, \text{(the identity function)} \) in theorem 3.1, then condition (2) reduced to the condition (34), and the proof follows the theorem (3.1).

**REFERENCES**