Annals of Pure and Applied Mathematics
Vol. 11, No. 2, 2016, 139-144
ISSN: 2279-087X (P), 2279-0888(online)
Published on 22 June 2016
www.researchmathsci.org

Common Fixed Point Theorem for Semi Compatible Pairs of Reciprocal Continuous Maps in Menger Spaces

***Preeti Malviya*[1], *Vandna Gupta*[2] and *V.H. Badshah*[3]**

[1] Govt. New Science College, Dewas (M.P.), India
Email: mpreeti2709@gmail.com
[2] Govt. Kalidas Girl's College, Ujjain (M.P.), India
Email: drvg1964@gmail.com
[3]School of Studies in Mathematics, Vikram University, Ujjain (M.P.), India.

Received 12 May 2016; accepted 3 June 2016

Abstract. The aim of this paper is to present some common fixed point theorem in Menger Space using the concept of semi compatible pairs of reciprocal continuous maps.

Keywords: Common fixed point, menger space, compatible maps, semi compatible maps, reciprocal continuous

AMS Mathematics Subject Classification (2010): 47H10, 54H25

1. Introduction

The idea of introducting probabilistic notion into geometry was one of the great thoughts of Menger. In 1942, Menger [1] has introduced the theory of probabilistic metric space.

In 1966 , Sehgal [2] initiated the study of contraction mapping theorem in PM-space. Altun and Turkoglu [3] proved two common fixed point theorems on complete PM- space with an implicit relation. Schweizer and Sklar [4] played major role in development of fixed point theory in PM - space .

In 1972, Sehgal and Bharucha- Reid [5] initiated the study of contraction mappings in the development of fixed point theorems. Singh et. al. [6] introduced the concept of weakly commuting mapping in PM- space. Kumar and Chugh [7] established some common fixed point theorem using the idea of reciprocal continuous of mappings.

Recently Al- Thagafi and Shahzad [8] weakned the notion of weakly compatible maps by introducing owc maps. Bouhadjera and Godet-thobie [9] introduced two new notions subsequential continuity and subcompability which are weaker than reciprocal continuity and compatibility respectively.

2. Preliminaries

Definition 2.1[10] A t-norm is a binary operation on the interval [0,1] such that for all a,b,c,d $\in$ [0,1] the following conditions are satisfied

(i). $a*1 = a$;
(ii). $a*b = b*a$;
(iii). $a*b \le c*d$, whenever $a \le c$ and $b \le d$;

(iv). a *(b*c) = (a*b)*c.

Definition 2.2. [10] A mapping $F : R \to R$, is called a distribution if it is non- decreasing left continuous with $\inf\{ F(t) : t \in R \} = 1$.

Definition 2.3. [10] A mapping $t: [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it is satisfies the following conditions :

(i) t is commutative and associative ;
(ii) $t(a,1) = a$, for all $a \in [0,1]$;
(iii) $t(a,b) \leq t(c,d)$, for $a \leq c$ and $b \leq d$.

Definition 2.4.[10] A Probabilistic metric space is an ordered pair (X, F) consisting of a non empty set X and a function $F : X\times X \to L$, where L is the collection of all distribution functions and the value of F at $(u,v) \in X\times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ is assumed to satisfy the following conditions;

(i) $F_{u,v}(x) = 1$, for all $x> 0$ if and only if $u = v$,
(ii) $F_{u,v}(0) = 0$,
(iii) $F_{u,v} = F_{v,u}$,
(iv) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$, then $F_{u,w}(x + y) = 1$,for all u,v,w in X ,x,y > 0.

Example 2.1. Let $X = [0,\infty)$ and d be the usual metric on X and for each $t \in [0,1]$, define

$$F_{x,y}(t) = \begin{cases} \frac{t}{\{t+|x-y|\}}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}$$

for all $x,y \in X$. Clearly (X,F,t) be a Menger space , where t- norm is defined by $t(c,d) = \min\{c,d\}$, for all $a,b \in [0,1]$.

Definition 2.5. [10] A sequence $\{x_n\}$ in a Menger space (X,F,t) is said to be converges to a point x in X if and only if for each $\varepsilon > 0$ and $t > 0$, there is an integer $M(\varepsilon) \in N$ such that $F_{x_n,x_m}(\varepsilon) > 1- t$, for all $n, m \geq M(\varepsilon)$.

Definition 2.6. [10] A Menger PM-space (X,F,t) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 2.7. [11] Self mappings P and S of a Menger space (X,F,t) are said to be compatible if $F_{Psx_n,SPx_n}(x) \to 1$, for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $PSx_n, SPx_n \to u$, for some u in X , as $n\to \infty$.

Definition 2.8. [12] Two maps P and S are said to be weakly compatible if they commute at a coincidence point.

Definition 2.9. [13] Two self maps P and S of a Menger space (X,F,t) are said to be reciprocally continuous if $PSx_n \to Pz$ and $SPx_n \to Sz$, Whenever $\{x_n\}$ is a sequence in X such that $Px_n , Sx_n \to z$, for some z in X as $n \to \infty$.

Definition 2.10. [14] Two self maps P and S of a Menger space (X,F,t) are said to be semi compatible if $F_{PSx_n,Tu}(x) \to 1$, for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Px_n, Sx_n \to u$ for some u in X as $n \to \infty$.

Lemma 2.1. [15] Let (X,F,*) be a Menger space with continuous t- norm * , if there exists a constant $h \in (0,1)$ such that $F_{x,y}(ht) \geq F_{x,y}(t)$, for all $x,y \in X$, and $t > 0$ then $x = y$.

Example 1.1. Let M = [2, 20] and d be usual metric on M. Define mappings P, S : M → M by

$$Pv = \begin{cases} 2, \text{ if } v = 2 \\ 3, \text{ if } v > 2 \end{cases} \quad \text{and} \quad Sv = \begin{cases} 2, \text{ if } v = 2 \\ 6. \text{ if } v > 2 \end{cases}$$

It is noted that P and S are reciprocally continuous mappings but they are not continuous.

Lemma 2.2. [15] Let $\{x_n\}$ be a sequence in a Menger space (X,F,t), where t is continuous and satisfies $t(x,y) \geq x$, for all $x \in [0,1]$. If there exists a constant $k \in (0,1)$ such that

$F_{u_n,u_{n+1}}(kx) \geq F_{u_{n-1},u_n}(x)$, n= 1,2,3…

then $\{x_n\}$ is a Cauchy sequence in X.

3. Main result

Theorem 3.1. Let P, Q, S and T be self mappings on a complete Menger space (X,F,t) with continuous t-norm $t(c,c) \geq c$, for some $c \in [0,1]$ satisfying :

(3.1) $P(X) \subseteq T(X)$, $Q(X) \subseteq S(X)$,

(3.2) (Q,T) is weak compatible ,

(3.3) For all $x,y \in X$, and $h > 1$,

$$F_{Px,Qy}(hx) \geq Min[F_{sx,Ty}(x), \{F_{Sx,Px}(x). F_{Qy,Ty}(x)\}, F_{Px,Sx}(x)]$$

If (P,S) is semi compatible pairs of reciprocal continuous maps then P,Q,S and T have a unique common fixed point .

Proof: Let $x_0 \in X$, be any arbitrary point.Then we can constract two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Px_{2n+1} = Tx_{2n}$, and $y_{2n+1} = Qx_{2n+2} = Sx_{2n+1}$, for n = 0,1,2,…

First , we will prove that $\{y_n\}$ is a Cauchy sequence in X.

Now, by inequality (3.3) , we have

$$F_{y_{2n+1},y_{2n+2}}(hx) = F_{Sx_{2n+1},Tx_{2n+2}}(hx)$$
$$\geq Min[F_{Sx_{2n+1},Tx_{2n+2}}(x), \{F_{Sx_{2n+1},Px_{2n+1}}(x) . F_{Qx_{2n+2},Tx_{2n+2}}(x)\}, F_{Px_{2n+1},Sx_{2n+1}}(x)]$$
$$\geq Min[F_{y_{2n+1},y_{2n+2}}(x), \{F_{y_{2n+1},y_{2n}}(x) . F_{y_{2n+1},y_{2n+2}}(x)\}, F_{y_{2n+1},y_{2n+2}}(x)]$$
$$F_{y_{2n+1},y_{2n+2}}(hx) \geq F_{y_{2n},y_{2n+1}}(x)$$

Similarly , we get

$$F_{y_{2n+2},y_{2n+3}}(hx) \geq F_{y_{2n},y_{2n+1}}(x)$$

In general, we have

$F_{y_{n+1},y_n}(hx) \geq F_{y_n,y_{n-1}}(x)$

Then by Lemma 2.2 , $\{y_n\}$ is a Cauchy sequence and it convergent to some point z in X.

Hence the subsequences convergent as follows :

$\{Px_{2n}\} \to z$, $\{Sx_{2n}\} \to z$, $\{Qx_{2n+1}\} \to z$ and $\{Tx_{2n+1}\} \to z$.

Now , since P and S are reciprocal continuous and semi- compatible then we have $\lim_{n\to\infty} PSx_{2n} = Pz$, $\lim_{n\to\infty} SPx_{2n} = Sz$, and $\lim_{n\to\infty} M(PSx_{2n}, Sz,t) = 1$. Therefore we get Pz = Sz .

Now we will show that Pz = z .

By inequality (3.3) , putting $x = z$, $y = x_{2n+1}$, we get

$F_{Pz, Qx_{2n+1}}(hx) \geq Min [F_{Sz, Tx_{2n+1}}(x) , \{ F_{Sz,Pz}(x) . F_{Qx_{2n+1},Tx_{2n+1}}(x) \}, F_{Pz,Sz}(x)]$

Taking limit $n\to\infty$, we get

$F_{Pz,z}(hx) \geq Min [F_{sz,z}(x) , \{ F_{Sz,Pz}(x) . F_{z,z}(x) \}, F_{Pz,Sz}(x)]$

Since Pz =Sz , then we get

$F_{Pz,z}(hx) \geq Min [F_{Pz,z}(x) , \{ F_{Pz,Pz}(x) . F_{z,z}(x) \}, F_{Pz,Pz}(x)]$

$F_{Pz,z}(hx) \geq F_{Pz,z}(x)$, then by Lemma 2.1, then we get z = Pz.

Since, Pz =Sz , combining both we get z = pz = Sz.

Now , $P(X) \subseteq T(X)$, therefore there exists a point $u \in X$ such that z= Pz = Tu.

Putting $x= x_{2n}$, y = u in inequality (3.3) , we get

$F_{Px_{2n}, Qu}(hx) \geq Min [F_{Sx_{2n},Tu}(x) , \{ F_{Sx_{2n},Px_{2n}}(x) . F_{Qu,Tu}(x) \}, F_{Px_{2n}, Sx_{2n}}(x)]$

Letting $n\to\infty$, we get

$F_{z, Qu}(hx) \geq Min [F_{z, Tu}(x) , \{ F_{z,z}(x) . F_{Qu,z}(x) \} , F_{z,z}(x)]$

$F_{Z,Qu}(hx) \geq F_{z,Tu}(x)$

Then, by Lemma 2.1, we get Qu = Tu .

Since z = Pz = Tu and we proved that Qu = Tu , combining both we get z = Qu = Tu.

Weak compatibility of (Q,T) gives TQu = QTu i.e. Qz = Tz .

Now , we will prove that Qz = Pz .

Again assuming $Qz \neq Pz$, By inequality (3.3), putting x = z , y = z , we get

$F_{Pz,Qz}(hx) \geq Min [F_{Sz,Tz}(x) , \{ F_{Sz,Pz}(x) . F_{Qz,Tz}(x) \} , F_{Pz,Sz}(x)]$

$F_{Pz,Qz}(hx) \geq Min [F_{Pz,Qz}(x) , \{ F_{Pz,Pz}(x) . F_{Qz,Qz}(x) \} , F_{Pz,Pz}(x)]$

$F_{Pz,Qz}(hx) \geq F_{Pz,Qz}(x)$,

which is a contradiction , thus we get Pz = Qz. Since Pz =Sz = z , and Qz = Tz

Hence finally we get z = Pz = Qz = Sz = Tz. i.e. z is a common fixed point of P,Q, S and T.

Uniqueness: Let w be another common fixed point of P,Q,S and T, then

w = Pw = Qw = Sw = Tw .

Putting x = z and y = w , in inequality (3.3) , we get $F_{Pz, Qw}(hx) \geq Min [F_{Sz,Tw}(x), \{ F_{Sz,Pz}(x) . F_{Qw,Tw}(x) \} , F_{Pz,Sz}(x)]$

$F_{z,w}(hx) \geq Min [F_{z,w}(x) , \{ F_{z,z}(x) . F_{w,w}(x) \} , F_{z,z}(x)]$

$F_{z,w}(hx) \geq F_{z,w}(x)$

Hence , from Lemma 2.1 , we get z = w .

Therefore z is a unique common fixed point of P,Q,S and T .

By setting P = Q in theorem 3.1 ,we can drive a corollary for three mappings

Corollary 3.2. Let P, S and T be self maps of a complete Menger space (X,F,t) , where t is continuous t-norm, satisfying following conditions :

1. The pair (P,T) is weak compatible,
2. For all $x,y \in X$ and $h > 1$,

$$F_{Px,Py}(hx) \geq \text{Min}\left[F_{Sx,Ty}(x) , \{ F_{Sx,Px}(x) . F_{Py,Ty}(x) \}, F_{Px,Sx}(x) \right]$$

If (P,S) is semi compatible pairs of reciprocally continuous maps Then , P,S and T have a unique common fixed point in X.

On taking P = Q and S = T , we get another corollary

Corollary 3.3. Let P and S be self maps of a complete Menger space (X,F,t) , where t is continuous t-norm, satisfying following conditions :

1. For all $x,y \in X$ and $h > 1$,

$$F_{Px,Py}(hx) \geq \text{Min}\left[F_{Sx,Sy}(x) , \{ F_{Sx,Px}(x) . F_{Py,Sy}(x) \}, F_{Px,Sx}(x) \right]$$

If (P,S) is semi compatible pairs of reciprocally continuous maps and weak compatible. Then, P and S have a unique common fixed point in X.

Acknowledgement. Authors thankful to the referees for their valuable comments for the improvement of the paper.

REFERENCES

1. K.Menger, Statistical metrics, *Poc. Nat. Acad. Sci.*, 28 (1942) 535-537.
2. V.M. Sehgal, Some fixed point theorems in functions analysis and probability, Ph.D. dissertation, Wayne State Univ. Michigan (1966).
3. I. Altun and D.Turkoglu , Some fixed point theorems on fuzzy metric spaces with implicit relations, *Commun. Korean Math. Soc.*, 23(1) (2008) 111-124.
4. B.Schweizer and A.Sklar, Probabilistic Metric Spaces, North Holland (Amsterdam 1983).
5. V.M.Sehgal and A.T.Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, *Math. Systems Theory,* 6(1972) 97-102.
6. S.L. Singh, B.D.Pant and R.Talwar, Fixed points of weakly commuting mappings on Menger spaces, *Jnanabha,* 23 (1993) 115-122.
7. S.Kumar and R.Chugh, Common fixed point theorems using minimal commutativity and reciprocal continuity conditions in metric spaces, *Sci. Math. Japan*, 56 (2002) 269-275.
8. M.A.Al-Thagafi and N.Shahzad, Generalized I- nonexpensive selfmaps and invariant approximations, *Acta Math. Sinica*, 24(5) (2008) 867-876.
9. H.Bouhadjera and C.Godet–Thobie, Common fixed point theorems for pairs of sub-compatible maps, arXiv : 0906 .3159v2, (2009).
10. S.N.Mishra, Common fixed points of compatible mappings in PM-space, *Math. Japon.*, 36(2) (1991) 283-289.
11. A.Jain and B.Singh, Common fixed point theorems in Menger space through compatible maps of type (A), *Chh. J. Sci.Tech.*, 2 (2005) 1-12.

12. G.Jungck and B.E.Rhoades, Fixed points for set valued functions without continuity, *Indian Pure Appl. Math.*, 29(3) (1998) 227-238.
13. S.Kumar and B.D.Pant, A common fixed point theorem in probabilistic metric space using implicit relatios, *Filomat*, 22(2) (2008) 43-52.
14. B.D.Pant and S.Chouhan, Common fixed point theorems for semi-compatibility maps using implicit relation, *Int. J. of Math. Analysis*, 3(28) (2009) 1389-1398.
15. S.L.Singh and B.D.Pant, Common fixed point theorems in probabilistic metric space and extension to uniform spaces, *Honam. Math. J.*, (1984) 1-12.