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On Best Approximation in $L^1(\mu, X)$ and $L^\phi(\mu, X)$

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Abstract. We introduce what we called proximinal additivity in Banach spaces, and prove that it holds in some major occasions .

For instance, closed subspaces of Hlibert spaces, ϕ -summands and in a sense, 1-complemented subspaces obey this property. As a result, and with this property we have proved the following result.

The subspace G is proximinal in the Banach space X if and only if $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$ if and only if $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$ for every modulus function ϕ and any finite measure space (T, μ) .

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1. Introduction

For the subset G of the normed linear space $(X, \|.\|)$. We define, for x $\in$ X , d (x , G) = inf $\{\|x-g\| : g \in G\}$ .If G is a subspace of X, an element $g_\circ \in$ G is called a best approximant of x in G if $\|x - g_\circ\|$ = d (x,G) .We shall denote the set of all best approximants of x in G as P(x ,G) . If for each x $\in$ X, the set P (x, G) $\neq \phi$, then G is said to be proximinal in X. and if P (x, G) is a singleton for each x$\in$ X than G is called a Chebychev subspace.

An increasing function ϕ : [0 , ∞) $\longrightarrow$ [0 , ∞) is said to be a modulus function if it vanishes at zero, and is subadditive. This means that $\phi(x+y) \le \phi(x) + \phi(y)$ for all x and y in[0 , ∞) . Examples of modulus functions are : x^p , 0 < p $\le$1 , and ln(1+x). Furthermore, if ϕ is a modulus function, then ϕ (x) = $\dfrac{\phi(x)}{1+\phi(x)}$ is again modulus.
It is also evident that the composition of two modulus functions is a modulus function, [2.p.159] .

Let X be a real Banach space and let (T, μ) be a finite measure space. For a modulus function ϕ, we define the Orlicz space $L^\phi(\mu, X)$, as the set

$$\left\{ f : T \to X \ \ such\,that \ \int_T \phi(\|f(t)\| d\mu(t) < \infty \right\} .$$

The function d : L$^{\phi}$ (µ , X)$\times L^{\phi}$ (μ,X) $\to$[0,∞) given by:
d(f,g) = $\int_T \phi(\|f(t)-g(t)\|)d\mu(t)$ turns L$^{\phi}$ (µ,X) into a complete metric space [3].

For f $\in L^{\phi}(\mu, X)$,we write $\|f\|_{\phi} = \int_T \phi(\|f(t)\| du(t)$. In what follows, when ϕ is mentioned, it is to be assumed a modulus function. We would also like to mention that in the literature, we did not find conditions under which the proximinality of G in X is equivalent to the proximinality of $L^{\phi}(\mu,G)$ in $L^{\phi}(\mu, X)$ and to the proximinality of $L^{1}(\mu,G)$ in $L^{1}(\mu, X)$. Here we give the condition of proximinal additivity under which we achieve the required equivalence.

Recently, authors seem to concentrate on the extensions of classical results in which they consider Haar subspaces for approximating sets, [6]. Past tries can also be found in [4,7].

2. Proximinal additivity

Definition 2.1. A subspace G of a Banach space X is said to proximinally additive if G is closed and $z_1 + z_2 \in P(x_1 + x_2, G)$ whenever $z_1 \in P(x_1, G)$ and $z_2 \in P(x_2, G)$.

Example 2.2. Let X = R2 , and let G = $\{(x,0) : x \in R\}$. Then G is proximinally additive in X, with the Euclidean norm.

Definition 2.3. A closed subspace G of a Banach space X is said to be a ϕ-summand if there is a bounded projection E : X$\to$ G such that, for all $x \in X$, $\phi(\|x\|) = \phi(\|E(x)\|) + \phi(\|x - E(x)\|)$. In [3] , it was shown that ϕ-summands are proximinal. Further, we prove that:

Propostion 2.4. If G is a ϕ-summand of a Banach space X , then G is Chebyshev.
Proof. Assume that G is a ϕ-summand of X. By [4.page72] , One has:
($\forall x \in X$) E(x) ∈ P(x,G). Now suppose that g* ∈ G is another closest element to x.
So,

$$\|x - g^*\| = \|x - E(x)\| \qquad (1)$$

But x-g* ∈ X , so:

$$\phi(\|x - g^*\|) = \phi(\|E(x - g^*)\|) + \phi(\|x - g^* - E(x - g^*)\|)$$
$$= \phi(\|E(x) - g^*\|) + \phi(\|x - E(x)\|).$$

By (1) , we conclude that $\phi(\|E(x)-g^*\|)=0$, hence g* = E (x) .
Since x and g* were arbitrary , G is Chebyshev.

Proposition 2.5. If G is a ϕ-summand of X , then G is proximinally additive.
Proof: Let $z_1 \in P(x_1, G)$ and $z_2 \in P(x_2, G)$ be arbitrary .

Since G is a ϕ-summand , choose an appropriate projection:
E : X $\rightarrow$ G such that E (x) is the unique best approximant of x $(\forall x \in X)$.
By proposition (2.4) , z_1 = E (x_1) and z_2 = E (x_2).
But $z_1 + z_2$ = E (x_1)+E(x_2)= E (x_1+x_2) since E is linear , so $z_1 + z_2 \in P(x_1 + x_2, G)$

Definition 2.6. [1] A subspace G of a Banach space X is said to be 1-complemented in X if there is a closed subspace W of X such that :
X=G$\oplus$W, and the projection E : X $\rightarrow$W is contractive.

Proposition 2.7. If G is 1-complemented and Chebyshev in X, then G is proximinally additive.
Proof: Let for i = 1,2, $z_i \in P(x_i, G)$. Since G is 1-complementd in X , choose an appropriate closed subspace W of X such that X = G $\oplus$ W. This implies that x_i can be written as $x_i = g_i + w_i$ where $g_i \in G$ and $w_i \in W$ (i = 1,2).
Since G is Chebyshev , $z_i = g_i$(i=1,2).
Now $x_1+x_2 = (g_1+g_2)+(w_1+w_2)$. But since G and W are subapaces , $(g_1 + g_2) \in G$ and $(w_1 + w_2) \in W$. It now follows that : $z_1 + z_2 = g_1 + g_2 \in P(x_1 + x_2, G)$. Since z_1 and z_2 were arbitrary, G is proximinally additive.

Proposition 2.8. Let G be a closed subspace of a Hilbert space $(X, <.>)$ Then G is proximinally additive.
Proof: Let $z_1 \in P(x_1, G), z_2 \in P(x_2, G)$. By [5 , P.92] , $x_i - z_i \perp$ G (for i=1,2) .
Hence, $<x_1 + x_2 - (z_1 - z_2), g> = 0$ for all g$\in$ G
Hence, $x_1+x_2 - (z_1 + z_2) \perp$ G which implies that $z_1 + z_2 \in$ P ($x_1 + x_2$, G) .
Thus G is proximinally additive in X.

3. Main results
We begin with the following two lemmas.
Lemma 3.1. Let G be a subspace of a normed space X , and let
$P_G^{-1}(0) = \{x \in X : 0 \in p(x, G)\}$. Then the following statements are equivalent:
(i) G is proximinal in X.
(ii) X =G + $P_G^{-1}(0)$

Proof: If G is proximinal, $x\in X$, and $g_\circ\in P(x,G)$, then

$x=g_\circ+(x-g_\circ)\in G+p_G^{-1}(0)$ since

$\|x-g_\circ-o\|=\|x-g_\circ\|\leq\|x-g_\circ-g\|\forall g\in G.$

Conversely, if (ii) holds , and if $x\in X$ then x=g∘ +y , where $g_\circ\in G$ and $y\in P_G^{-1}(0).$

Thus $0\in P(y,G)=p(x-g_\circ,G)$, which implies that d(x-g∘,G) = $\|x-g_\circ\|$. It now follows that $g_\circ\in P(x,G),$ and so G is proximinal in X.

Lemma 3.2. Let X be a Banach space, and G a closed subspace of X which is proximinally additive. Then $p_G^{-1}(0)$ is a closed subspace of X and $P_G^{-1}(0)\cap G=\{0\}$.

Proof: Let x1 and x2 be elements in $p_G^{-1}(0)$. So $0\in P(x_1,G)\cap P(x_2,G).$

Since G is proximinally additive, $0\in P(x_1+x_2,G)$, and so $x_1+x_2\in P_G^{-1}(0)$

Let $x\in P_G^{-1}(0)$ and α be any scalar. Then by [5,page 147], One has:

d (αx , G) = $\alpha d(x,G), and\ so |\alpha|\|x\|=\|\alpha x\|.$

Hence , $0\in P(\alpha x,G)$,which in turn implies that $\alpha x\in P_G^{-1}(0)$ (2)

By (1) and (2), $P_G^{-1}(0)$ is a subspace of X.

Now, let (x_n) be a sequence in $P_G^{-1}(0)$ which converges to x. Since $0\in G$,

One has that $d(x,G)\leq\|x\|$ (3)

For $\in>0$, choose $n_\circ\in N$ such that, $\forall n\geq n_\circ, \|x_n-x\|<\frac{\in}{2}$.

Fixing $n\geq n_\circ$, we have that:

$$\|x\|=\|x-x_n+x_n\|\leq\|x-x_n\|+\|x_n\|$$

$$<\frac{\in}{2}+\left|\,\|x_n\|-d(x,G)+d(x,G)\right|$$

$$<\frac{\in}{2}+\left|\,\|x_n\|-d(x,G)\right|+d(x,G)$$

$$=\frac{\in}{2}+|d(x_n,G)-d(x,G)|+d(x,G)|\,(\text{since } x_n\in P_G^{-1}(0)\,|)$$

$$<\frac{\in}{2}+\|x_n-x\|+d(x,G)$$

$$< \frac{\in}{2} + \frac{\in}{2} + d(x,G)$$
$$= \in + d(x,G)$$

Hence $\|x\| \le d(x,G)$ (4)

By (3) and (4), $\|x\| = d(x,G)$, so $0 \in P(x,G)$. Thus $x \in P_G^{-1}(0)$, so $P_G^{-1}(0)$ is closed.

Finally, let
$g \in P_G^{-1}(0) \cap G$. Therefore $g \in P_G^{-1}(0)$ and $g \in G$. Thus, $0 \in P(g,G)$ and $g \in G$. So $\|g\| = d(g,G)$ *and* $g \in G$, which implies that $\|g\| = 0$, so $g = 0$. Therefore $P_G^{-1}(0) \cap G = \{0\}$

Theorem 3.3. Let X be a Banach space , and G a closed subspace of X which is proximinally additive. Then G is proximinal in X if and only if G is 1- complemented in X.

Proof: Suppose that G is proximinal in X . By lemma (3.1) X = G + $P_G^{-1}(0)$.

By Lemma 3.2, since $P_G^{-1}(0)$. is a closed subspace of X , and meets G in exactly $\{0\}$,

so $X = G \oplus P_G^{-1}(0)$.

For $x \in X$, $g \in G$ and $z \in P_G^{-1}(0)$ with $x = g + z$, we define $E(x) = z$.

Now E is a contraction : $X \to P_G^{-1}(0)$. To see this ,

for $x \in X$, $g \in G$ and $z \in P_G^{-1}(0)$ with x =g+z , we have by [5, page 147] that:

$$\|x\| \ge d(x,G) = d(g+z,G)$$
$$= d(z,G)$$
$$= \|z\|$$

The converse was done in [1,page 529]
The proof of our main result we will be broken into a few lemmas. We begin with:

Lemma 3.4. Let G be 1-complemented in the Banach space X, and let (T, μ) be a finite measure space. Then $L^1(\mu, X)$ is 1-complemented in $L^1(\mu, X)$.

Proof: Let $X = G \oplus W$ and let E : $X \to W$ be a contractive projection. So $\forall x \in X$, $x = (I - E)(x) + E(x)$, and $\|E(x)\| \le \|x\|$ where I is the identity map.

For $f \in L^1(\mu, X)$, set $f_1 = (1 - E)_\circ f$ and

$$f_2 = E_\circ f \text{ (a.e.on T)}$$

Now , $\left\| f_2 \right\|_1 = \int_T \left\| f_2(t) \right\| d\mu(t) = \int_T \left\| E(f(t)) \right\| d\mu(t) \le \int_T \left\| f(t) \right\| d\mu(t) = \left\| f_1 \right\|_1 < \infty$

Thus $f_2 \in L^1(\mu, W)$.

Furthermore , we have:

$$\begin{aligned}
\| f_1 \|_1 &= \int_T \| f_1(t) \| d\mu(t) = \int_T \| (I-E)(f(t)) \| d\mu(t) \\
&= \int_T \| f(t) - E(f(t)) \| d\mu(t) \\
&\le \int_T \| f(t) \| d\mu(t) + \int_T \| E(f(t)) \| d\mu(t) \\
&\le \int_T \| f(t) \| d\mu(t) + \int_T \| (f(t)) \| d\mu(t) \\
&= 2 \| f \|_1 \\
&< \infty
\end{aligned}$$

Hence, $f_1 \in L^1(\mu, G)$. Clearly f=f_1+f_2 (a.e. on T).

Since W is a closed subspace of X , $L^1(\mu, W)$ is a closed subspace of $L^1(\mu, X)$. Also , if $f \in L^1(\mu, G) \cap L^1(\mu, W)$ then $f(t) \in G \cap W = \{0\} \forall t \in T$, so f is the zero function. Hence:

$$L^1(\mu, X) = L^1(\mu, G) \oplus L^1(\mu, W).$$

Finally , the map $\hat{E}(f) = f_2$ is a contractive projection form $L^1(\mu, X) \to L^1(\mu, W)$.

Thus, $L^1(\mu, G)$ is 1-complemented in X .

Corollary 3.5. If G is 1-complemented in X, then $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$.

Proof: By lemma (3.4) , $L^1(\mu, G)$ is 1-complemented in $L^1(\mu, X)$ .By [1,p.529] $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$.

Lemma 3.6. Let G be a closed subspace of a Banach space X and let (T, μ) be a finite measure space . If $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$, then G is proximinal in X.

Proof: Let $x \in X$ be arbitrary . For all $t \in T$, let f (t) = x. Then $f \in L^\phi(\mu, X)$. By proximinality of $L^\phi(\mu, G)$ in $L^\phi(\mu, X)$, choose $g \in L^\phi(\mu, G)$.

Such that $\|f-g\|_\phi = d(f, L^\phi(\mu, G))$. By[4,p.73], One has :

$\|f(t)-g(t)\| \le \|f(t)-y\|$ *a.e. on* T, *and* $\forall y \in G$

Hence for some t, $\|x-g(t)\| \le \|x-y\| \forall y \in G$

Consequently, G is proximinal in X. Now we are ready to prove our main result.

Theorem 3.7. Let X be a Banach space and G be a subapace which is proximinally additive in X, then the followings are equivalent, for any finite measure space (T, μ):

(a) G is proximinal in X

(b) $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$

(c) $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$

Proof: $(a) \Rightarrow (b)$: Assume G is proximinal in X. By Theorem 3.3 G is 1-complemented in X. By Cor.(3.5) , $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$.

$(b) \Rightarrow (c)$: Assume $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$. By [4,p.73], $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$.

$(c) \Rightarrow (a)$: Assume $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$. By lemma (3.6) , G is proximinal in X.

Corollary 3.10. Let X be a Banach space and G be be a ϕ-summand of X. Then $L^\phi(\mu, G)$ is proxminal in $L^\phi(\mu, X)$.

Proof: By [4,p.72] G is proximinal in X . By Proposition (2.5),G is proximinally additve. Now, by theorem (3.7) , $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$.

4. A note on optimization theory

Optimization is a mathematical technique that concerns the finding of maxima or minima of functions within some feasible region. A diversity of optimization techniques fight for the best solution. Particle Swarm Optimization (PSO) is a comparatively new, current, and dominant method of advanced optimization technique that has been empirically shown to perform well on many of these optimization problems. It is lucidly and widely used to find the global optimum solution in a complex search space. This, in a sense, is another face of best approximation theory, each in its field of application. The difference is in the fact that, optimal solutions occur as values of functions while proximinal maps have the basic problem of non-being linear. This in part shortens the scope of invoking such maps in the theory of best approximation. For further development, we would like to refer the reader to [8,9,10].

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