Annals of Pure and Applied Mathematics
Vol. 12, No. 1, 2016, 85-94
ISSN: 2279-087X (P), 2279-0888(online)
Published on 3 August 2016
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Injective Gamma Modules

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Received 13 July 2016; accepted 27 July 2016

Abstract. In this paper we extend the concept of injectivity from the category of modules to that gamma modules. An $R_\Gamma-$ module M is called injective if for any $R_\Gamma-$module B and $R_\Gamma-$submodule A of B , any $R_\Gamma-$homomorphism f from A to M can be extended to an $R_\Gamma-$homomorphism from B to M. We show that every gamma module can be embedding in injective gamma module.

Keywords: Gamma ring; Gamma module; gamma submodule; essential gamma submodule; divisible gamma module; injective gamma module; injective gamma hull.

AMS Mathematics Subject Classification (2010): 17D20

1. Introduction

The notion of Γ-ring was first introduced by N. Nobusawa[4] and then Barnes [2] generalized the definition of Nobusawa's gamma rings. Let R and Γ two additive abelian groups, R is called a $\boldsymbol{\Gamma-ring}$ (in the sense of Barnes), if there exists a mapping $\cdot : R\times\Gamma\times R\longrightarrow R,\ \cdot(r,\gamma,s)\mapsto r\gamma s$ such that $(a+b)\alpha c=a\alpha c+b\alpha c, a(\alpha+\beta)c=a\alpha c+a\beta c, a\alpha(b+c)=a\alpha c+b\alpha c$ and $(a\alpha b)\beta c=a\alpha(b\beta c)$ where $a,b,c\in R$ and $\alpha,\beta\in\Gamma$ [2]. A subset A of $\Gamma-$ring R is said to be a ***right(left) ideal*** of R if A is an additive subgroup of R and $A\Gamma R\subseteq A(R\Gamma A\subseteq A)$, where $A\Gamma R=\{a\alpha r : a\in A,\ \alpha\in\Gamma, r\in R\}$. If A is both right and left ideal, we say that A is an ***ideal*** of R [2]. An element 1 in $\Gamma-$ ring R is ***unity*** if $1\gamma_\circ r=r$ for each $r\in R$ and some $\gamma_\circ\in\Gamma$, unities in $\Gamma-$ rings differ from unities in rings are not necessarily unique [3]. Principle ideal $\Gamma-$domain is $\Gamma-$ ring with unity and each ideal is principle. Let R be a $\Gamma-$ring and let M be an additive abelian group. Then M together with a mapping $\cdot : R\times\Gamma\times M\longrightarrow M$, $\cdot(r,\gamma,m)\mapsto r\gamma m$ such that $(r_1+r_2)\alpha m=r_1\alpha m+r_2\alpha m,\ r\alpha(m_1+m_2)=r\alpha m_1+r\alpha m_2,\ r(\alpha+\beta)m=r\alpha m+r\beta m$ and $(r_1\alpha r_2)\beta m=r_1\alpha(r_2\beta m)$ for each $r,r_1,r_2\in R,\alpha,\beta\in\Gamma$ and $m,m_1,m_2\in M$, is called a ***lef*** $\boldsymbol{R_\Gamma-module}$ [6]. An $R_\Gamma-$module M is ***unitary*** if there exists element, say 1 in R and $\gamma_\circ\in\Gamma$ such that $1\gamma_\circ m=m$ for every $m\in M$. Let M be an $R_\Gamma-$module , a nonempty subset N of M is said to be an $\boldsymbol{R_\Gamma-submoule}$ of M (denoted by $N\le M$) if N is a subgroup of M and $R\Gamma N\subseteq N$, where $R\Gamma N=\{r\alpha n : r\in R,\ \alpha\in\Gamma,\ n\in N\}$ [6]. If X is a nonempty subset of M, then the $R_\Gamma-$submodule of M ***generated*** by X is $\cap\{N\le M: X\subseteq N\}$ and denoted by $\langle X\rangle$, X is called the ***generator*** of $\langle X\rangle$ and $\langle X\rangle$ is finitely generated if $|X|<\infty$. In particular , if $X=\{x\}$,

$\langle X\rangle$ is called the ***cyclic*** $\boldsymbol{R_\Gamma -}$ ***submodule*** of M generated by x, in general $\langle X\rangle = \{\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{k} r_j\gamma_j x_j : k, m \in \mathbb{N}, n_i \in \mathbb{Z}, \gamma_j \in \Gamma, r_j \in R, x_i, x_j \in X\}$ and if M is unitary, then $\langle X\rangle = \{\sum_{i=1}^{n} r_i\gamma_i x_i : n \in \mathbb{N}, \gamma_i \in \Gamma, r_i \in R, x_i \in X\}$ [1]. Let M and N two $R_\Gamma -$modules , A mapping $f: M \to N$ is called homomorphism of $R_\Gamma -$ modules (or $\boldsymbol{R_\Gamma -}$ ***homomorphism***) if $f(x+y) = f(x) + f(y)$ and $f(r\gamma x) = r\gamma f(x)$ for each $x, y \in M, r \in R$ and $\gamma \in \Gamma$. An $R_\Gamma -$ homomorphism is $R_\Gamma -$ monomorpism if it is one-to-one and $R_\Gamma -$epimorphism if it is onto. Set of all $R_\Gamma -$ homomorphism from M into N denote by $Hom_{R_\Gamma}(M,N)$ in particular if M=N denote by $End_{R_\Gamma}(M)$[1]. $End_{R_\Gamma}(M)$ is a $\Gamma -$ ring and if M is left $R_\Gamma -$ module , then M is right $End_{R_\Gamma}(M)$ – module [1, proposition 5.6]. All modules in this paper are unitary left $R_\Gamma -$modules and $\gamma_\circ \in \Gamma$ denote to the element such that $1\gamma_\circ$ is the unity.

2. Basic structure of injective gamma modules :

An R-module A is called $N -$ injective if for every submodule X of N , any homomorphism f from X to A can be extended to a homomorphism from N to A [5]. In this section we introduce the concept of injectivegamma module,many characterizations and properties of injective gamma modules are given.

Definition 1.1. Let M and N be two $R_\Gamma -$ modules. Then M is called $N -$ ***injective*** $R_\Gamma -$ module if for any $R_\Gamma -$ submodule A of N and for any $R_\Gamma -$ homomorphism $f: A \to M$ there exists an $R_\Gamma -$ homomorphism $g: N \to M$ such that $gi = f$ where i is the $R_\Gamma -$ inclusion mapping.

Proposition 1.2. If M is $N -$ injective $R_\Gamma -$ module and A is $R_\Gamma -$ submodule of N , then M is $A -$ injective and $N/A -$ injective.

Proof. It is clear that M is $A -$ injective $R_\Gamma -$ module if A=N . Let X/A be an $R_\Gamma -$ submodule of N/A and $f: X/A \to M$ is an $R_\Gamma -$ homomorphism, let $\pi : N \to N/A$ be the natural $R_\Gamma -$ homomorphism and $\pi' = \pi_{|X}$, since M is $N -$ injective then there exists an $R_\Gamma -$ homomorphism $\alpha: N \to M$ such that $\alpha_{|X} = f\pi'$. Now $\alpha(A) = f\pi'(A) = f(0) = 0$, then $A = ker\pi \subseteq ker\alpha$, hence by [1,proposition 5.20] there exists $\theta: N/A \to M$ such that $\theta\pi = \alpha$ and for any $x \in X$ we have $\theta(x+A) = \theta\pi(x) = \alpha(x) = f\pi'(x) = f(x+A)$, thus θ extends f and therefore M is $N/A -$ injective.

In the following proposition the concept of $N -$ injective can be reduced to elements of N.

Proposition 1.3. If M and N are two $R_\Gamma -$modules , then M is $N -$ injective if and only if M is $\langle a\rangle -$ injective $R_\Gamma -$ module for each $a \in N$.

Proof. For any $R_\Gamma -$submodule A of an $R_\Gamma -$module N and a $R_\Gamma -$homomorphism f from A to M, by Zorn's lemma there exists maximal element $(A_\circ, f_\circ)$ such that $A \le A_\circ$ and $f_\circ$ extends of f to $A_\circ$. If $A_\circ = N$, then the proof is complete, if not there exists $x \in N - A_\circ$, let $L = \{r \in R: r\Gamma x \subseteq A_\circ\}$, then L is an ideal of R, define an $R_\Gamma -$homomorphism $\theta: I\gamma_\circ x \to M$ by $\theta(r\gamma_\circ x) = f_\circ(r\gamma_\circ x)$ for each $r\gamma_\circ x \in I\gamma_\circ x$, by assumption θ can extended to an $R_\Gamma -$ homomorphism $\lambda: \langle x\rangle \to M$. Let $\psi: C = A_\circ + \langle x\rangle \to M$ by $\psi(a_\circ + \sum_{i=1}^{n} r_i\gamma_i x) = f_\circ(a_\circ) + \lambda(\sum_{i=1}^{n} r_i\gamma_i x)$ for each $a_\circ \in A_\circ$ and $\sum_{i=1}^{n} r_i\gamma_i x \in \langle x\rangle$. Then ψ is

$R_\Gamma -$ homomorphism which is contradiction with maximality of $(A_\circ, f_\circ)$, hence $A_\circ = N$ and $f_\circ$ extends f to N, thus M is $N-$ injective $R_\Gamma -$ module.

Proposition 1.4. An $R_\Gamma -$ module M is $(\bigoplus_{i\in I} N_i)-$injective $R_\Gamma -$ module if and only if M is N_i-injective $R_\Gamma -$ module for each $i \in I$.

Proof.(⇒) by proposition(1.2). (⇐) Let $=\bigoplus_{i\in I} N_i$, for any $R_\Gamma -$submodule A of an $R_\Gamma -$module N and a $R_\Gamma -$homomorphism f from A to M, by Zorn's lemma there exists maximal element $(N_\circ, f_\circ)$ such that $N \leq N_\circ$ and $f_\circ$ extends of f to $N_\circ$. If $N_\circ = N$, then the proof is complete, if not there exists $x \in N - N_\circ$, since M is $N_i -$ injective then M is $\langle x\rangle -$ injective , thus $f_\circ$ can extended to an $R_\Gamma -$ homomorphism $\psi: C = A_\circ + \langle x\rangle \to M$ which is contradiction, thus M is $N-$injective $R_\Gamma -$ module.

Lemma 1.5. Let $\{E_\lambda : \lambda \in \Lambda\}$ be family of $R_\Gamma -$ modules.Then $\prod_{\lambda\in\Lambda} E_\lambda$ is $N-$ injective $R_\Gamma -$ module if and only if E_λ is $N-$ injective $R_\Gamma -$ module for each $\lambda \in \Lambda$ and each $R_\Gamma -$ module N.

Proof. Put E=$\prod_{\lambda\in\Lambda} E_\lambda$ and denote the injections and projections by: $\phi_\lambda: E_\lambda \to E$ and $\pi_\lambda: E \to E_\lambda$ respectively. Assume E_λ is injective $R_\Gamma -$ module for each $\lambda \in \Lambda$, for any $R_\Gamma -$submodule A of an $R_\Gamma -$module N and a $R_\Gamma -$homomorphism f from A to E , there exists $g_\lambda \in Hom_{R_\Gamma}(N, E_\lambda)$ such that $g_\lambda i = \pi_\lambda f$, define an $R_\Gamma -$homomorphism $g: N \to E$ by $g(b) = (g_\lambda(b))_{\lambda\in\Lambda}$ for each $b \in N$ and $gi(a) = \big(g_\lambda(a)\big)_{\lambda\in\Lambda} = \big(\pi_\lambda f(a)\big)_{\lambda\in\Lambda} = f(a)$ for each $a \in A$, thus E is injective $R_\Gamma -$ module. Conversely, if E is $N-$ injective $R_\Gamma -$ module and for $\lambda \in \Lambda$, let A be an $R_\Gamma -$submodule of N and $\mu: A \to E_\lambda$ be an $R_\Gamma -$ homomorphism, since E is $N-$ injective then there exists $h \in Hom_{R_\Gamma}(N,E)$ such that $hi = \varphi_\lambda \mu$, define an $R_\Gamma -$ homomorphism $h': N \to E_\lambda$ by $h'(b) = \pi_\lambda h(b)$ for each $b \in N$. Then $h'(a) = \pi_\lambda h(a) = \pi_\lambda \varphi_\lambda \mu(a) = \mu(a)$ for each $a \in N$. Thus E_λ is $N-$ injective $R_\Gamma -$ module for each $\lambda \in \Lambda$.

Definition 1.6. An $R_\Gamma -$ module M is called ***injective*** $R_\Gamma -$ module if for any $R_\Gamma -$ submodule A of an $R_\Gamma -$ module B and for any $R_\Gamma -$ homomorphism $f: A \to M$ there exists an $R_\Gamma -$ homomorphism $g: N \to M$ such that $gi = f$ where i is the $R_\Gamma -$ inclusion mapping. An $R_\Gamma -$ module M is injective if it is $N-$ injective for any $R_\Gamma -$ module N.

The following proposition shows that in order M to be gamma injective $R_\Gamma -$module, it's enough to be injective gamma relative to the $\Gamma -$ring R.

Proposition 1.7. Let M be an $R_\Gamma -$ module. Then the following statements are equivalent:

(a) M is injective $R_\Gamma -$ module.

(b) For any ideal I of $\Gamma -$ring R and $R_\Gamma -$homomorphism $f: I \to M$, there exists $R_\Gamma -$ homomorphism $g: R \to M$ such that $gi = f$ where i is inclusion mapping of I into R.

(c) For any exact sequence $0 \to A \to B \to C \to 0$ of $R_\Gamma -$ modules, the sequence $0 \to Hom_{R_\Gamma}(C,M) \to Hom_{R_\Gamma}(B,M) \to Hom_{R_\Gamma}(A,M) \to 0$ is exact.

Proof. (a)⟹(b) Clear.(b) ⟹(a) Let A be a $R_\Gamma -$ submodule of an $R_\Gamma -$ modules B and f is $R_\Gamma -$ homomorphisms from A to M, let $\Omega = \{(A', f'): A \leq A' \leq B\ , f'$ extend of $f\ to\ A'\}$, then by Zorn's lemma Ω has a maximal element $(A_\circ, f_\circ)$ say. If $A_\circ \neq B$ then there exists $x \in B - A_\circ$, let $C = A_\circ + R\gamma_\circ x$ which is $R_\Gamma-$submodule of B contains $A_\circ$ properly , define an ideal $I = \{r \in R: r\gamma_\circ x \in A_\circ, \gamma_\circ \in \Gamma\}$. Define an $R_\Gamma-$homomorphism $\theta: I \to M$ by $\theta(r) = f_\circ(r\gamma_\circ x)$ for each $\in I$, by assumption there exists $R_\Gamma-$ homomorphism $\lambda: R \to M$ such that $\lambda i = \theta$. Define $\psi: C = A_\circ + R\gamma_\circ x \to M$ by $\psi(a_\circ + r\gamma_\circ x) = f_\circ(a_\circ) + \lambda(r)$ for each $a_\circ \in A_\circ, r \in R$, then $(C, \psi) \in \Omega$ contradiction with maximal of $(A_\circ, f_\circ)$, thus $A_\circ = B$ and $f_\circ$ extends f to B. (a)⇔(c) Easy to show that (c) equivalent to, if $0 \to A \overset{\theta}{\to} B$ exact then the sequence $Hom_{R_\Gamma}(B,M) \overset{\Phi}{\to} Hom_{R_\Gamma}(A,M) \to 0$ is exact where $\Phi(g) = g\theta$, $\forall g \in Hom_{R_\Gamma}(B,M)$, for any $R_\Gamma-$submodule A of an $R_\Gamma-$module B and an $R_\Gamma-$homomorphism f from A to M, there exists $g \in Hom_{R_\Gamma}(B,M)$ such that $gi = f = \Phi(g)$.

Theorem(Bear's gamma condition) 1.8. Let M be unitary $R_\Gamma-$ module. Then M is injective gamma module if and only if for each left ideal I of $\Gamma-$ring R and $R_\Gamma-$homomorphism $f: I \to M$ there is $m \in M$ such that $f(x) = x\gamma_\circ m$ for each $x \in I$, for some $\gamma_\circ \in \Gamma$.

Proof. Suppose M is injective gamma module , I an ideal of $\Gamma-$ring R and $f: I \to M$ is an $R_\Gamma-$ homomorphism, then there exists $R_\Gamma-$ homomorphism $g: R \to M$ extends f, put $g(1) = m$ then $f(x) = f(x\gamma_\circ 1) = g(x\gamma_\circ 1) = x\gamma_\circ g(1) = x\gamma_\circ m$. Conversely, for any ideal I of R and $R_\Gamma-$ homomorphism $f: I \to M$, there exists $m \in M$ such that $f(x) = x\gamma_\circ m$ for each $x \in I$, for some $\gamma_\circ \in \Gamma$, define $g: R \to M$ by $g(r) = r\gamma_\circ m$ for each $r \in R$, it's clear that g is $R_\Gamma-$ homomorphism and g extends to f, hence M is injective $R_\Gamma-$ module.

An $R_\Gamma-$ submodule N of $R_\Gamma-$ module M is a direct summand if there exists an $R_\Gamma-$ submodule K of M such that $M = N \oplus K$ and $N \cap K = 0$.

Proposition 1.9. An $R_\Gamma-$ module M is injective if and only if it is a direct summand of every extension of itself.

Proof. Assume that M is injective and E is an extension of M, then there exists an $R_\Gamma-$homomorphism θ from E to M such that $\theta_{|M} = id_M$,for each $e \in E$ we have $\theta(e) \in M$ and $\theta(e) = \theta(\theta(e))$ so $\theta(e) - \theta\big(\theta(e)\big) = 0$ hence $\theta\big(e - \theta(e)\big) = 0$ then $e - \theta(e) \in \ker(\theta)$ so $e \in M + \ker(\theta)$ hence $E = M + \ker(\theta)$ but $M \cap \ker(\theta) = 0$, then $E = M \oplus \ker(\theta)$. Conversely, M can be embedded in *E(M)*, then M is a direct summand of *E(M)* and by example(1.10) M is injective.

Examples and Remarks 1.10.

1- Every injective R-module is injective R_R- module.

2- Let Q be a ring, $R = \{(x \quad y): x, y \in Q\}$, and $\Gamma = \{\binom{\gamma}{0}: \gamma \in Q\}$. Define $\cdot : R \times \Gamma \times R \longrightarrow R$ by $(x \quad y)\binom{\gamma}{0}(a \quad b) = (x\gamma a \quad x\gamma b)$, let $I = \{(0 \quad x): x \in Q\}$ which is a left ideal of R, define an $R_\Gamma-$ homomorphism $f: I \to R$ by $f(0 \quad x) = (x \quad 0)$, if R is injective gamma $R_\Gamma-$ module, then for each $A \in I$ there exits $B \in R$ such that $f(A) = A\gamma_\circ B$, take $A = (0 \quad a) \neq 0$, $\gamma_\circ = \binom{\gamma}{0}$ and $B = (z \quad y)$, but $A\gamma_\circ B = (0 \quad a)\binom{\gamma}{0}(z \quad y) = (0 \quad 0) \neq f(A)$ a contradiction.

3- A direct summand of injective $R_\Gamma -$ module is injective $R_\Gamma -$ module, for any direct summand N of $R_\Gamma -$ module M, let A be $R_\Gamma -$ submodule of an $R_\Gamma -$ module B and f is $R_\Gamma -$ homomorphism from A to N, if $\pi : M \to N$ is the projection $R_\Gamma -$ homomorphism, then f can extended to $R_\Gamma -$ homomorphism λ from B to M, define $\alpha : B \to N$ by $\alpha = \pi\lambda$, so $\alpha(a) = \pi\lambda(a) = \pi\big(\lambda(a)\big) = \pi(f(a)) = f(a)$ for each $a \in A$.
4- A $\Gamma -$ ring R is called simple $\Gamma -$ ring if the only non zero ideal is itself, by theorem(1.8) every $R_\Gamma -$ module over simple $\Gamma -$ ring is injective. In particular, Z_2 is simple $Z -$ ring, so every $Z_{2_Z} -$ module is injective.

2. Divisiblegamma modules

In this part we will introduce the concept of gamma divisible $R_\Gamma -$ module and discusses the relation between gamma divisible $R_\Gamma -$ module and injective gamma $R_\Gamma -$ module.

Definition 2.1. Let R be a Γ-ring an element $r(\neq 0) \in R$ is called ***$R_\Gamma -$ zero-divisor*** if there exists $\gamma(\neq 0) \in \Gamma, s(\neq 0) \in R$ such that $r\gamma s = 0$, that is, $r(\neq 0) \in R$ is $R_\Gamma -$ zero-divisor if for each $s \in R$ and for each $\gamma(\neq 0) \in \Gamma$ such that if $r\gamma s = 0$, then $s = 0$.

Definition 2.2. Let M be an $R_\Gamma -$ module , an element $a \in M$ is called ***$R_\Gamma -$ divisible*** if for each $r \in R$ which is not $R_\Gamma -$ zero-divisor and for each $\gamma(\neq 0) \in \Gamma$, there exists $a' \in M$ such that $a = r\gamma a'$, M is called ***gamma divisible $R_\Gamma -$module*** (shortily $R_\Gamma -$ divisible) if each element in M is $R_\Gamma -$ divisible, that is , for each not $R_\Gamma -$ zero-divisor $r \in R$ then $M = r\Gamma M$. An abelian group is ***$R_\Gamma -$divisible group*** if and only if it is $Z_z -$ divisible module.

Examples 2.3.

1- Q as $Z_Z -$ module is $R_\Gamma -$divisible.
2- Let $M = Z_2$, $\Gamma = 2Z$ and $R=Z$, then M is not $R_\Gamma -$divisible, since $1 \neq n.(2m).x$ for any $n \in Z, 2m \in 2Z$ and $x \in Z_2$.
3- Let $= Q$, $\Gamma = Z$ and $M = \{\begin{pmatrix} x \\ y \end{pmatrix}, x, y \in Q\}$, then for any $\begin{pmatrix} x \\ y \end{pmatrix} \in M, \frac{p}{q} \neq 0 \in Q$ and $n \neq 0$, we have $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{p}{q}.n.\begin{pmatrix} \frac{xq}{pn} \\ \frac{yq}{pn} \end{pmatrix}$, thus M is $R_\Gamma -$divisible.

Let M be an $R_\Gamma -$ module , annihilator of an element $r \in R$ denoted by $Ann_R(r)$= $\{s \in R : s\Gamma r = 0\}$ which is ideal of R [1]. It's easy to proof the following lemmas:

Lemma 2.4. An $R_\Gamma -$module M is $R_\Gamma -$divisible if and only if for each $m \in M$ and a not $R_\Gamma -$ zero-divisor $r \in R$ such that $Ann_R(r) \subseteq Ann_R(m)$, then there exists $m' \in M$ such that $m = r\Gamma m'$.

Lemma 2.5. Homomorphic image of $R_\Gamma -$divisible is $R_\Gamma -$divisible.

Corollary 2.6. If M is $R_\Gamma-$divisible $R_\Gamma-$module and N is $R_\Gamma-$submodule, then M/N also $R_\Gamma-$divisible.

Lemma 2.7. Let $\{M_\lambda : \lambda \in \Lambda\}$ be family of $R_\Gamma-$divisibles, then $\prod_{\lambda\in\Lambda} M_\lambda$ is $R_\Gamma-$divisible if and only if M_λ is $R_\Gamma-$divisible for each $\lambda \in \Lambda$.

A generator set of an $R_\Gamma-$module M is basis if each element $m \in M$ can be written in only one way as $m = \sum_{i=1}^{n} r_i \gamma_i x_i$ where $r_i \in R$ and $\gamma_i \in \Gamma$, M is free if it has basis [6].

Lemma 2.8. Every $R_\Gamma-$ module is epimorphic image of free $R_\Gamma-$ module.
Proof. Let Y be generator set for M , then $R \cong R\gamma\circ x_i$ for each $x_i \in Y$ by $\varphi_{x_i}: R \to R\gamma\circ x_i$, $\varphi_{x_i}(r) = r\gamma\circ x_i$ for each $r \in R$. since $\varphi_{x_i}(1) = 1\gamma\circ x_i = x_i$, then $R^{(Y)} = \oplus_{i\in I} R\gamma\circ x_i = \oplus_{i\in I} R\gamma\circ\varphi_{x_i}(1)$ and if $\sum_{i\in I'} m_i\gamma\circ\varphi_{x_i}(1) = \sum_{i\in I'} s_i\gamma\circ\varphi_{x_i}(1)$, then $\sum_{i\in I'}(m_i - s_i)\gamma\circ\varphi_{x_i}(1) = 0$ and hence $m_j - s_j = \sum_{\substack{i\in I' \\ i\neq j}}(m_i - s_i) \in R\gamma\circ x_j \cap \sum_{\substack{i\in I' \\ i\neq j}} R\gamma\circ x_i = 0$, thus $R^{(Y)} = \oplus_{i\in I} R\gamma\circ x_i$ has a basis $\{\varphi_{x_i}(1): i \in I\}$ and so $R^{(Y)}$ is free [6], define an $R_\Gamma-$homomorphism $\Psi: R^{(Y)} \to M$ by $\Psi\left(\sum r_i\gamma\circ\varphi_{x_i}(1)\right) = \sum r_i\gamma\circ x_i$, since for each $m \in M$, $m = \sum_{i\in I} r_i\gamma\circ x_i$ then Ψ is $R_\Gamma-$epimorphism.

Lemma 2.9. Every abelian group can be embedded in Z_Z- divisible group.
Proof. By lemma(2.8) M as abelian group is epimorphic image of a free Z_Z- module F, so there exists an Z_Z- epimorphism $\varphi: F \to M$,by [1,proposition 5.20] $F/_{Ker(\varphi)} \cong M$, hence there exists an Z_Z- isomorphism $\beta: F/_{Ker(\varphi)} \to M$, if Y is a basis for F , then $F = \oplus_{b\in Y} Z.1.b$, let $D = Q^{(Y)} = \oplus_{b\in Y} Q.Z.b$, since $Q.Z.b \cong Q$, then by lemma(2.5)Q is Z_Z- divisible and F is a subgroup of D , hence by Corollary(2.6) $\overline{D} = D/Ker(\varphi)$ is Z_Z- divisible , then $i\beta^{-1}: M \to \overline{D}$ is Z_Z- monomorphism where i: $F/_{Ker(\varphi)} \to \overline{D}$ inclusion map.

Lemma 2.10. Every injective $R_\Gamma-$ module M is $R_\Gamma-$ divisible.
Proof. Assume that $a \in M$, for each non $R_\Gamma-$ zero-divisor $r \in R$ and $\gamma(\neq 0) \in \Gamma$, let $I = R\gamma r$, hence I is an ideal of R, define$R_\Gamma-$ homomorphism $f: I \to M$ by $f(s\gamma r) = s\gamma\circ a$ for each $r \in R$, since M is injective, then by theorem(1.8) there exists $a' \in M$ such that $f(x) = x\gamma\circ a'$ for each $x \in I$, define an $R_\Gamma-$ submodule $N = R\gamma\circ a'$ of M and $g: N \to M$ by $g(s\gamma\circ a') = s\gamma a'$, g is well define, since r is not $R_\Gamma-$ zero-divisor and $R_\Gamma-$ homomorphism, hence $a = 1\gamma\circ a = f(1\gamma\circ r) = f(r) = g(r\gamma\circ a') = s\gamma a'$, thus M is a $R_\Gamma-$ divisible.

Lemma 2.11. If R is principle ideal $\Gamma-$domain and M is an $R_\Gamma-$ module, then M is injective $R_\Gamma-$ module if and only if M is a $R_\Gamma-$ divisible.
Proof. Let $I = r\Gamma R$ and f any $R_\Gamma-$homomorphisms from I to M, since M is a $R_\Gamma-$divisible, then there exists $m \in M$ such that $f(r) = r\gamma m$ for fixed $\gamma \in \Gamma$, define $g: R \to M$ by $g(s) = s\gamma m$ for each $s \in R$.

Ameri and Sadeghi [1] proved if M is $R_\Gamma -$module, then $Hom_{R_\Gamma}(R,M)$ is $R_\Gamma -$module by $\cdot: R \times \Gamma \times Hom_{R_\Gamma}(R,M) \to Hom_{R_\Gamma}(R,M)$ such that $\cdot(r,\gamma,f) \mapsto r\gamma f$ where $(r\gamma f)(s) = f(r\gamma s)$. For each $s,r \in R, \gamma \in \Gamma$ and $f \in Hom_{R_\Gamma}(R,M)$, another proved in next example and we will used this example in next lemma.

Example 2.12. If M is a left $R_\Gamma -$ module , then $Hom_{R_\Gamma}(R,M)$ is a left unitary $R_\Gamma -$ module by $\cdot: R \times \Gamma \times Hom_{R_\Gamma}(R,M) \longrightarrow Hom_{R_\Gamma}(R,M)$ such that $\cdot(r,\gamma,f) \longmapsto r\gamma f$ where $(r\gamma f)(s) = f(s\gamma_\circ r\gamma 1)$, for each $s,r \in R, \gamma, \gamma_\circ \in \Gamma$ and $f \in Hom_{R_\Gamma}(M,N)$ where $1\gamma_\circ$ is the identity element, for each $r,s,r_1,r_2 \in R, \gamma, \beta \in \Gamma$ and $f \in Hom_{R_\Gamma}(R,M)$, then $(r\gamma f)(r_1 + r_2) = f\big((r_1 + r_2)\gamma_\circ r\gamma 1\big) = f(r_1\gamma_\circ r\gamma 1) + f(r_2\gamma_\circ r\gamma 1) = (r\gamma f)(r_1) + (r\gamma f)(r_2)$ and $(r\gamma f)(s\beta r_2) = f((s\beta r_2)\gamma_\circ r\gamma 1) = f(s\beta(r_2\gamma_\circ r\gamma 1)) = s\beta(r\gamma f)(r_2)$, thus $r\gamma f \in Hom_{R_\Gamma}(R,M)$, easy to verify that $Hom_{R_\Gamma}(R,M)$ is $R_\Gamma -$ module with unitary $1\gamma_\circ$.

Lemma 2.13. If M is $R_\Gamma -$ divisible , then $Hom_{z_z}(R,M)$ is injective $R_\Gamma -$module.
Proof. For any $R_\Gamma -$ submodule A of B and $R_\Gamma -$ homomorphism f from A to $Hom_{z_z}(R,M)$, define $\varphi: Hom_{z_z}(R,M) \to M$ by $\varphi(h) = (h)(1)$ if we regarded f and φ only as $Z_Z -$ homomorphism, then there exists an $Z_z -$homomorphism $\theta: B \to M$ such that $\varphi f = \theta_{|A}$. Define a $R_\Gamma -$homomorphism $g: B \to Hom_{z_z}(R,M)$ by $\big(g(b)\big)(r) = \theta(r\gamma \circ b)$ for each $r \in R, \gamma_\circ \in \Gamma$and $b \in B$, then $\big(g(a)\big)(r) = \theta(r\gamma_\circ a) = \varphi f(r\gamma_\circ a) = \varphi(f(r\gamma_\circ a) = (f(r\gamma_\circ a))(1) = \big(r\gamma_\circ f(a)\big)(1)$, by example(2.12) $\big(r\gamma_\circ f(a)\big)(1) = f(a)(1\gamma_\circ r\gamma_\circ 1) = f(a)(r)$, so $f = g_{|A}$, therefore $Hom_{z_z}(R,M)$ is injective $R_\Gamma -$ module.

The following our main result show that there is enough gamma injective $R_\Gamma -$ module.

Theorem 2.14. Every $R_\Gamma -$ module M can be embedding in injective $R_\Gamma -$ module
Proof. Assume M is an $R_\Gamma -$ module, then by lemma (2.9) M as abelian group can be embedded in $Z_Z -$divisible group M', so there exists an $R_\Gamma -$monomorphism $\alpha: M \to M'$, define an $R_\Gamma -$monomorphism $\beta: Hom_{Z_Z}(R,M) \to Hom_{Z_Z}(R,M')$ by $\beta(f) = \alpha f$ for each $f \in Hom_{Z_Z}(R,M)$. Now define an $R_\Gamma -$monomorphism $\varphi: M \to Hom_{Z_Z}(R,M)$ by $[\varphi(e)](r) = r\gamma_\circ e$, for each $r \in R$ where $\gamma_\circ \in \Gamma$, then $\beta\varphi: M \to Hom_{Z_Z}(R,M')$ is an $R_\Gamma -$monomorphism , since M' is $Z_Z -$divisible , then by lemma(2.11) $Hom_{Z_Z}(R,M')$ is injective $R_\Gamma -$ module.

3. Essential extension R_Γ −module

In this part we extend the concept essential extension from category of $R -$ module to the category of $R_\Gamma -$ module and investigate their properties.

Definition 3.1. Let N be an $R_\Gamma -$ submodule of an $R_\Gamma -$ module M , we say that N is an ***essential extension*** of M if every nonzero $R_\Gamma -$ submodule of M has nonzero intersection with N. We also say that N ***essential $R_\Gamma -$ submodule*** (or ***large $R_\Gamma -$ submodule***) of M and write $N \leq_e M$, in this case M is essential extention of N. Any $R_\Gamma -$ module is always an essential extension of itself, this essential extension is called

trivial. Other essential extensions are called ***proper***. The field of all rational numbers Q considered as a $Z_Z -$ module is an essential extension of Z.

In above definition to see $R_\Gamma -$ submodule N of $R_\Gamma -$ module M is essential , it is enough to show that any nonzero cyclic $R_\Gamma -$ submodule of M has nonzero intersection with N as shows in the following proposition:

Proposition 3.2. An $R_\Gamma -$ submodule N of $R_\Gamma -$ module M is essential if and only if for each nonzero element m in M, there is $r_1, r_2, \dots, r_n \in R$ and $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$ such that $\sum_{i=1}^{n} r_i \gamma_i m(\neq 0) \in N$.

Proof. Assume that $N \leq_e M$ and $m(\neq 0) \in M$, then there exists $x(\neq 0) \in \langle m \rangle \cap N$, so $x = \sum_{i=1}^{n} r_i \gamma_i m(\neq 0) \in N$. Conversely, let W be nonzero $R_\Gamma -$ submodule of M and $w_\circ(\neq 0) \in W$, then there is $r_1, r_2, \dots, r_n \in R$ and $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$ such that $\sum_{i=1}^{n} r_i \gamma_i w_\circ(\neq 0) \in N$, hence $0 \neq \langle w_\circ \rangle \cap N \subseteq W \cap N$.

The proof of the following lemma similar to those on modules.

Lemma 3.3. Let A,B and C be $R_\Gamma -$ modules with $A \leq B \leq C$, then:

1- $A \leq_e C$ if and only if $A \leq_e B$ and $B \leq_e C$.
2- $A \leq_e B \leq C$ and $A' \leq_e B' \leq C$, then $A \cap A' \leq_e B \cap B'$.
3- If $f: C \to B$ is an $R_\Gamma -$ homomorphism and $A \leq_e B$, then $f^{-1}(A) \leq_e C$.
4- If $\{A_\lambda : \lambda \in \Lambda\}$ family of $R_\Gamma -$ modules and if for each $\in \Lambda$, A_λ has essential extension B_λ, then $\bigoplus_{\lambda\in\Lambda} B_\lambda$ is essential extension of $\bigoplus_{\lambda\in\Lambda} A_\lambda$.

Proposition 3.5. An $R_\Gamma -$ module M is injective if and only if M has no proper essential extensions.

Proof. Assume that M is injective and E is a proper essential extension of M, then from Proposition(1.9) M is direct summand of E, hence there exists an $R_\Gamma -$ submodule N such that $E = M \oplus N$ contradiction. Conversely, by theorem(2.14) M can be embedded in injective $R_\Gamma -$ module Q, define $\Omega = \{S \leq Q : S \cap M = 0\}$ then $0 \in \Omega \neq \phi$, Ω partially order set with respect to inclusion, then by Zorn's lemma Ω has maximal member say N, we shall show that $Q = M + N$. Assume $Q \neq M + N$ then $M + N \subsetneq Q$ so $(M+N)/N \subsetneq Q/N$, consider $0 \neq X/N \subsetneq Q/N$, then $N \subset X$ and $N \neq X$, since N maximal of Ω we have $M \cap X \neq 0$ and $M \cap X \nsubseteq N$ therefore $N \subsetneq N + (M \cap X) = X \cap (M+N)$, so there exists $e \in X \cap (M+N)$ and $e \notin N$, hence $e + N \in X/N$ and $e + N \in M + N/N$ but $e + N \neq N$, therefore $e + N(\neq 0) \in X/N \cap (M+N)/N$, so $(M+N)/N \leq_e Q/N$, since $M/(M \cap N) \cong (M+N)/N \leq_e Q/N$ but M has no proper essential extension so $(M+N)/N = Q/N$ hence $Q = M \oplus N$, thus M is injective.

The following lemma show that every essential extension of $R_\Gamma -$ module M contining in injective gamma extension of M.

Proposition 3.6. Let M be an $R_\Gamma -$ module, E an essential extension of M and Q an injective extension of M, then E can be embedding in Q.

Proof. Let $i_1 : M \to E$ and $i_2 : M \to Q$ are inclusions, then there exists $R_\Gamma -$ homomorphism $\theta : E \to Q$ such that $\theta i_1 = i_2$, so $M \cap ker\theta = 0$, since E essential extension of M, then $ker\theta = 0$ thus θ is $R_\Gamma -$ monomorphism.

Definition 3.7. Let E be an extension $R_\Gamma -$ module of an $R_\Gamma -$ module M, then E is said to be a ***maximal essential extension*** of M if:

1- E essential extension of M.
2- If E' is a proper extension of E, then E' is not essential extension of M.

Definition 3.8. Let Q be an extension $R_\Gamma -$ module of an $R_\Gamma -$ module M, then Q is said to be a ***minimal injective extension*** of M if:

1- Q is injective.
2- If Q' is a proper $R_\Gamma -$ submodule of Q which contains M, then Q' is not injective.

Proposition 3.9. Let M be an $R_\Gamma -$module and Q be an injective extension $R_\Gamma -$ module of M, then Q has an $R_\Gamma -$ submodule E which is maximal essential extension of M.

Proof. Let $\Omega = \{N \leq Q: M \leq_e N \leq Q\}\}$, then $M \in \Omega \neq \phi$, Ω partially order set with respect to inclusion, then by Zorn's lemma Ω has maximal member say N', let K be an essential extention of N', then there exists an $R_\Gamma -$ monomorphism $\theta: K \to Q$ which extends the inclusion map $i: N' \to Q$, for each $L(\neq 0) \leq \theta(K)$, there exists $0 \neq x = \theta(x_\circ) \in L$, so $M \cap \langle x_\circ \rangle \neq 0$, let $m_\circ(\neq 0) \in M \cap \langle x_\circ \rangle$, hence $m_\circ = \sum_{i\in I} r_i \gamma_i x_\circ$, so $(0 \neq) m_\circ = \theta(m_\circ) = \theta(\sum_{i\in I} r_i \gamma_i x_\circ) = \sum_{i\in I} r_i \gamma_i \theta(x_\circ) = \sum_{i\in I} r_i \gamma_i x \in L$, so $M \cap L \neq 0$, so $\theta(K) \in \Omega$, by maximality of N' we have $\theta(K) = N'$, thus $K = N'$.

Proposition 3.10. Let M be an $R_\Gamma -$ module and E be an essential extension $R_\Gamma -$ module of M, the following statements are equivalent :

1- E is an essential injective extension of M.
2- E is a maximal essential extension of M.
3- E is a minimal injective extension of M.

Proof. (1)⇔(2) by proposition(3.6). (2)⇒(3) Assume (2) then E has no proper essential extension and by proposition(3.5) E is injective.(3)⇒(2) Assume (3) by proposition(3.9) E has an $R_\Gamma -$ submodule E' which is maximal essential extension of M, so injective, hence $E = E'$, then E is maximal essential extension.

Definition 3.11. Any $R_\Gamma -$ module satisfying the conditions of proposition(3.10) is called an ***injective gamma hall*** (or ***injective gamma envelope***) of M and we use $E(M)$ to stand for it. The injective hull always exist for any $R_\Gamma -$ module which is unique up to isomorphism as we show .

Proposition 3.12. Every $R_\Gamma -$ module M has an injective gamma hull which is unique up to isomorphism.

Proof. M can be embedded in injective $R_\Gamma -$ module Q and by proposition(3.9) Q has maximal essential extension $R_\Gamma -$submodule E, by proposition(3.10) $E=E(M)$. Now we shall prove the uniqueness of E up to isomorphism. Let E and E_1 are two injective gamma hulls of M, then by proposition(3.6) there exists a $R_\Gamma -$ monomorphism $\theta: E \to E_1$, so $E = Im(\theta) \subseteq E_1$, from injectivity of E and by proposition(1.9) $E_1 = Im(\theta) \bigoplus N$. Since $M \subseteq Im(\theta)$, then $M \cap N = 0$ but E_1 essential extension of M we have $N=0$, hence $Im(\theta) \cong E_1$, that means θ is $R_\Gamma -$ epimorphism, thus θ is $R_\Gamma -$ isomorphism.

Proposition 3.13. An $R_\Gamma -$ module M is $N -$ injective $R_\Gamma -$ module if and only if $f(N)$ is $R_\Gamma -$ submodule of M for every $f \in Hom_{R_\Gamma}(E(N), E(M))$.

Proof. Let X be an $R_\Gamma -$ submodule of N and $f: X \to M$ is an $R_\Gamma -$homomorphism, since *E(M)* is injective, then f extended to an $R_\Gamma -$homomorphism $\varphi: N \to E(M)$ and by assumption $\varphi(N)$ is an $R_\Gamma -$ submodule of M, hence $\varphi: N \to M$ extends f, therefore M is N-injective $R_\Gamma -$ module. Conversely , for any $f \in Hom_{R_\Gamma}(E(N), E(M))$, define $X = \{a \in N: f(a) \in M\}$, for each $r \in R$, $\gamma \in \Gamma$ and $a \in X$, $r\gamma a \in N$ and $f(r\gamma a) = r\gamma f(a) \in M$, so X is an $R_\Gamma -$ submodule of N , since M is $N -$ injective , then $f_{|X}$ can extended to an $R_\Gamma -$homomorphism $g: N \to M$. We claim that $M \cap (g - f)(A) = 0$, let $m \in M$ and $a \in A$ such that $m = (g - f)(a)$, then $f(a) = g(a) - m \in M$, thus $a \in X$ and so $m = g(a) - f(a) = f(a) - f(a) = 0$, therefore $M \cap (g - f)(A) = 0$ but M is essential in *E(M)*, then $(g - f)(A) = 0$, hence $g(A) = f(A)$ is $R_\Gamma -$ submodule of M.

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