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b-Coloring of Corona and Shadow in C_n^k

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Abstract. A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has neighbor in all other color classes. Any such vertex is called as a colorful vertex. The b-chromatic numbers b(G) is the largest integer k such that G admits a b-coloring with k-colors. b(G) is same as φ(G). In this paper, we obtain b-chromatic number of corona graph and shadow graph in C_n^k.

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Keywords: b-coloring, b-continuous Corona graph and Shadow graph

1. Introduction

A k-vertex coloring of a graph G is a assignment of k-colors 1,2,…, k to the vertices. The coloring is proper if no two district adjacent vertices share the same color. A graph G is k-colorable if it has a proper k-vertex coloring [1]. The chromatic number $\chi(G)$ is the minimum number k such that G is k-colorable. Color of a vertex v is denoted by c(v). A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes. In other words each color class contains a color dominating vertex [2]. The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 was defined by Frucht and Harrary [4] as the graph G obtained by taking one copy of G_1 (which has P_1 points) and p_1 copies of G_2 and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2 [6] is the corona of two graphs and the corona of a single graph is defined as follows. The corona cor(G) of a graph G is that graph obtained from G by adding a new vertex w^1 to G for each vertex w of G and joining w^1 to w [5]. This may be viewed as corona graph of G with K_1 note that if G has order n and size m then cor(G) has order 2n and size m+n [5].

The shadow graph shad (G) of G is the graph with vertex Sect V(G) ∪ $\{u_1, u_2,\dots u_n\}$, where u_i is called the shadow vertex of v_i u_i is adjacent to u_j if v_i is adjacent to v_j u_i adjacent to v_j if v_i is adjacent to v_j for $1 \leq i, j \leq n$ [5].

A graph is said to be a power of cycle, denoted by C_n^k. If $V(C_n^k)=\{v_0(=v_n), v_1, v_2,\dots, v_{n-1}\}$ and $E(C_n^k)=E^1 \cup E^2 \cup \dots \cup E^k$, where $E^i = \{(v_j, v_{(j+1)(\text{mod } n)} : 0 \leq j \leq n-1\}$ and $1\leq k\leq \lfloor\frac{n-1}{2}\rfloor$ [3].

The graph C_n^k is a 2k-regular graph for all $1\leq k \leq j \leq \lfloor\frac{n-1}{2}\rfloor$. We take $(v_o, \dots, v_{n-1})$ to be a cyclic order on the vertex set of G, and always perform modular operations on the vertex indices [3].

2. b-coloring of corona and shadow of C_n^k

The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 was defined by Frucht and Harary [4], as the graph G obtained by taking one copy of G_1 (which has p_1 points) and p_1 copies of G_2, and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2 [6].

The above definition is the corona of two graphs and the corona of a single graph is defined as follows: The corona cor(G) of a graph G is that graph obtained from G by adding a new vertex w′ to G for each vertex w of G and joining w′ to w [5]. Note that, if G has order n and size m, then cor(G) has order 2n and size m+n [5].

The b-chromatic number of corona graphs have been studied recently by Vernold Vivin and Venkatachalam [9].

Actually, they find the b-chromatic number on corona graph of any graph G with path P_n, cycle C_n and complete graph K_n.

Finally, they generalized the b-chromatic number on corona graph of any two graphs, each one on n vertices. They proved the following results:

Let φ: G→H be a covering projection from a graph G onto another graph H. If the graph H is b-colorable with k-Colors when so is G[1].

The b-chromatic number of Corona of two graph is denoted by $\varphi(G \circ P_n)$, $\varphi(G \circ C_n)$, $\varphi(G \circ K_n)$ etc.

Theorem 2.1. [9] Let G be a simple graph on n vertices.

Then

1. $\varphi(G \circ P_n) = n+1$ if $n \leq 3$ and $\varphi(G \circ P_n) = n$ if $n > 3$
2. $\varphi(G \circ C_n) = n$ for all $n > 3$
3. $\varphi(G \circ K_n) = n+1$
4. $\varphi(K_{1,n} \circ P_n) = n+1$

In this section, we find the b-coloring number of corona of $G = C_n^K$. Also we study the b-coloring of the shadow of G.

Example 2.2. The corona of C_8^3 given below.

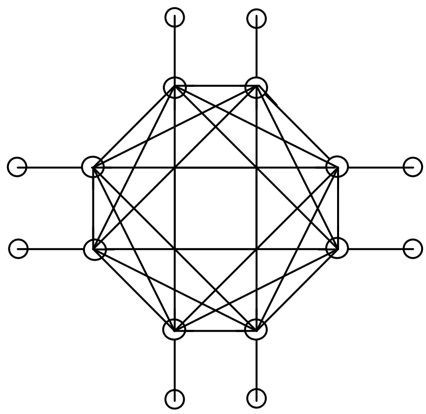

Figure 2.1:

Note that $|V(C_8^3)| = 8 = n$ and $|E(C_8^3)| = 24 = m$. The corona of C_8^3 has 2n=16 vertices and m+n=32 edges (as shown in the figure). That is, $|V(cor(C_8^3))| = 16$ and $|E(cor(C_8^3))\} = 32$.

Theorem 2.3. Let G be the corona of C_n^k and $n \geq 2(2k+2)$. Then G is b-colorable with 2k+1 colors and 2k+2 colors and φ(G)=2k+2.

Proof. Let $V(C_n^k)=(1,2,\ldots,n-1, n(=0)\}$ be the vertex set of C_n^k and $(a_1, a_2,\ldots, a_n\}$ be the newly added vertices to make corona of C_n^k such that $(i, a_i) \in E(G)$ for $1 \leq i \leq n$.

Claim 1.
G is b-colorable with 2k+2 colors.
Let n=k(2k+2)+g, where $h \geq 2$ and $1 \leq g \leq 2k+2$.
Color of the first h(2k+2) vertices of C_n^k are defined as follows:
c(i)=i(mod(2k+2)), for $1 \leq i \leq h(2k+2)$. The remaining g vertices are colored in two cases as given below:
Case 1. $1\leq g \leq k$

C(h(2k+2)+1) = k+1, c(h(2k+2)+2)=k+2,…, c(h(2k+2)+g)=k+g

Case 2. $k+1 \leq g \leq 2k + 2$
In this case, we color the vertex I with $h (2k+2)+1 \leq i \leq n$, as c(i)=I (mod (2k+2)).

The newly added vertices are colored as follows:
$c(a_1)=c(a_2)=..=c(a_{k+1}) = 2k +2$, $c(a_{(k+1)+1}) = 1$, $c(a_{(k+1)+2}) = 2,\ldots, c(a_{k+1+2k+1}) = 2k +1$. The remaining vertices colored as follows: $c(a_j)=c(j) \oplus_{2k+2} 1$. Note that these two colorings (discussed in Cases 1 and 2) are proper colorings with 2k+2 colors and the vertices k+1, k+2,…, (2k+1)+(k+1) are colorful vertices with colors k+1, k+2, …, 2k+2, 1,2, …, k respectively.
Claim 2. G is b-colorable with 2k+1 colors.
Let n=h(2k+1)+g, where $h \geq 2$ and $1 \leq g \leq 2k+1$
Coloring of the first h(2k+1) vertices of C_n^k are defined as follows: c(i)=i (mod(2k+1)). The remaining g vertices are colored in two cases as given below:
Case 3. $1\leq g \leq k$
C(h(2k+1)+1=k+1 c(h(2k+1)+2) = k+2, …, c(h(2k+1)+g)=k+g.
Case 4. $k+1 \leq g \leq 2k+1$

In this case, we color the vertex i with $h(2k+1)+1 \leq i \leq n$, as c(i)=i(mod(2k+1)). The newly added vertices are colored as follows: The color of vertex a_i may be colored with any color except c(i).

Note that these two colorings (discussed in cases 3 and 4) are proper coloring with 2k+1 colors and the vertices k+1, k+2, …, (2k+1)+k are colorful vertices with colors k+1, k+2,…, 2k+1, 1, 2, …, k respectively.

Note that Δ(G)=2k+1 and hence φ(G) ≤2k+2. We gave the method of coloring the graph G with 2k+2 colors. Hence φ(G)=2k+2.

Remark 2.4. Note that in the corona of C_n^k, the vertices 1,2, …., k+1 are mutually adjacent vertices and hence $\chi(G) \geq k+1$. Suppose k+1 does not divide n, then the graph is not b-colorable with k+1 colors. With this information, we propose the following conjecture.

Conjecture 2.5. If n is a multiple of k+1 and $n \geq 2 (2k+2)$, then the corona of C_n^k, is b-continuous.

Let G be a graph with vertex set $V(G)=\{v_1, v_2, \ldots, v_n\}$. The shadow graph shad(G) of G is the graph with vertex set $V(G) \cup \{u_1,u_2, \ldots, u_n\}$, where u_i is called the shadow vertex of v_i; u_i is adjacent to u_j if v_i is adjacent to v_j; u_i is adjacent to u_j if u_i is adjacent to u_j for $1 \le i, j \le n$[5].

Next we study the b-coloring of shadow graph of C_n^k. Here, we take the vertex set of C_n^k as $V(C_n^k) = \{b_1, b_2, \ldots, b_{n-1}, b_n(=a_0)\}$ and $\{a_1, a_2, \ldots, a_n\}$ be the newly added vertices such that a_i is the shadow of bi for each i. $1 \le i \le n$. Note that, if G is the shadow graph of C_n^k then $\Delta(G)= \delta(G)=4k+1$.

Remark 2.6. Let G be the shadow graph of C_n^k. The vertices a_i and b_i are adjacent to $a_{i\oplus n}1$, $a_{i\oplus n}2$, …, $a_{i\oplus n}k$, $a_{i\oplus n}(n-1)$, $a_{i\oplus n}(n-2)$…, $a_{i\oplus n}(n-k)$, $b_{i\oplus n}1$, $b_{i\oplus n}2$,…, $b_{i\oplus n}k$, $b_{i\oplus n}(n-1)$, $b_{i\oplus n}(n-2)$.,,, $b_{i\oplus n}(n-k)$.

a) We say that b_i' is are type 1 vertices and a_i's are type 2 vertices.
b) Note that each vertex has exactly 2k neighbors of type 1 and 2k neighbors of type 2.

Example 2.7. The shadow graph of C_6 is given below:
Here the vertices $\{b_1, b_2,\ldots, b_6\}$ is a cycle C_6 and a_i is the shadow of b_i for each i with $1 \le i \le 6$. Since a_2 is adjacent with a_1 and a_3, we have the following edges: (b_1, b_2), (b_3, b_2), (a_1, b_2), (a_3,b_2).

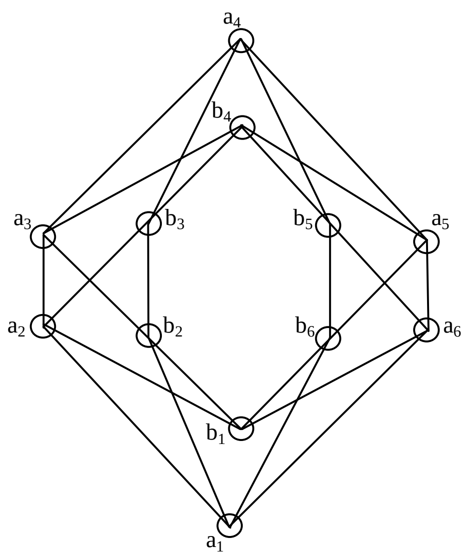

Figure 2.2:

Lemma 2.8. Let G be the shadow of C_n^k and n is a multiple of (4k+1)(k+1). Then G is b-colorable with 4k+1 colors and $\varphi(G)=4k+1$.

Proof. Let $V(C_n^k)=\{b_1, b_2,\ldots,b_{n-1}, b_n(=b_0)\}$ and $\{a_1, a_2, \ldots,a_n\}$ be the newly added vertices such that a_i is the shadow of b_i. If $\varphi(G)=4k+1$ and a_i (or b_i) is colorful, then a_i and b_i receive the same color for each i,$1 \le i \le n$ (since both the vertices have the same open neighborhood).

[For example, consider the following figure, the shadow of C_n^2. To make the vertex u as a colorful vertex with 4k+1=9 colors, we should color all the vertices adjacent with u by different colors, namely, 1,2,…, 9.

Without loss of generality, assume the neighbors of u are colored with different colors as shown in the figure.

Now the vertex v, which is the shadow of u, is adjacent with all the colors except 3. Hence u should be colored with color 3.

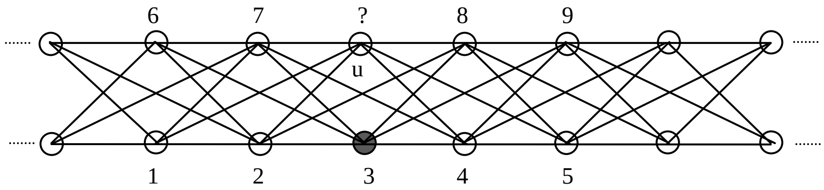

Figure 2.3:

Also any k+1 consecutive vertices and their shadows cannot have two colorful vertices with different colors.

Hence $n \geq (k+1)(4k+1)$. To prove the theorem, it remains to show that G is b-colorable with 4k+1 colors.

We first prove this for the graph H with (k+1) (4k+1) vertices.

The coloring of vertices of H is given below:

$c(a_{k+1}) = c(b_{k+1}) = 1$, $c(a_{2(k+1)}) = c(b_{2(k+1)}) = (2k+1)+1$,
$c(a_{3(k+1)}) = c(b_{3(k+1)}) = 2$, $c(a_{4(k+1)}) = c(b_{4(k+1)}) = (2k+1)+2$,
$c(a_{5(k+1)}) = c(b_{5(k+1)}) = 3, \ldots, c(a_{4k(k+1)}) = c(b_{4k(k+1)}) = 2k$,
$c(a_{(4k+1)(k+1)}) = c(b_{(4k+1)(k+1)}) = 2k+1$; the set of all vertices say, Box 1=$\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ are colored with different colors, namely, 2,3,…, 2k+1; the set of all vertices say, Box 2 = $\{a_{k+2}, a_{k+3}, \ldots, a_{2k+1}, b_{k+2}, b_{k+3}, \ldots, b_{2k+1}\}$ are colored with different colors, namely, 2k+2, 2k+3,…, 4k+1.

Note that the remaining vertices are colored by using the colored vertices of Box 1 and Box 2.

Note that, the vertices lying in Box 1 are actually the 2k vertices which are lying to the left side of the colored vertices a_{k+1} and b_{k+1}; and the vertices lying in Box 2 are actually the 2k vertices which are lying to the left side of the colored vertices $a_{2(k+1)}$ and $b_{2(k+1)}$.

Similarly for each i with $1 \leq i \leq 4k+1$, we denote the 2k vertices which are lying to the left side of the colored vertices $a_{i(k+1)}$ and $b_{i(k+1)}$ as Box i.

These boxes are colored as follows:

For each i with $3 \leq i \leq 4k+1$, the vertices in Box i are colored by adding 1 (the operation addition modulo 4k+1 is used here to add the colors) to the existing colors of Box i-2.

[Consider the b-coloring of C_n^2 with 4k+1 = 9 colors (as given in the following figure.)

Consider the Boxes 1,2,3,4. Box 1 is colored with colors 6,7,8 and 9. Box 2 is colored with 2,3,4 and 5. Now Box 3 is colored by adding 1 to the existing colors of Box 1, that is, 6+1=7, 7+1=8, 8+1=9 and 9+1=1.

Similarly, the Box 4 is colored by adding 1 to the existing colors of box 2, that is, 2+1=3, 3+1=4, 4+1=5 and 5+1=6.]

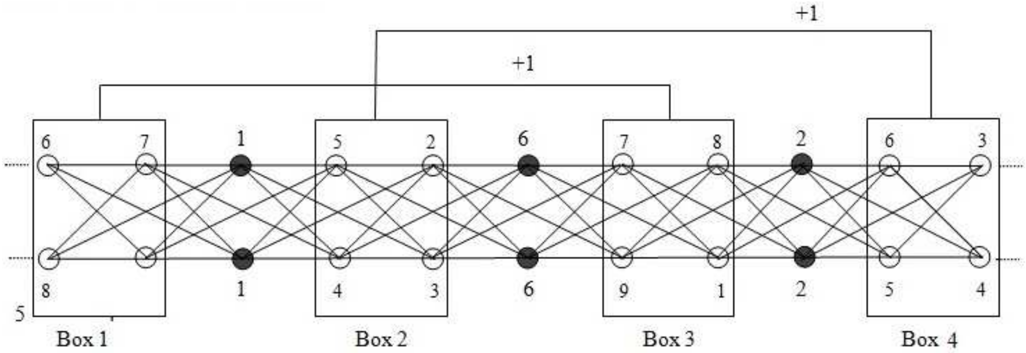

Figure 2.4:

Note that, the above discussed coloring is proper and for each i with $1 \leq i \leq 4k + 1$, the vertices $a_{i(k+1)}$ and $b_{i(k+1)}$ are colorful with color i.

Thus the above coloring is a b-coloring of the shadow graph of C_n^k and hence $\varphi(H) \geq 4k+1$.

Note that, if H is the shadow graph of C_n^k then $\Delta(H)= 4k$.

Hence $\varphi(H) \leq 4k+1$ and hence $\varphi(H)=4k+1$.

Now we illustrate the method of b-coloring the shadow of C_n^k with 4k+1 colors, where n is a multiple of (k+1) (4k+1).

Define a function $f:V(G) \to V(H)$ by $f(_{ai(\bmod((k+1)(4k+1)))}$, $f(b_i)=b_{i\ (\bmod((k+1)(4k+1)))}$ for each i with $1 \leq i \leq n$. Then f is a covering projection from G onto H. By Lemma [3] G is also b-colorable with 4k+1 colors.

Since $\Delta(G)=4k$, we can easily conclude that $\varphi(G)=4k+1$.

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