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Some Special Properties of I-rough Topological Spaces

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Abstract. This paper extends some essential topological properties in general topological spaces in to the I-rough topological spaces, the topology of the rough universe. I-rough compactness and I-rough Hausdorffness are introduced and several properties are investigated. I-rough compactness of subsets of I-rough Hausdorff spaces are studied. Also the paper establishes I-rough connectedness in an I-rough topological space.

Keywords: I-rough topological spaces; I-rough continuous functions; I-rough compactness; I-rough Hausdorffness; I-rough connectedness

Mathematics Subject Classification (2010): 54A05, 54D05, 54D30, 54D70, 03Exx

1. Introduction

Rough set theory was introduced by Pawlak [6]. There are many approaches in rough sets. Some reviews can be seen in [12, 13]. Several applications of rough set models can be seen in [7, 8]. Iwinski [1] introduced the set oriented view of rough set in an algebraic method. A topology on a non-empty set is a collection of subsets of it, satisfying certain axioms [5, 11]. Mathew & Sunil [3] introduced I-rough topological spaces. The notion of I-rough continuous functions can be seen in [4]. Some related works in generalized topological spaces and fuzzy topological spaces can be seen in [2, 9, 10]. This paper is an attempt to strengthen the topology of the rough universe by extending the concepts of compactness, Hausdorffness and connectedness of general topological spaces in to I-rough topological spaces.

2. Preliminaries

Some of the basic definitions for our further study need to be quoted before introducing the new concepts. Let U be any non-empty set and let β be a complete sub-algebra of the Boolean algebra 2^U of subsets of U. Then the pair (U,β) is called a rough universe [1]. Let (U,β) be a given fixed rough universe. Let R be a relation on β defined by $A=(A_1,A_2)\in R$ iff $A_1,A_2\in\beta$ and $A_1\subseteq A_2$. The elements of R are called rough sets and the elements of β are called exact sets [1]. In order to distinguish this definition of rough sets from Pawlak's definition, this rough set is named as an I-rough set [12]. The element $(X, X) \in R$ is identified with the element $X \in \beta$ and hence an exact set is a rough set in the sense of the above definition. But a rough set need not be exact. Set theoretic operators on the rough sets are defined component wise using ordinary set operations as follows [1]. Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be any two I-rough sets in the rough universe (U, β). Then,

$$X \cup Y = (X_1 \cup Y_1, X_2 \cup Y_2)$$

$$X \cap Y = (X_1 \cap Y_1, X_2 \cap Y_2)$$

$X \subseteq Y$ if $X \cap Y = X$ .That is $X \subseteq Y$ if $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$

$$X - Y = (X_1 - Y_2, X_2 - Y_1).$$

Hence $X^C = (U, U) - (X_1, X_2) = (U - X_2, U - X_1) = (X_2{}^C, X_1{}^C)$.

The above set operations on I-rough sets are named as I-rough union, I-rough intersection, I-rough inclusion, I-rough difference and I-rough complement respectively [3]. Let (U, β) be a fixed rough universe. Then a sub collection τ of R is an I-rough topology on (U, β) if the following 1, 2 and 3 hold [3].

1. $(\phi, \phi) \in \tau$ and $(U, U) \in \tau$
2. τ is closed under finite I-Rough intersection
3. τ is closed under arbitrary I-Rough union.

If τ be an I-rough topology on the rough universe (U, β). Then the triple (U, β, τ) is called an I-rough topological space [3]. An I-rough set (A_1, A_2) is an I-rough open set in an I-rough topological space (U, β, τ) if $(A_1, A_2) \in \tau$ and an I-rough set (A_1, A_2) is an I-rough closed set if its I-rough complement $A^C = (U - A_2, U - A_1)$ is I-rough open [3]. Let A be any subset of U. Then $\tau / A = \{G \cap A = (G_1 \cap A, G_2 \cap A) / G = (G_1, G_2) \in \tau\}$ is an I-rough topology on the rough universe $(A, \beta / A)$ induced by τ, where $\beta / A = \{X \cap A / X \in \beta\}$ is the complete sub-algebra β restricted to A. Then τ / A is called the relative I-rough topology on A or the subspace I-rough topology on A and $(A, \beta / A, \tau / A)$ is called an I-rough subspace of the I-rough topological space (U, β, τ) [3].

Let (X, β_1) and (Y, β_2) be any two rough universes and let $f : X \to Y$ be any function. Then the function f is an I- rough function if both f and f^{-1} maps exact sets on to exact sets. That is f is an I-rough function if $f(A) \in \beta_2, \ \forall A \in \beta_1$ and $f^{-1}(A) \in \beta_1, \ \forall A \in \beta_2$ [4]. Let (X, β_1, τ_1) and (Y, β_2, τ_2) are any two I-rough topological spaces and let $f : (X, \beta_1) \to (Y, \beta_2)$ be any I-rough function. Then f is an

I-rough continuous function if $f^{-1}(A) = \left(f^{-1}(A_1), f^{-1}(A_2)\right) \in \tau_1$ for every $A = (A_1, A_2) \in \tau_2$ [4]. An I-rough function $f:(X, \beta_1) \to (Y, \beta_2)$ is an I-rough embedding if f is one-one and both f and f^{-1} are I-rough continuous functions [4]. Also $f:(X, \beta_1) \to (Y, \beta_2)$ is an I-rough homeomorphism if f is one-one, onto and both f and f^{-1} are I-rough continuous functions [4].

3. I-rough compactness

In this section, the paper studies the compactness of the I-rough topological spaces. This is an attempt to generalize the concept of compactness in general topological space in to I-rough topological spaces.

Definition 3.1. A family $V = \{(A_{i_1}, A_{i_2}) : i \in I\}$ of I-rough subsets of a rough universe (X, β) is an I-rough covering of an I-rough set (E_1, E_2) if $(E_1, E_2) \subseteq \bigcup_{i \in I} (A_{i_1}, A_{i_2}) = \left(\bigcup_{i \in I} A_{i_1}, \bigcup_{i \in I} A_{i_2}\right)$. The I-rough covering V is finite I-rough covering if V contains only finitely many I-rough sets. Also the family V of I-rough subsets of a rough universe (X, β) is an I-rough covering of (X, β) if X is the I-rough union of all members of V.

Definition 3.2. Let (X, β, τ) be an I-rough topological space. Then an I-rough covering V of I-rough subsets of a rough universe (X, β) is an I-rough open covering if all the members of V are I-rough open sets.

Definition 3.3. Let (X, β) be a rough universe and let $U = \{(A_{i_1}, A_{i_2}) : i \in I\}$ and $V = \{(B_{i_1}, B_{i_2}) : i \in J\}$ are any two I-rough covering of (X, β). If for each $i \in I$, $(A_{i_1}, A_{i_2}) = (B_{j_1}, B_{j_2})$ for some $j \in J$, then the I-rough covering
$U = \{(A_{i_1}, A_{i_2}) : i \in I\}$ is an I-rough sub covering of the I-rough cover
$V = \{(B_{i_1}, B_{i_2}) : i \in J\}$.

Definition 3.4. An I-rough topological space (X, β, τ) is I-rough compact if for every I-rough open covering of (X, β) has a finite I-rough sub covering.

Definition 3.5. Let (X, β, τ) be an I-rough topological space and let Y be any subset of X. Then Y is I-rough compact if the relative I-rough subspace $(Y, \beta/Y, \tau/Y)$ is I-rough compact.

Remark 3.1. Next theorem relates the I-rough compactness of an I-rough subspace $(Y, \beta/Y, \tau/Y)$ of an I-rough topological space (X, β, τ) to the I-rough topology τ of (X, β, τ).

Theorem 3.1. Let (X, β, τ) be an I-rough topological space and let Y be any subset of X. Then Y is I-rough compact iff for each I-rough open covering $\{(A_{i_1}, A_{i_2}): i \in I\}$ of Y such that for every $i \in I$, $(A_{i_1}, A_{i_2}) \in \tau$, there is a finite I-rough sub covering of Y.

Proof: Let (X, β, τ) be an I-rough topological spaces and let Y be any subset of X. Suppose Y is I-rough compact. Let $\{(A_{i_1}, A_{i_2}): i \in I\}$ be an I-rough open covering of Y using I-rough open sets of (X, β, τ). Which implies that for every $i \in I$, $(A_{i_1}, A_{i_2}) \in \tau$. Then $(A_{i_1}, A_{i_2}) \cap (Y, Y) \in \tau/Y$ for every $i \in I$. Hence $\{(A_{i_1}, A_{i_2}) \cap (Y, Y): i \in I\}$ is an I-rough open covering of Y using I-rough open sets in the relative I-rough topology of $(Y, \beta/Y, \tau/Y)$. Since Y is I-rough compact, this I-rough covering has a finite I-rough open covering of Y. Let this finite I-rough sub covering be $\{(A_{j_1}, A_{j_2}) \cap (Y, Y): j = 1, 2, 3, \cdots, n\}$. Then clearly $\{(A_{j_1}, A_{j_2}): j = 1, 2, 3, \cdots, n\}$ covers Y. Hence for each I-rough open covering $\{(A_{i_1}, A_{i_2}): i \in I\}$ of Y such that for every $i \in I$, $(A_{i_1}, A_{i_2}) \in \tau$, there is a finite I-rough sub covering of Y.

Conversely suppose for each I-rough open covering $\{(A_{i_1}, A_{i_2}): i \in I\}$ of Y such that for every $i \in I$, $(A_{i_1}, A_{i_2}) \in \tau$, there is a finite I-rough sub covering of Y. Let $\{(A_{i_1}, A_{i_2}): i \in I\}$ be an I-rough open covering of Y using I-rough open sets in the relative I-rough topology on Y. That is for every $i \in I$, $(A_{i_1}, A_{i_2}) \in \tau/Y$. Hence $(A_{i_1}, A_{i_2}) = (G_{i_1}, G_{i_2}) \cap (Y, Y)$ where $(G_{i_1}, G_{i_2}) \in \tau$ for every $i \in I$. Thus $\{(G_{i_1}, G_{i_2}): i \in I\}$ is an I-rough open covering of Y using I-rough open sets in (X, β, τ). Then by our supposition, there is a finite I-rough sub covering $\{(G_{j_1}, G_{j_2}): j = 1, 2, 3, \cdots, n\}$ of Y. Then $\{(G_{j_1}, G_{j_2}) \cap (Y, Y): j = 1, 2, 3, \cdots, n\}$ is a finite I-rough sub covering of Y. Hence Y is I-rough compact.

Definition 3.6. Let (X, β, τ) be an I-rough topological space and let (Y_1, Y_2) be any I-rough set of the rough universe (X, β). Then (Y_1, Y_2) is I-rough compact set if for every I-rough open covering of (Y_1, Y_2) using I-rough open sets of (X, β, τ) has a finite I-rough sub covering.

Theorem 3.2. Let (X, β_1, τ_1) and (Y, β_2, τ_2) are any two I-rough topological spaces and let $f : (X, \beta_1) \rightarrow (Y, \beta_2)$ be any I-rough continuous function. Then if an I-rough set

(A_1, A_2) of the rough universe (X, β_1) is I-rough compact subset of (X, β_1, τ_1) then $f(A_1, A_2)$ is an I-rough compact subset of (Y, β_2, τ_2).

Proof: Let $f:(X,\beta_1)\to(Y,\beta_2)$ be any I-rough continuous function and let (A_1, A_2) be an I-rough set of the rough universe (X, β_1) is I-rough compact subset of (X, β_1, τ_1). Let $\{(G_{i_1}, G_{i_2}): i\in I\}$ be an I-rough open covering of $f(A_1, A_2)$ using I-rough open sets in (Y, β_2, τ_2). This implies that

$$f(A_1, A_2)=(f(A_1), f(A_2)) \subseteq \bigcup_{i\in I}(G_{i_1}, G_{i_2})=\left(\bigcup_{i\in I} G_{i_1}, \bigcup_{i\in I} G_{i_2}\right).$$

Hence, $(A_1, A_2)\subseteq \bigcup_{i\in I} f^{-1}(G_{i_1}, G_{i_2})$. Thus $\{f^{-1}(G_{i_1}, G_{i_2}): i\in I\}$ be an I-rough covering of (A_1, A_2). Since $f:(X,\beta_1)\to(Y,\beta_2)$ be an I-rough continuous function and $(G_{i_1}, G_{i_2})\in \tau_2$ for every $i\in I$, clearly $f^{-1}(G_{i_1}, G_{i_2}) \in \tau_1$ for every $i\in I$. Hence $\{f^{-1}(G_{i_1}, G_{i_2}): i\in I\}$ be an I-rough open covering of (A_1, A_2). Since (A_1, A_2) is I-rough compact set of (X, β_1, τ_1), it has a finite I-rough sub covering $\{f^{-1}(G_{i_1}, G_{i_2}): i=1, 2, 3, \cdots n\}$ of (A_1, A_2). That is $(A_1, A_2)\subseteq \bigcup_{i=1}^{n} f^{-1}(G_{i_1}, G_{i_2})$. Which implies that $f(A_1, A_2)=(f(A_1), f(A_2))\subseteq \bigcup_{i=1}^{n} f^{-1}(G_{i_1}, G_{i_2})$. Since $\{(G_{i_1}, G_{i_2}): i\in I\}$ be an arbitrary I-rough open covering of $f(A_1, A_2)$ the theorem follows by theorem 3.1.

Theorem 3.3. Let (X, β_1, τ_1) and (Y, β_2, τ_2) are any two I-rough topological spaces, where (X, β_1, τ_1) is I-rough compact and $f:(X,\beta_1)\to(Y,\beta_2)$ be any I-rough continuous and onto function. Then (Y, β_2, τ_2) is I-rough compact.

Proof: Since $f:(X,\beta_1)\to(Y,\beta_2)$ is an onto function, clearly $f(X, X)=(Y, Y)$. Then the proof follows from theorem 3.2.

Theorem 3.4. Let (X, β_1, τ_1) and (Y, β_2, τ_2) are I-rough homeomorphic I-rough topological spaces. Then (X, β_1, τ_1) is I-rough compact iff (Y, β_2, τ_2) is I-rough compact.

Proof: Proof follows directly from theorem 3.3 and the definition of I-rough homeomorphism.

Theorem 3.5. Let (X, β, τ) be an I-rough compact I-rough topological space then every I-rough closed subsets of (X, β, τ) are also I-rough compact.

Proof: Let (D_1, D_2) be any I-rough closed subset of the I-rough topological space (X, β, τ) where (X, β, τ) is I-rough compact. Let $\{(A_{i_1}, A_{i_2}): i\in I\}$ of (D_1, D_2) such that for every $i\in I$, $(A_{i_1}, A_{i_2})\in \tau$. Since (D_1, D_2) is an I-rough closed subset of the I-rough topological space (X, β, τ) implies $(X - D_2, X - D_1) \in \tau$. Now adjoin $(X - D_2, X - D_1)$ to $\{(A_{i_1}, A_{i_2}): i \in I\}$ to get an I-rough open covering of (X, β, τ). But since (X, β, τ) is I-rough compact it has a finite I-rough sub covering $\{(A_{j_1}, A_{j_2}): j = 1, 2, 3, \cdots, n\}$, which may or may not contains $(X - D_2, X - D_1)$. If it contains $(X - D_2, X - D_1)$, delete it from the finite I-rough sub covering to get a finite I-rough sub covering of $(X - D_2, X - D_1)^C = (D_1, D_2)$. Since $\{(A_{i_1}, A_{i_2}): i \in I\}$ is arbitrary, the theorem follows by theorem 3.1.

Remark 3.2. Next theorem characterizes the I-rough compactness of an I-rough topological space in terms of its I-rough closed sets instead of I-rough open sets.

Theorem 3.6. An I-rough topological space (X, β, τ) is I-rough compact iff whenever a family $\{(F_{\alpha_1}, F_{\alpha_2}), \alpha \in I\}$ of I-rough closed sets of (X, β, τ) such that $\left(\bigcap_{\alpha \in I} F_{\alpha_1}, \bigcap_{\alpha \in I} F_{\alpha_2}\right) = (\phi, \phi)$ then there is a finite subset of indices $\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\}$ such that $\left(\bigcap_{i=1}^{n} F_{\alpha_i 1}, \bigcap_{i=1}^{n} F_{\alpha_i 2}\right) = (\phi, \phi)$.

Proof: Suppose that (X, β, τ) be an I-rough topological space which is I-rough compact. Let $\{(F_{\alpha_1}, F_{\alpha_2}), \alpha \in I\}$ be a family of I-rough closed sets of (X, β, τ) such that $\left(\bigcap_{\alpha \in I} F_{\alpha_1}, \bigcap_{\alpha \in I} F_{\alpha_2}\right) = (\phi, \phi)$. Then $\bigcup_{\alpha \in I} (F_{\alpha 1}, F_{\alpha 2})^C = \left(\bigcap_{\alpha \in I} F_{\alpha_1}, \bigcap_{\alpha \in I} F_{\alpha_2}\right)^C = (\phi, \phi)^C = (X, X)$. Since each $(F_{\alpha 1}, F_{\alpha 2})$ is I-rough closed, $(F_{\alpha 1}, F_{\alpha 2})^C$ is I-rough open in (X, β, τ). Hence $\bigcup_{\alpha \in I} (F_{\alpha 1}, F_{\alpha 2})^C$ is an I-rough open covering of X, which is I-rough compact. Hence there is a finite I-rough sub covering $(F_{\alpha_1 1}, F_{\alpha_1 2})^C$, $(F_{\alpha_2 1}, F_{\alpha_2 2})^C$, $(F_{\alpha_3 1}, F_{\alpha_3 2})^C, ...,$ $(F_{\alpha_n 1}, F_{\alpha_n 2})^C$, such that $\bigcup_{i=1}^{n} (F_{\alpha_i 1}, F_{\alpha_i 2})^C = (X, X)$. Then $\left(\bigcup_{i=1}^{n} (F_{\alpha_i 1}, F_{\alpha_i 2})^C\right)^C = (X, X)^C = (\phi, \phi)$. But $\left(\bigcup_{i=1}^{n} (F_{\alpha_i 1}, F_{\alpha_i 2})^C\right)^C = \left(\bigcap_{i=1}^{n} F_{\alpha_i 1}, \bigcap_{i=1}^{n} F_{\alpha_i 2}\right) = (\phi, \phi)$.

Conversely suppose that for each family $\{(F_{\alpha_1}, F_{\alpha_2}), \alpha \in I\}$ of I-rough closed sets of (X, β, τ) such that $\left(\bigcap_{\alpha \in I} F_{\alpha_1}, \bigcap_{\alpha \in I} F_{\alpha_2}\right) = (\phi, \phi)$, there is a finite subset of indices

$\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\}$ such that $\left(\bigcap_{i=1}^{n} F_{\alpha_i 1}, \bigcap_{i=1}^{n} F_{\alpha_i 2}\right) = (\phi, \phi)$. Let $\{(A_{\beta_1}, A_{\beta_2}), \beta \in J\}$ be any I-rough open covering of (X, β, τ). Then $\bigcup_{\beta \in J}(A_{\beta_1}, A_{\beta_2}) = (X, X)$. Then $\{(A_{\beta_1}, A_{\beta_2})^C, \beta \in J\}$ be a family of I-rough closed sets such that

$$\left(\bigcup_{\beta \in J}(A_{\beta_1}, A_{\beta_2})\right)^C = (X, X)^C = (\phi, \phi).$$

That is $\left(\bigcup_{\beta \in J}(A_{\beta_1}, A_{\beta_2})\right)^C = \bigcap_{\beta \in J}(A_{\beta_1}, A_{\beta_2})^C = \left(\bigcap_{\beta \in J} A_{\beta 2}{}^C, \bigcap_{\beta \in J} A_{\beta 1}{}^C\right) = (\phi, \phi).$

Hence by our assumption there is a finite subset of indices $\{\beta_1, \beta_2, \beta_3, ..., \beta_n\}$ such that $\left(\bigcap_{i=1}^{n} A_{\beta_i 2}{}^C, \bigcap_{i=1}^{n} A_{\beta_i 1}{}^C\right) = (\phi, \phi)$. Then $\left(\bigcap_{i=1}^{n} A_{\beta_i 2}{}^C, \bigcap_{i=1}^{n} A_{\beta_i 1}{}^C\right)^C = (\phi, \phi)^C = (X, X)$. That is $\left(\bigcap_{i=1}^{n} A_{\beta_i 2}{}^C, \bigcap_{i=1}^{n} A_{\beta_i 1}{}^C\right)^C = \left(\left(\bigcap_{i=1}^{n} A_{\beta_i 1}{}^C\right)^C, \left(\bigcap_{i=1}^{n} A_{\beta_i 2}{}^C\right)^C\right) = \left(\bigcup_{i=1}^{n} A_{\beta_i 1}, \bigcup_{i=1}^{n} A_{\beta_i 2}\right) = (X, X)$.

That is $\bigcup_{i=1}^{n}(A_{\beta_i 1}, A_{\beta_i 2}) = (X, X)$. Hence the I-rough open covering $\{(A_{\beta_1}, A_{\beta_2}), \beta \in J\}$ has a finite I-rough sub covering. Since the I-rough open covering is arbitrary (X, β, τ) is I-rough compact.

4. I-rough Hausdorff spaces

This section introduces the Hausdorff property in I-rough topological spaces and studies the I-rough compactness of subsets of I-rough Hausdorff spaces.

Definition 4.1. An I-rough topological space (X, β, τ) is I-rough Hausdorff space if for any two distinct points x, y in X, there exist $(U_1, U_2), (V_1, V_2) \in \tau$, such that $x \in (U_1, U_2)$, $y \in (V_1, V_2)$ and $(U_1, U_2) \cap (V_1, V_2) = (U_1 \cap V_1, U_2 \cap V_2) = (\phi, \phi)$.

Theorem 4.1. Let (X, β, τ) be an I-rough Hausdorff space and $x \in X$. Let (F_1, F_2) be an I-rough set of the I-rough universe (X, β), which is an I-rough compact subset not containing x. Then there exist I-rough open sets (U_1, U_2) and (V_1, V_2) such that $x \in (U_1, U_2)$, $(F_1, F_2) \subset (V_1, V_2)$ and $(U_1, U_2) \cap (V_1, V_2) = (U_1 \cap V_1, U_2 \cap V_2) = (\phi, \phi)$.

Proof: Since (F_1, F_2) is I-rough compact subset not containing x, for each $y \in (F_1, F_2)$, there exist I-rough open sets (U_{y_1}, U_{y_2}) and (V_{y_1}, V_{y_2}) such that $x \in (U_{y_1}, U_{y_2})$, $y \in (V_{y1}, V_{y2})$ and $(U_{y_1}, U_{y_2}) \cap (V_{y1}, V_{y2}) = (U_{y1} \cap V_{y1}, U_{y2} \cap V_{y2}) = (\phi, \phi)$. Then the family $\{(V_{y1}, V_{y2}) : y \in (F_1, F_2)\}$ is an I-rough open covering of (F_1, F_2). Since (F_1, F_2) is I-rough compact, there is a finite I-rough sub cover for (F_1, F_2) say $\{(V_{y_1 1}, V_{y_1 2}), (V_{y_2 1}, V_{y_2 2}), ..., (V_{y_n 1}, V_{y_n 2})\}$. Let $(U_1, U_2) = \bigcap_{i=1}^{n} (U_{y_i 1}, U_{y_i 2})$ and $(V_1, V_2) = \bigcup_{i=1}^{n} (V_{y_i 1}, V_{y_i 2})$. Then clearly (U_1, U_2) and (V_1, V_2) are I-rough open sets and $x \in (U_1, U_2)$, $(F_1, F_2) \subset (V_1, V_2)$ and $(U_1, U_2) \cap (V_1, V_2) = (U_1 \cap V_1, U_2 \cap V_2) = (\phi, \phi)$.

Theorem 4.2. Let (X, β, τ) be an I-rough Hausdorff space and (Y_1, Y_2) be an I-rough compact subset of the rough universe (X, β). Then (Y_1, Y_2) is I-rough closed.

Proof: Let (Y_1, Y_2) be an I-rough compact subset of an I-rough Hausdorff space (X, β, τ). Then by theorem 4.1, for any $x \in (X, X) - (Y_1, Y_2)$ there exist I-rough open sets (U_1, U_2) and (V_1, V_2) such that $x \in (U_1, U_2)$, $(Y_1, Y_2) \subset (V_1, V_2)$ and $(U_1, U_2) \cap (V_1, V_2) = (U_1 \cap V_1, U_2 \cap V_2) = (\phi, \phi)$. In particular $(U_1, U_2) \cap (Y_1, Y_2) = (\phi, \phi)$, and $(U_1, U_2) \subseteq (X - Y_2, X - Y_1)$. That is $x \in (U_1, U_2) \subset (X - Y_2, X - Y_1)$. Thus $(X, X) - (Y_1, Y_2)$ is an I-rough neighbourhood of each of its points. Hence $(X, X) - (Y_1, Y_2)$ is I-rough open and then (Y_1, Y_2) is I-rough closed.

Theorem 4.3. Let (X, β, τ) be an I-rough compact Hausdorff space. Then an I-rough set (Y_1, Y_2) of the rough universe (X, β) is I-rough compact iff (Y_1, Y_2) is I-rough closed.

Proof: Proof follows from theorem 3.5 and theorem 4.2.

Theorem 4.4. Let $f : (X, \beta_1) \to (Y, \beta_2)$ be an I-rough continuous function from an I-rough compact topological space (X, β_1, τ_1) on to an I-rough Hausdorff topological space (Y, β_2, τ_2). Then an I-rough set (A_1, A_2) of the rough universe (Y, β) is I-rough closed in (Y, β_2, τ_2) iff $f^{-1}(A_1, A_2)$ is I-rough closed in (X, β_1, τ_1).

Proof: First suppose that the I-rough set (A_1, A_2) of the rough universe (Y, β) is I-rough closed in (Y, β_2, τ_2). Since $f : (X, \beta_1) \to (Y, \beta_2)$ be an I-rough continuous function, clearly $f^{-1}(A_1, A_2)$ is I-rough closed in (X, β_1, τ_1). Conversely suppose $f^{-1}(A_1, A_2)$ is I-rough closed in the I-rough compact topological space (X, β_1, τ_1). Then by theorem 3.5, $f^{-1}(A_1, A_2)$ is I-rough compact. Then by theorem 3.2, $f(f^{-1}(A_1, A_2)) = (A_1, A_2)$ is I-rough compact subset of (Y, β_2, τ_2). Then being an I-rough compact I-rough set of an I-rough Hausdorff topological space (Y, β_2, τ_2), (A_1, A_2) is I-rough closed in (Y, β_2, τ_2) by theorem 4.3.

Theorem 4.5. Let $f:(X,\beta_1)\to(Y,\beta_2)$ be an I-rough continuous function from an I-rough compact topological space (X,β_1,τ_1) on to an I-rough Hausdorff topological space (Y,β_2,τ_2). Then $f:(X,\beta_1)\to(Y,\beta_2)$ is I-rough closed mapping.

Proof: Let (A_1, A_2) be any I-rough closed subset of (X,β_1,τ_1). Then by theorem 3.5, (A_1, A_2) is I-rough compact subset of (X,β_1,τ_1). Then by theorem 3.2, $f(A_1, A_2)$ is an I-rough compact subset of (Y,β_2,τ_2). Now since (Y,β_2,τ_2) is I-rough Hausdorff, $f(A_1, A_2)$ is I-rough closed by theorem 4.3. Hence $f:(X,\beta_1)\to(Y,\beta_2)$ maps I-rough closed sets in to I-rough closed sets and hence it is an I-rough closed mapping.

Theorem 4.6. Let $f:(X,\beta_1)\to(Y,\beta_2)$ be an I-rough continuous bijective function from an I-rough compact topological space (X,β_1,τ_1) on to an I-rough Hausdorff topological space (Y,β_2,τ_2). Then $f:(X,\beta_1)\to(Y,\beta_2)$ is an I-rough homeomorphism.

Proof: Let $f:(X,\beta_1)\to(Y,\beta_2)$ be an I-rough continuous bijective function where (X,β_1,τ_1) is an I-rough compact topological space and (Y,β_2,τ_2) is an I-rough Hausdorff topological space. Let (G_1, G_2) be an I-rough open subset of (X,β_1,τ_1). Then $(G_1, G_2)^C=(X, X)-(\mathrm{G}_1, \mathrm{G}_2)$ is I-rough closed in (X,β_1,τ_1). Then since $f:(X,\beta_1)\to(Y,\beta_2)$ is a bijective I-rough function, $f\left((G_1, G_2)^C\right)= f((X, X)-(\mathrm{G}_1, \mathrm{G}_2))=(Y,Y)-f(G_1,G_2)=\ (f(G_1,G_2))^C$ is I-rough closed in (Y,β_2,τ_2) by theorem 4.5. So $f(G_1,G_2)$ is I-rough open in (Y,β_2,τ_2). Hence $f:(X,\beta_1)\to(Y,\beta_2)$ is an I-rough open function. Thus $f:(X,\beta_1)\to(Y,\beta_2)$ is I-rough continuous, I-rough open bijective function and hence an I-rough homeomorphism.

Corollary 4.1. Every I-rough continuous, one-one I-rough function from an I-rough compact topological space in to an I-rough Hausdorff topological space is an I-rough embedding.

Proof: Proof follows from theorem 4.6 and definition of I-rough embedding.

Theorem 4.7. Let (U,β,τ) be an I-rough topological space and let (X_1, X_2) and (Y_1, Y_2) are any two I-rough compact subsets of (U,β,τ). Then $(X_1, X_2)\cup(Y_1, Y_2)$ is again an I-rough compact subset of (U,β,τ).

Proof: Let $\{(A_{i_1}, A_{i_2}): i\in I\}$ be an arbitrary I-rough open covering of $(X_1, X_2)\cup(Y_1, Y_2)$ using I-rough open sets of (U,β,τ). Then clearly $\{(A_{i_1}, A_{i_2}): i\in I\}$ is an I-rough open covering of (X_1, X_2) and (Y_1, Y_2) using I-rough open sets of (U,β,τ). Since (X_1, X_2) and (Y_1, Y_2) are I-rough compact subspaces of (U,β,τ), there exist finite I-rough open sub covers $\{(A_{k_1}, A_{k_2}): k=1, 2, 3,\cdots, n\}$ covers (X_1, X_2) and

$$\left\{\left(A_{j_1}, A_{j_2}\right) : j = 1, 2, 3, \cdots, m\right\}$$

covers $\left(Y_1, Y_2\right)$, by theorem 3.1. Then clearly the union of these two sub collections covers $\left(X_1, X_2\right) \cup \left(Y_1, Y_2\right)$. Hence the union of these two sub collections is a finite I-rough open sub covering of $\left\{\left(A_{i_1}, A_{i_2}\right) : i \in I\right\}$. Since $\left\{\left(A_{i_1}, A_{i_2}\right) : i \in I\right\}$ is arbitrary I-rough open covering, $\left(X_1, X_2\right) \cup \left(Y_1, Y_2\right)$ is I-rough compact.

Theorem 4.8. Finite I-rough union of I-rough compact subspaces of an I-rough topological space is again I-rough compact.
Proof: The argument in theorem 4.7 is valid for a finite number of I-rough compact subspaces of an I-rough topological space.

Theorem 4.9. Arbitrary I-rough intersection of I-rough compact subsets of an I-rough Hausdorff topological space is again I-rough compact.
Proof: Let (U, β, τ) be an I-rough topological spaces and let $Y_i = \left(Y_{i1}, Y_{i2}\right)$ where $i \in I$ be any I-rough compact subsets of (U, β, τ). Then by theorem 4.3, each $Y_i = \left(Y_{i1}, Y_{i2}\right)$, where $i \in I$ is I-rough closed in (U, β, τ). Let $\left(Y_1, Y_2\right) = \bigcap_{i \in I} \left(Y_{i1}, Y_{i2}\right)$. Being arbitrary I-rough intersection of I-rough closed subsets of an I-rough topological spaces, $\left(Y_1, Y_2\right)$ is again I-rough closed. That is $\left(Y_1, Y_2\right)$ is an I-rough closed subset of an I-rough Hausdorff topological space (U, β, τ). Then by theorem 4.3, $\left(Y_1, Y_2\right)$ is I-rough compact.

Remark 4.1. The I-rough Hausdorff property in theorem 4.9 is very important. Since I-rough intersection of I-rough compact subspaces of an I-rough topological space need not be I-rough compact in general. Consider the following example.

Example 4.1. Let N be the set of natural numbers and let $x, y \notin N$ be any two real numbers. Let $U = N \cup \{x, y\}$. Let $\beta = P(U)$. Also let τ be the collection of I-rough sets of the rough universe (U, β) obtained by adding the following four I-rough sets to the discrete I-rough topology on N such as $(N, N \cup \{x\})$, $(N, N \cup \{y\})$, $(N, N \cup \{x, y\})$, $(N \cup \{x, y\}, N \cup \{x, y\})$. Then clearly τ is an I-rough topology on the rough universe (U, β). Let $A = (N, N \cup \{x\})$. Note that the only I-rough open sets containing $A = (N, N \cup \{x\})$ are $(N, N \cup \{x\})$, $(N, N \cup \{x, y\})$, $(N \cup \{x, y\}, N \cup \{x, y\})$. Hence any I-rough open covering of A has a finite I-rough open sub covering. Hence $A = (N, N \cup \{x\})$ is I-rough compact. Similarly $B = (N, N \cup \{y\})$ is also I-rough compact. But $A \cap B = (N, N)$ is not I-rough compact, since $\{(n, N), n \in N\}$ is an I-rough open covering of $A \cap B = (N, N)$ which does not have any finite I-rough open sub covering. Also note that $A^C = (N, N \cup \{x\})^C = \left((N \cup \{x\})^C, N^C\right) = (\phi, \{x\})$ and $B^C = (N, N \cup \{y\})^C = \left((N \cup \{y\})^C, N^C\right) = (\phi, \{y\})$

are not I-rough open in (U, β, τ). Hence A and B are not I-rough closed in (U, β, τ). Hence by theorem 4.3, the I-rough topological space (U, β, τ) is not I-rough Hausdorff.

5. I-rough connectedness

This section generalizes the connectedness property in general topological spaces in to the I-rough connectedness in an I-rough topological spaces.

Definition 5.1. An I-rough topological spaces (X, β, τ) is I-rough connected if it is impossible to find two non-empty exact I-rough open sets (A, A) and (B, B) such that $(X, X) = (A, A) \cup (B, B)$ and $(A, A) \cap (B, B) = (\phi, \phi)$. If an I-rough topological space is not I-rough connected it is called I-rough disconnected.

Remark 5.1. It is clear that the I-rough open sets in the definition of I-rough connected spaces can be replaced by I-rough closed sets. Hence an I-rough topological spaces (X, β, τ) is I-rough connected if it is impossible to find two non-empty exact I-rough closed sets (A, A) and (B, B) such that $(X, X) = (A, A) \cup (B, B)$ and $(A, A) \cap (B, B) = (\phi, \phi)$.

Theorem 5.1. An I-rough topological spaces (X, β, τ) is I-rough connected space iff the only exact I-rough clopen sets of (X, β, τ) are (ϕ, ϕ) and (X, X).

Proof: First suppose that (X, β, τ) is an I-rough connected space. If possible there exist an I-rough clopen subset (A, A) other than (ϕ, ϕ) and (X, X). Now (A, A) is I-rough open implies $(A, A)^C = (X - A, X - A)$ is I-rough closed. Again (A, A) is I-rough closed implies $(A, A)^C = (X - A, X - A)$ is I-rough open. That is (A, A) and $(X - A, X - A)$ are two exact I-rough open sets of (X, β, τ) such that $(X, X) = (A, A) \cup (X - A, X - A)$ and $(A, A) \cap (X - A, X - A) = (\phi, \phi)$. Which is a contradiction since (X, β, τ) is an I-rough connected space.

Conversely suppose that the only exact I-rough clopen sets of (X, β, τ) are (ϕ, ϕ) and (X, X). If (X, β, τ) is I-rough connected then there are two non-empty exact I-rough open sets (A, A) and (B, B) such that $(X, X) = (A, A) \cup (B, B)$ and $(A, A) \cap (B, B) = (\phi, \phi)$. Then clearly (A, A) and $(B, B) = (X - A, X - A)$ are exact I-rough clopen sets of (X, β, τ) other than (ϕ, ϕ) and (X, X). Which is a contradiction to our assumption.

Example 5.1. Let (X, β) be any I-rough universe. Then (X, β) disconnected in the discrete I-rough topology and (X, β) is connected in the indiscrete I-rough topology.

Theorem 5.2. The image of an I-rough connected space under an I-rough continuous function is I-rough connected.

Proof: Let (X,β_1,τ_1) and (Y,β_2,τ_2) are two I-rough topological spaces, where (X,β_1,τ_1) is I-rough connected and let $f:(X,\beta_1)\to(Y,\beta_2)$ be an I-rough continuous bijective function. If (Y,β_2,τ_2) is not I-rough connected then there are two non-empty exact I-rough open sets (A, A) and (B, B) of (Y,β_2,τ_2) such that $(Y,Y)=(A,A)\cup(B,B)$ and $(A,A)\cap(B,B)=(\phi,\phi)$. Since $f:(X,\beta_1)\to(Y,\beta_2)$ is an I-rough continuous bijective function $f^{-1}(A,A)$ and $f^{-1}(B,B)$ are exact I-rough open subsets of (X,β_1,τ_1) and $f^{-1}(A,A)\cup f^{-1}(B,B)$ $=\left(f^{-1}(A\cup B), f^{-1}(A\cup B)\right)$ $=\left(f^{-1}(Y), f^{-1}(Y)\right)$ $=(X,X)$.

Also $f^{-1}(A,A)\cap f^{-1}(B,B)$ $=\left(f^{-1}(A\cap B), f^{-1}(A\cap B)\right)=\left(f^{-1}(\phi), f^{-1}(\phi)\right)$ $=(\phi,\phi)$. Which is a contradiction to the fact that (X,β_1,τ_1) is I-rough connected. Hence our assumption is wrong and (Y,β_2,τ_2) is also I-rough connected.

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