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# Second Hankel Determinant for Starlike Functions Using Al-Oboudi Operator

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***Abstract.*** This paper focuses on attaining the upper bounds for the functional

$$|a_2a_4 - a_3^2|$$

belonging to the subclass $S^{**}$ defined in the unit disc $D = \{z: |z| < 1\}$ which is introduced here by means of Al-Oboudi Operator [2].

***Keywords:*** Starlike functions**,** Hankel determinant

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## 1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the unit disc D $= \{z: |z| < 1\}$. Let S denote the subclass of A that is univalent in D. Let $S^*$ denote the starlike subclass of $S$. It is well known that $f \in S^*$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in D.$$

Let $K$ denote the class of all functions $f \in A$ that are convex. Further, $f$ is convex if and only if $zf'$is starlike. It is well known that $K \subset S^* \subset S$.

In 1966, Pommerenke [9] stated the $qth$ Hankel determinant for $q \geq 1$ and $n \geq 0$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{(n+1)} & \cdots & a_{(n+q+1)} \\ a_{(n+1)} & \ddots & & \vdots \\ \vdots & & & \\ a_{(n+q-1)} & \cdots & & a_{(n+2q-2)} \end{vmatrix}$$

where $a_n$'s are the coefficient of various powers of $z$ in $f(z)$ defined by (1.1).

This determinant has also been considered by several authors. For example, Noor [8] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f$ given by (1.1) with bounded boundary.

Easily one can observe that the Fekete and Szegö [5] functional is $H_2(1)$. Fekete and Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ where $\mu$ is real and $f \in S$. We consider the Hankel determinant for the case $q = 2$ and $n = 2$.

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2| \tag{1.2}$$

Let $D^n$ be the Salagean differential operator [10]$D^n: A \to A, n \in N$, defined by

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)$$
$$\vdots$$
$$D^n f(z) = D(D^{n-1} f(z))$$

Let $n \in N$ and $\lambda \geq 0$. Let $f \in S$ denote with $D_\lambda^n: A \to A$ the Al-oboudi operator [2] defined by

$$D_\lambda^0 f(z) = f(z)$$
$$D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)$$
$$\vdots$$
$$D_\lambda^n f(z) = D_\lambda \left(D_\lambda^{n-1} f(z)\right).$$

Acu and Owa [1] considered the operator $D_\lambda^\beta$ which is defined as follows.

**Definition 1.1. [1]** Let $\beta, \lambda \in R, \beta \geq 0, \lambda \geq 0$ and $f(z)$ defined as (1.1). We denote by $D_\lambda^\beta$ the linear operator defined by $D_\lambda^\beta: A \to A$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^\beta a_j z^j. \tag{1.3}$$

We now define the following class $S^{**}$ .

**Definition 1.2.** Let $f(z)$ be given by (1.1). Then $f(z) \epsilon S^{**}$ if and only if

$$Re\left\{\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)}\right\} > 0, \ z \in D. \tag{1.4}$$

When $\lambda = 1, \ \beta = 0$ the subclass S**=S*. In the present paper, we obtain an upper bound for functional $|a_2 a_4 - a_3^2|$ in the class $S^{**}$.

## 2.Preliminary results

The following lemmas are required to prove our main results.

Let $P$ be the family of all functions $p$ analytic in the unit disk $D$ for which $Re\ p(z) > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.1}$$

**Lemma 2.1.** [4] Let $p \in P$, then $|c_k| \leq 2, k = 1,2, \ldots$ and the inequality is sharp.

**Lemma 2.2.** [6,7]

Let $p \in P$, then $2c_2 = c_1^2 + x(4 - c_1^2)$ (2.2)

and $4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2 c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2)$ (2.3)

for some $x, y$ such that $|x| \leq 1$ and $|y| \leq 1$.

## 3. Main results

**Theorem 3.1.**

Let $f \in S^{**}$, then $\left|a_2a_4 - a_3^2\right| \le \frac{1}{\lambda^2[1+2\lambda]^{2\beta}}$.

The inequality is sharp for the function f(z) = $z + \frac{1}{\lambda(1+2\lambda)^{\beta}} z^3 + \cdots$

**Proof:** Since $f \in S^{**}$, it follows from (2.1) that there exists $p \in P$such that

$$D_\lambda^{\beta+1} f(z) = \left(D_\lambda^{\beta} f(z)\right) p(z) \tag{3.1}$$

for some $z \in D$.

Equating coefficients in (3.1) yields,

$$a_2 = \frac{c_1}{[(1+\lambda)^{\beta+1} - (1+\lambda)^{\beta}]}$$

$$a_3 = \frac{c_2}{[(1+2\lambda)^{\beta+1} - (1+2\lambda)^{\beta}]} + \frac{(1+\lambda)^{\beta} c_1^2}{[(1+\lambda)^{\beta+1} - (1+\lambda)^{\beta}][(1+2\lambda)^{\beta+1} - (1+2\lambda)^{\beta}]}$$

$$a_4 = \frac{c_3}{[(1+3\lambda)^{\beta+1} - (1+3\lambda)^{\beta}]} + \frac{(1+\lambda)^{\beta} c_1 c_2}{[(1+\lambda)^{\beta+1} - (1+\lambda)^{\beta}][(1+3\lambda)^{\beta+1} - (1+3\lambda)^{\beta}]} + \frac{(1+2\lambda)^{\beta} c_1 c_2}{[(1+3\lambda)^{\beta+1} - (1+3\lambda)^{\beta}][(1+2\lambda)^{\beta+1} - (1+2\lambda)^{\beta}]} + \frac{(1+\lambda)^{\beta}(1+2\lambda)^{\beta} c_1^3}{[(1+\lambda)^{\beta+1} - (1+\lambda)^{\beta}][(1+2\lambda)^{\beta+1} - (1+2\lambda)^{\beta}][(1+3\lambda)^{\beta+1} - (1+3\lambda)^{\beta}]}$$

Substituting the values of $a_2, a_3$ and $a_4$, we have

$$\left|a_2a_4 - a_3^2\right| = \left| \frac{c_1c_3}{3\lambda^2(1+\lambda)^{\beta}(1+3\lambda)^{\beta}} - \frac{c_2^2}{4\lambda^2[1+2\lambda]^{2\beta}} + \frac{\left[2(1+2\lambda)^{2\beta} - 3(1+3\lambda)^{\beta}\right]c_1^4}{12\lambda^4(1+2\lambda)^{2\beta}(1+3\lambda)^{\beta}} + \frac{\left((1+2\lambda)^{2\beta+1} - 3\lambda(1+\lambda)^{\beta}(1+3\lambda)^{\beta}\right)c_1^2c_2}{12\lambda^4(1+\lambda)^{\beta}(1+2\lambda)^{2\beta}(1+3\lambda)^{\beta}} \right|$$

$$= \left| \frac{c_1c_3}{p_1} - \frac{c_2^2}{p_2} + \frac{c_1^4}{p_3} + \frac{c_1^2c_2}{p_4} \right| \tag{3.2}$$

where, $p_1 = 3\lambda^2(1+\lambda)^{\beta}(1+3\lambda)^{\beta}$, $p_2 = 4\lambda^2[1+2\lambda]^{2\beta}$

$p_3 = \frac{12\lambda^4(1+2\lambda)^{2\beta}(1+3\lambda)^{\beta}}{2(1+2\lambda)^{2\beta} - 3(1+3\lambda)^{\beta}}$ and $p_4 = \frac{12\lambda^4(1+\lambda)^{\beta}(1+2\lambda)^{2\beta}(1+3\lambda)^{\beta}}{(1+2\lambda)^{2\beta+1} - 3\lambda(1+\lambda)^{\beta}(1+3\lambda)^{\beta}}$

Substituting the values of $c_2$and $c_3$ using the equations (2.2) and (2.3) from lemma 2.2, we obtain

$$\left|a_2a_4 - a_3^2\right| = \left| \frac{c_1\{c_1^2 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2y(4-c_1^2)(1-x^2)\}}{4p_1} + \frac{[c_1^2 + x(4-c_1^2)]^2}{4p_2} + \frac{c_1^4}{p_3} + \frac{c_1^2[c_1^2 + x(4-c_1^2)]}{2p_4} \right|$$

Suppose that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying the triangle inequality with $\rho = |x| \leq 1$, we get

$$\left|a_2 a_4 - a_3^2\right| \leq \left\{\frac{c\{c^3 + 2c(4-c^2)\rho - c(4-c^2)\rho^2 + 2(4-c^2)(1-\rho^2)\}}{4p_1} + \frac{\{c^2 + \rho(4-c^2)\}^2}{4p_2} + \frac{c^4}{|p_3|} + \frac{c^2[c^2 + \rho(4-c^2)]}{2|p_4|}\right\}$$

$$\leq k_1 c^4 + k_2 c(4-c^2) + k_3 \rho c^2(4-c^2) + (4-c^2)\rho^2\left[\frac{c^2-2c}{4p_1} + \frac{4-c^2}{4p_2}\right]$$

$$= k_1 c^4 + k_2 c(4-c^2) + k_3 \rho c^2(4-c^2) + (4-c^2)\rho^2\left[k_4 c^2 - \frac{c}{2p_1} + \frac{1}{p_2}\right]$$

$$= F(\rho) \text{ for } \rho > 0.$$

$k_1 = \frac{1}{4p_1} + \frac{1}{4p_2} + \frac{1}{|p_3|} + \frac{1}{2|p_4|}$, $k_2 = \frac{1}{2p_1}$, $k_3 = \frac{1}{2p_1} + \frac{1}{4p_2} + \frac{1}{2|p_4|}$ and $k_4 = \frac{1}{4}\left\{\frac{1}{p_1} - \frac{1}{p_2^2}\right\}$.

Differentiating $F(\rho)$, we get

$$F'(\rho) = k_3 c^2(4-c^2) + 2\rho(4-c^2)\left[k_4 c^2 - \frac{c}{2p_1} + \frac{1}{p_2}\right]$$

We note that $F'(\rho) > 0$ and consequently $F$ is an increasing function of $\rho$ on a closed interval [0,1]. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0,1]$.

$$F(\rho) \leq k_1 c^4 + k_2 c(4-c^2) + k_3 c^2(4-c^2) + (4-c^2)\left[k_4 c^2 - \frac{c}{2p_1} + \frac{1}{p_2}\right]$$

$$= G(c)$$

then $G'(c) = -c\left\{\left|\frac{2}{p_2} - 8k_3 - 8k_4\right| + 4c^2|k_3 + k_4 - k_1|\right\} = 0$

implies $c = 0$. We observe that $G''(c) \leq 0 \; for \; 0 \leq c \leq 2$ and $G(c)$ has real critical point at $c = 0$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Therefore the maximum of $G(c)$ occurs at $c = 0$. Hence, $\left|a_2 a_4 - a_3^2\right| \leq \frac{1}{\lambda^2[1+2\lambda]^{2\beta}}$.

By setting $c_1 = 0$, and x= 1 in (2.1) and (2.2) we find that $c_2 = 2$ and $c_3 = 0$. Using these in (3.2), we get the equality

$$\left|a_2 a_4 - a_3^2\right| = \frac{1}{\lambda^2[1+2\lambda]^{2\beta}}$$

The inequality is sharp for the function f(z) $= z + \frac{1}{\lambda(1+2\lambda)^\beta} z^3 + \cdots$

**Corollary 2.2.** [3] When $\lambda = 1, \; \beta = 0$ then it reduces to [3] , $\left|a_2 a_4 - a_3^2\right| \leq 1$.

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