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Common Fixed Point Theorem in Complex Valued Metric Spaces Satisfying Rational Inequality

Antima Sindersiya[1*], Aklesh Pariya[2], Nirmala Gupta[3]* and *V. H. Badshah[1]

[1]School of Studies in Mathematics, Vikram University, Ujjain (M.P.), India.
[2]Department of Applied Mathematics and Statistics, Medi-Caps University
Indore (M.P.), India.
[3]Department of Mathematics, Govt. Girls P. G. College, Ujjain (M.P.), India.
Corresponding Author. Email: *antimakumrawat@gmail.com

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Abstract. In this paper, we prove a common fixed point theorem in complex valued metric spaces satisfying rational inequality using compatible mappings. Our result generalized the recent results of Nashine et al. [5] and Azam et al. [1].

Keywords: Common fixed point, compatible mapping, continuity, complex valued metric spaces.

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1. Introduction

Azam et al. [1], introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed the definition of a cone metric space banks on the underlying Banach space which is not a division Ring. However, in complex valued metric spaces, one can study improvements of a host of result of analysis involving division. One can refer related results in [3,9]. Jungck [4] generalized the concept of weak commuting mapping given by Sessa [8], by introducing the concept of compatible mappings. Many authors [2, 6, 7] proved fixed point theorems for compatible mapping in different types.

The aim of paper is to prove a common fixed point theorem using compatible mappings and continuity in complex valued metric spaces.

2. Basic definitions and preliminaries

The following definition is recently introduced by Azam et al. [1].

We recall some notations and definitions that will be utilize in our subsequent discussion.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(Z_2)$, $Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \precsim z_2$, $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 2.1. [1] Let X be a nonempty set, whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \rightarrow \mathbb{C}$, satisfies the following conditions:

(d_1) $0 \precsim d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d_3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X,d) is called a complex valued metric space.

Definition 2.2. [1] Let (X, d) be a complex valued matric space and $B \subseteq X$.

(i) $b \in B$ is called an interior point of a set B whenever there is $0 \prec r \in \mathbb{C}$ such that $N(b, r) \subseteq B$, where $N(b, r) = \{y \in X: d(b, y) \prec r\}$.

(ii) A point $x \in X$ is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$, $N(x, r) \cap (B \setminus \{X\}) \neq \phi$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A whereas a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{N(x, r) : x \in X, 0 \prec r\}$$

is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 2.3. [1] Let (X, d) be a complex valued metric space and $\{x_n\}_{n\geq1}$ be a sequence in X and $x \in X$. We say that

(i) the sequence $\{x_n\}_{n\geq1}$ converges to x if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_n x_n$ or $x_n \rightarrow x$, as $n \rightarrow \infty$,

(ii) the sequence $\{x_n\}_{n\geq1}$ is Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$,

(iii) the metric space (X, d) is a complete complex valued metric space if every Cauchy sequence is convergent.

Definition 2.4. If f and g are mappings from a metric space (X,d) into itself, are called commuting on X, if $d(fgx, gfx) = 0$ for all $x \in X$.

Definition 2.5. [8] If f and g are mappings from a metric space (X,d) into itself, are called weakly commuting on X, if $d(fgx,gfx) \leq d(fx,gx)$ for all $x \in X$.

Definition 2.6. [4] If f and g are mappings from a metric space (X,d) into itself, are called compatible on X, if $\lim_{n\to\infty} d(fgx_n, \ gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$, for some point x ∈ X.

Lemma 2.1. [1] Let (X, d) be a complex valued metric space and $\{x_n\}$be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n\to\infty$.

Lemma 2.2. [1] Let (X, d) be a complex valued metric space and $\{x_n\}$be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n, m\to\infty$.

Lemma 2.3. [4] Let f and g be compatible mappings from a metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x ∈ X.
Then $\lim_{n\to\infty} gfx_n = fx$, if f is continuous.

3. Main results

Our result generalize the recent result of Nashine et al. [5] and Azam et al. [1].

Theorem 3.1. Let (X, d) be a complete complex valued metric space and mappings f,g,S,T:X→ X satisfying:

(3.1.1) $S \subset g, \quad T \subset f$;

(3.1.2) and $d(Sx, Ty) \precsim \alpha d(fx, gy) + \beta \left\{\frac{d(fx, Sx)\, d(gy, Ty)}{d(fx, Ty) + d(gy, Sx) + d(fx, gy)}\right\}$

for all x,y in X such that x≠y, d(fx, Ty)+d(gy, Sx)+d(fx, gy) $\neq 0$ where α , β are nonnegative reals with
$\alpha + \beta < 1$.

(3.1.3) Suppose that one of f, g,S and T is continuous,

(3.1.4) and pairs (S, f) and (T, g) are compatible on X.

Then f, g, S and T have unique common fixed point in X.

Proof. Suppose x_0 be an arbitrary point in X. We define a sequence $\{y_{2n}\}$in X such that

$$y_{2n} = Sx_{2n} = gx_{2n+1}$$
$$y_{2n+1} = Tx_{2n+1} = fx_{2n+2} \qquad ; n=0,1,2,\dots$$

Then,

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\precsim \alpha\, d(fx_{2n}, gx_{2n+1}) + \beta \left\{\frac{d(fx_{2n}, Sx_{2n})\, d(gx_{2n+1}, Tx_{2n+1})}{d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n}) + d(fx_{2n}, gx_{2n+1})}\right\} \\
&\precsim \alpha\, d(y_{2n-1}, y_{2n}) + \beta \left\{\frac{d(y_{2n-1}, y_{2n}) d(y_{2n}, y_{2n+1})}{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n})}\right\} \\
&\precsim \alpha\, d(y_{2n-1}, y_{2n}) + \beta \left\{\frac{d(y_{2n-1}, y_{2n}) d(y_{2n}, y_{2n+1})}{d(y_{2n}, y_{2n+1})}\right\} \\
&\precsim \alpha\, d(y_{2n-1}, y_{2n}) + \beta d(y_{2n-1}, y_{2n})
\end{aligned}$$

$$d(y_{2n}, y_{2n+1}) \precsim (\alpha + \beta)\, d(y_{2n-1}, y_{2n})$$

Similarly, we can show that

$$d(y_{2n+1}, y_{2n+2}) \precsim (\alpha + \beta)\, d(y_{2n}, y_{2n+1}).$$

If $(\alpha + \beta) = \delta < 1$, then

$$|d(y_{2n+1}, y_{2n+2})| \precsim \delta |d(y_{2n}, y_{2n+1})| \precsim \cdots \precsim \delta^{2n+1} |d(y_0, y_1)|$$

so that for any m > n,

$$|d(y_{2n}, y_{2m})| \precsim |d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + \ldots + d(y_{2m-1}, y_{2m})|$$
$$\precsim (\delta^{2n} + \delta^{2n+1} + \ldots + \delta^{2m-1}) \, |d(y_0, y_1)|$$
$$\precsim \frac{\delta^{2n}}{1-\delta} |d(y_0, y_1)| \to 0 \text{ as m, n} \to \infty.$$

Hence $\{y_{2n}\}$ is a Cauchy sequence and since X is complete, sequence $\{y_{2n}\}$ converges to point t in X and its subsequences Sx_{2n}, Tx_{2n+1}, fx_{2n+2}, and gx_{2n+1} of sequence $\{y_{2n}\}$ also converges to point t.

Suppose that f is continuous and since the mappings S and f are compatible on X. Then by lemma (2.3), we have

f^2x_{2n} and Sf $x_{2n} \to$ft as n$\to\infty$.

Consider

$$d(Sfx_{2n}, Tx_{2n+1}) \precsim \alpha \, d(f^2x_{2n}, gx_{2n+1}) + \beta \left\{ \frac{d(f^2x_{2n}, Sfx_{2n}) \, d(gx_{2n+1}, Tx_{2n+1})}{d(f^2x_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sfx_{2n}) + d(f^2 x_{2n}, gx_{2sn+1})} \right\}.$$

Letting n$\to\infty$, we get

$$d(ft,t) \precsim \alpha \, d(ft,t) + \beta \left\{ \frac{d(ft,ft) d(t,t)}{d(ft,t) + d(t,ft) + d(ft,t)} \right\}$$

$(1-\alpha)$ d(ft,t) $\precsim 0$

This yields d(ft,t) $\precsim 0$ so that ft = t.

Again consider,

$$d(St, Tx_{2n+1}) \precsim \alpha \, d(ft, gx_{2n+1}) + \beta \left\{ \frac{d(ft,St) \, d(gx_{2n+1}, Tx_{2n+1})}{d(ft, Tx_{2n+1}) + d(gx_{2n+1}, St) + d(ft, gx_{2n+1})} \right\}.$$

Letting n$\to\infty$, we get

$$d(St,t) \precsim \alpha \, d(t,t) + \beta \left\{ \frac{d(t,St) d(t,t)}{d(t,t) + d(t,t) + d(t,t)} \right\}$$

d(St, t) $\precsim 0$ so that St = t.

Now since S $\subset g$ and there exists another point u in X, such that t = St = gu.

Consider

d(t, Tu) = d(St, Tu)

$$d(t, Tu) \precsim \alpha \, d(ft, gu) + \beta \left\{ \frac{d(ft,St) d(gu,Tu)}{d(ft,Tu) + d(gu,St) + d(ft,gu)} \right\}$$

$$d(t, Tu) \precsim \alpha \, d(t, t) + \beta \left\{ \frac{d(t,t) d(t,Tu)}{d(t,Tu) + d(t,t) + d(t,t)} \right\}$$

d(t,Tu) $\precsim 0$ so that Tu = t.

Since T and g are compatible on X and Tu = gu = t and d(gTu, Tgu) = 0 implies that gt = gTu= Tgu= Tt.

Consider

d(t, gt)=d(St, Tt)

$$d(t, gt) \precsim \alpha \, d(ft, gt) + \beta \left\{ \frac{d(ft,St) d(gt,Tt)}{d(ft,Tt) + d(gt,St) + d(ft,gt)} \right\}$$

$(1-\alpha)$ d(t, gt) $\precsim 0$ so that gt = t.

Hence ft = gt = St = Tt = t.

Thus t is a common fixed point of f, g, S and T.

Similarly, we can show that t is a common fixed point of f, g, S and T, when g is continuous.

Suppose that S is continuous and since mappings S and f are compatible on X. Then by lemma (2.3), we have

$S^2\, x_{2n}$ and fSx_{2n} = St as n→ ∞.

Consider

$$d(S^2x_{2n,}Tx_{2n+1}) \precsim \alpha\, d(fSx_{2n}, gx_{2n+1}) + \beta \left\{\frac{d(fSx_{2n}, S^2x_{2n})d(gx_{2n+1}, Tx_{2n+1})}{d(fSx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, S^2x_{2n}) + d(fSx_{2n}, gx_{2n+1})}\right\}.$$

Letting n→∞, we get

$$d(St, t) \precsim \alpha\, d(St, t) + \beta \left\{\frac{d(St,St)d(t,t)}{d(St,t)+d(t,St)+d(St,t)}\right\}$$

(1- α) d(St,t) $\precsim$ 0 so that St = t.

Now S⊂g, there exists a point v in X, such that t = St = gv.

Consider

$$d(S^2x_{2n,}\, Tv) \precsim \alpha\, d(fSx_{2n}, gv) + \beta \left\{\frac{d(fSx_{2n}, S^2x_{2n})d(gv,Tv)}{d(fSx_{2n}, Tv) + d(gv, S^2x_{2n}) + d(fSx_{2n}, gv)}\right\}.$$

Letting n→∞, we get

$$d(St, Tv) \precsim \alpha\, d(St, t) + \beta \left\{\frac{d(St,St)d(t,Tv)}{d(St,Tv)+d(t,St)+d(st,t)}\right\}$$

d(t, Tv) $\precsim$ α d(t, t)

d(t, Tv) $\precsim$ 0 so that Tv = t.

Since T and g are compatible on X and Tv = gv = t, d(gTv, Tgv) = 0

impiles that gt = gTv = Tgu = Tt.

Now consider

$$d(Sx_{2n}, Tt) \precsim \alpha\, d(fx_{2n}, gt) + \beta \left\{\frac{d(fx_{2n}, Sx_{2n})d(gt,Tt)}{d(fx_{2n}, Tt)+d(gt,Sx_{2n})+d(fx_{2n},gt)}\right\}.$$

Letting n→∞, we get

$$d(t, Tt) \precsim \alpha\, d(t, Tt) + \beta \left\{\frac{d(t,t)\, d(Tt,Tt)}{d(t,Tt)+d(Tt,t)+d(Tt,t)}\right\}$$

(1- α) d(t, Tt) $\precsim$ 0 so that Tt = t.

Now since T⊂ f, there exists a point w in X, such that t = Tt = fw.

Now

$$\begin{aligned} d(Sw, t) &= d(Sw, Tt) \\ &\precsim \alpha\, d(fw, gt) + \beta \left\{\frac{d(fw,Sw)d(gt,Tt)}{d(fw,Tt)+d(gt,Sw)+d(fw,gt)}\right\} \\ &\precsim \alpha\, d(t, t) + \beta \left\{\frac{d(t,Sw)d(t,t)}{d(t,t)+d(t,Sw)+d(t,t)}\right\} \end{aligned}$$

d(Sw, t) $\precsim$ 0 so that Sw = t.

Since S and f are compatible on X and Sw = fw = t and d(fSw, Sfw) = 0.

Implies that ft = fSw = Sfw = St.

Therefore, t is common fixed point of f, g, S and T.

Similarly, we can show that t is also common fixed point of f, g, S and T, when T is continuous.

Now, we prove the uniqueness of t.

Suppose that w ≠ t be another common fixed point of f, g, S and T.

Then

d(t, w) = d(St, Tw)

$$d(t, w) \precsim \alpha\, d(ft, gw) + \beta \left\{\frac{d(ft,St)d(gw,Tw)}{d(ft,Tw)+d(gw,St)+d(ft,gw)}\right\}$$

$$\precsim \alpha \mathrm{d}(\mathrm{t},\mathrm{w}) + \beta \left\{ \frac{d(t,t)\, d(w,w)}{d(t,w)+d(w,t)+d(t,w)} \right\}$$

$\mathrm{d}(\mathrm{t},\mathrm{w}) \precsim \alpha \mathrm{d}(\mathrm{t},\mathrm{w})$

$(1-\alpha)$d(t, w) $\precsim 0$, which is a contradiction. Hence t = w.

Therefore, t is unique common fixed point of f, g, S and T.

By setting f = g = I, we get the following corollary:

Corollary 3.2. Let (X, d) be a complete complex valued metric space and mappings S, T:X→X satisfy:

(3.2.1) $S \subset T$

(3.2.2) $d(Sx,Ty) \precsim \alpha d(x,y) + \beta \left\{ \frac{d(x,Sx)\, d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} \right\}$

for all x,y in X such that x≠y, d(x,Ty)+d(y,Sx)+d(x,y) $\neq 0$. where α, β are non-negative reals with $\alpha + \beta < 1$. If pair (S, T) is compatible. Then S and T have a unique common fixed point in X.

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