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Common Fixed Point Theorems for Six Mappings Satisfying Almost Generalized (S, T)-Contractive Condition in Partially Ordered Metric Spaces

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Abstract. The existence of coincidence point and common fixed point theorems for six self-mappings satisfying almost generalized (S, T)-contractive condition in the setup of partially ordered complete metric spaces have been proved. Our result unify and generalize the earlier results of Aghajani et al. and others. A related example is also furnished.

Keywords: Common fixed points, partially ordered metric spaces, almost contraction, weakly annihilator, dominating map

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1. Introduction

In 1968, Kannan [10] was the first who proved the existence of a fixed point for a map that can have a discontinuity in a domain, however the maps involved in every case were continuous at the fixed point. On the other hand after the classical result of Jungck [7] of common fixed point of two commuting maps, Sessa [11] initiated the weaker condition than that of commutativity namely weak commutativity of maps and proved the result regarding common fixed point consideration of such maps. Of course two commuting mappings are weakly commuting but the converse is not true always. Further a weaker condition of these notions namely, compatibility of maps has been introduced by Jungck [8] and proved result regarding common fixed points of such maps. Jungck [8] also demonstrated that commuting mappings are weakly commuting and weakly commuting are compatible but neither implication is reversible.

Recently Jungck & Rhoades [9] has introduced a weaker class among all commutative conditions namely weakly compatibility maps or coincidently commutativity of maps and gave results regarding common fixed points of such maps.

Abbas et al. [2] introduced the notions namely weak annihilator and dominating maps and proved some results of common fixed points by using these notions in the framework of partially ordered metric space.

Berinde in [4, 5] initiated the concept of almost contractions. Further by introducing the concept of almost generalized contractive condition Ćirić et al. [6] extend the concept of almost contractions to a pair of self-maps. Also Aghajani et al. [3] generalized the notion of almost generalized contraction by introducing the notion of almost generalized (S, T)-contraction.

The aim of this paper is to establish coincidence point and common fixed point results for six mappings which satisfy almost generalized (S, T)-contractive condition in the setting of partially ordered metric spaces. Infact, we have generalized the results of Aghajani et al. [3] and many others.

2. Preliminaries

We start this section by some basic definitions and results which are used in sequel.

Definition 2.1. Let $(X, \leq)$ be a partially ordered set and $f, g: X \to X$ then

(2.1.1) an ordered pair (f, g) are called partially weakly increasing if $fx \leq gfx$ for all $x \in X$. (cf. [2])

(2.1.2) a map f is called weak annihilator of g if $fgx \leq x$ for all $x \in X$. (cf. [2])

(2.1.3) a map f is called dominating if $x \leq fx$ for all $x \in X$. (cf. [2])

Definition 2.2. [9] Let (X, d) be a metric space then mappings $f, g: X \to X$ are said to be weakly compatible if they commute at their coincidence points, that is, if $fx = gx$ for some $x \in X$ then $fgx = gfx$.

Berinde [5] introduced the notion of "weak contraction" which further renamed as "almost contraction" by Berinde [4] defined as:

Definition 2.3 [4, 5] A self-map f on a metric space X is said to an almost contraction or $(\delta, L) -$ contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

(2.3.1) $$d(fx, fy) \leq \delta d(x, y) + Ld(y, fx), \quad \forall\ x, y \in X.$$

Remark 2.4. [4, 5] Due to the symmetry of the distance, the almost contraction condition (2.3.1) implicitly includes the following dual one

$d(fx, fy) \leq \delta . d(x, y) + Ld(x, fy), \quad \forall\ x, y \in X$, obtained from (2.3.1) by formally replacing $d(fx, fy)$ and $d(x, y)$ by $d(fy, fx)$ and $d(y, x)$, respectively and then interchanging x and y.

Berinde in [5] established some fixed point theorems for almost contractions in complete metric spaces and shown that any strict contraction such as Kannan [10] mapping as well as a large class of quasi-contractions are all almost contractions.

By generalizing the notion of almost contraction Ćirić et al. [6] gave:

Definition 2.5. [6] Let (X, d) be a metric space and $f: X \to X$ is said to be an almost contraction with respect to $g: X \to X$ if there exists a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(fx, fy) \leq \delta d(gx, gy) + Ld(fx, gy), \quad \forall\ x, y \in X.$$

Remark 2.6. [6] If one take $g =$ Identity map on X, then the above definition reduced to the notion almost contraction.

Further, Ćirić et al. [6] introduced the notion namely “almost generalized contraction” defined as:

Definition 2.7. [6] Let (X, d) be a metric space and $f, g: X \to X$ are said to satisfy almost generalized contractive condition if there exists a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that

(2.7.1) $$d(fx, gy) \leq \delta max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}\right\} + Lmin\{d(x,fx), d(y,gy), d(x,gy), d(y,fx)\}, \quad \forall\, x, y \in X.$$

Theorem 2.8. [6] Let $(X, \leq)$ be a partially ordered set and there exists a metric d on X such that (X, d) is a complete metric space. Let $f, g : X \to X$ be strictly weakly increasing mappings with respect to $\leq$ satisfying (2.7.1) for every comparable $x, y \in X$. If either f or g is continuous then f and g have a common fixed point in X.

Aghajani et al. [3] generalized the notion of almost generalized contraction by introducing the notion namely “almost generalized (S, T)-contraction” as:

Definition 2.9. [3] Let f, g, S and T be self-maps on a metric space (X, d), then f and g are said to satisfy almost generalized (S, T)-contractive condition if there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that

(2.9.1) $d(fx, gy) \leq \delta M(x, y) + LN(x, y)$, where

$$M(x,y) = max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2}\right\},$$
$$N(x,y) = min\{d(fx,Sx), d(gy,Ty), d(Sx,gy), d(fx,Ty)\}, \quad \forall\, x, y \in X.$$

Remark 2.10. [3] If one take $S = T =$ Identity map on X, then above definition reduced to almost generalized contractive condition.

Theorem 2.11. [3] Let $(X, \leq)$ be a partially ordered set and there exists a metric d on X such that (X, d) is a complete metric space. Let $f, g, S, T: X \to X$ satisfying the condition (2.9.1) for each pair of comparable elements $x,\ y \in X$ and

(2.11.1) $fX \subseteq TX$ and $gX \subseteq SX$.

(2.11.2) f and g are dominating, and weak annihilators of T and S, respectively.

(2.11.3) there exists a non-decreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies that $x_n \leq u$.

(2.11.4) pairs (f, S) and (g, T) are weakly compatible.

(2.11.5) one of fX, gX, SX and TX is a closed subspace of X, then f, g, S and T have a unique common fixed point.

3. Main result

Our main result is generalization of result of [3] for six self-maps as opposed to four maps satisfying almost generalized (S, T)-contractive condition in partially ordered complete metric space.

Theorem 3.1. Let $(X, \leq, d)$ be an ordered complete metric space. Let $A, B, L, Q, S, T: X \to X$ satisfying (2.11.3) and

(3.1.1) $ABX \subseteq TX$ and $LQX \subseteq SX$.

(3.1.2) $d(ABx, LQy) \leq \delta M(x, y) + LN(x, y)$, where

$$M(x,y) = max\left\{d(Sx,Ty), d(ABx,Sx), d(LQy,Ty), \frac{d(Sx,LQy)+d(ABx,Ty)}{2}\right\},$$

$$N(x,y) = min\{d(ABx,Sx), d(LQy,Ty), d(Sx,LQy), d(ABx,Ty)\},$$

for each pair of comparable elements $x, y \in X$, $\delta \in [0,1)$ and some $L \geq 0$.

(3.1.3) (i) the pairs (T, AB) and (S, LQ) are partially weakly increasing.

(ii) AB and LQ are dominating, and weak annihilators of T and S, respectively.

(3.1.4) one of ABX, LQX, SX and TX is a closed subspace of X, then

(i) LQ and T have a coincidence point in X,

(ii) AB and S have a coincidence point in X.

(3.1.5) pairs (AB, S) and (LQ, T) are weakly compatible then

(iii) AB, LQ, S and T have a unique common fixed point in X.

Furthermore if

(3.1.6) the pairs (A,B), (A,S), (B,S), (L,Q), (L,T) and (Q,T) commute at the common fixed point of AB, LQ, S and T then A, B, L, Q, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X, since $ABX \subseteq TX$ then there exists $x_1 \in X$ such that $ABx_0 = Tx_1$. Also since $LQX \subseteq SX$ then there exists $x_2 \in X$ such that $LQx_1 = Sx_2$. Inductively we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n} = ABx_{2n} = Tx_{2n+1}$ and $y_{2n+1} = LQx_{2n+1} = Sx_{2n+2}$ for all $n = 0, 1, 2, 3 \ldots$.

From (3.1.3), we have

$x_{2n} \leq ABx_{2n} = Tx_{2n+1} \leq (AB)Tx_{2n+1} \leq x_{2n+1}$ and

$x_{2n+1} \leq LQx_{2n+1} = Sx_{2n+2} \leq (LQ)Sx_{2n+2} \leq x_{2n+2}$.

Thus $\forall\ n \geq 0$, we obtain $x_0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \ \ldots$.

Now we claim that $\{y_n\}$ is a Cauchy sequence in X. If $y_{2n} = y_{2n+1}$, for some n, then from (3.1.2), we have

$$d(y_{2n+1}, y_{2n+2}) = d(ABx_{2n+2}, LQx_{2n+1}) \leq \delta M(x_{2n+2}, x_{2n+1}) + LN(x_{2n+2}, x_{2n+1}),$$

where

$$\begin{aligned}
M(x_{2n+2}, x_{2n+1}) = \max\{&d(Sx_{2n+2}, Tx_{2n+1}), d(ABx_{2n+2}, Sx_{2n+2}),\\
&d(LQx_{2n+1}, Tx_{2n+1}), \frac{d(Sx_{2n+2}, LQx_{2n+1}) + d(ABx_{2n+2}, Tx_{2n+1})}{2}\}\\
= max\{&d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n+1}),\\
&\frac{d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}{2}\}\\
= max\{&0, d(y_{2n+2}, y_{2n+1}), 0, \frac{0 + d(y_{2n+2}, y_{2n})}{2}\} \leq d(y_{2n+1}, y_{2n+2}),
\end{aligned}$$

and

$$\begin{aligned}
N(x_{2n+2}, x_{2n+1}) = \min\{&d(ABx_{2n+2}, Sx_{2n+2}), d(LQx_{2n+1}, Tx_{2n+1}),\\
&d(Sx_{2n+2}, LQx_{2n+1}), d(ABx_{2n+2}, Tx_{2n+1})\}\\
= \min\{&d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+1}), d(y_{2n+2}, y_{2n})\} = 0.
\end{aligned}$$

Hence $d(y_{2n+1}, y_{2n+2}) \leq \delta d(y_{2n+1}, y_{2n+2})$, since $\delta \in [0,1)$ yields that $y_{2n+1} = y_{2n+2}$. Further by using the similar arguments, we have $y_{2n+2} = y_{2n+3}$ and so on. Thus $\{y_n\}$

turns out to be a constant sequence and y_{2n} is the common fixed point of AB, LQ, S and T.

If we suppose $d(y_{2n}, y_{2n+1}) > 0$, for every n and since $x = x_{2n}$ and $y = x_{2n+1}$ are comparable, then from (3.1.2), we have

$$d(y_{2n}, y_{2n+1}) = d(ABx_{2n}, LQx_{2n+1}) \leq \delta M(x_{2n}, x_{2n+1}) + LN(x_{2n}, x_{2n+1}) \quad (3.1)$$

where

$$M(x_{2n}, x_{2n+1}) = max\{d(Sx_{2n}, Tx_{2n+1}), d(ABx_{2n}, Sx_{2n}), d(LQx_{2n+1}, Tx_{2n+1}), \frac{d(Sx_{2n}, LQx_{2n+1}) + d(ABx_{2n}, Tx_{2n+1})}{2}\}$$
$$= max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \frac{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})}{2}\}$$
$$\leq max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n})\}.$$

and

$$N(x_{2n}, x_{2n+1}) = \min\{d(ABx_{2n}, Sx_{2n}), d(LQx_{2n+1}, Tx_{2n+1}), d(Sx_{2n}, LQx_{2n+1}), d(ABx_{2n}, Tx_{2n+1})\}$$
$$= \min\{d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n+1}), 0\} = 0.$$

Therefore from (3.1), we have

$$d(y_{2n}, y_{2n+1}) = d(ABx_{2n}, LQx_{2n+1}) \leq \delta max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n})\} \quad (3.2)$$

Now

$$max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n})\} = \text{either } d(y_{2n-1}, y_{2n}) \text{ or } d(y_{2n+1}, y_{2n})$$

If $max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n})\} = d(y_{2n+1}, y_{2n})$ then from (3.2), we have $d(y_{2n}, y_{2n+1}) \leq \delta d(y_{2n}, y_{2n+1})$, which is a contradiction, since $\delta \in [0, 1)$. Therefore $d(y_{2n}, y_{2n+1}) \leq \delta d(y_{2n-1}, y_{2n})$. Similarly it can be proved that $d(y_{2n-1}, y_{2n}) \leq \delta d(y_{2n-2}, y_{2n-1})$. Therefore for all n ≥ 1, we have $d(y_n, y_{n+1}) \leq \delta d(y_{n-1}, y_n)$.
Inductively for all n ≥ 1, we have

$$d(y_n, y_{n+1}) \leq \delta(d(y_{n-1}, y_n)) \leq \delta^2(d(y_{n-2}, y_{n-1})) \leq \cdots \leq \delta^n(d(y_0, y_1))$$

By triangle inequality for $m > n$, we have

$$d(y_m, y_n) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \cdots + d(y_{m-1}, y_m)$$
$$\leq \frac{\delta^n}{1-\delta} d(y_0, y_1) \to 0$$ as $n \to \infty$ (since $\delta \in [0, 1)$), yields that $\{y_n\}$ is a Cauchy sequence in X. By the completeness of X, the Cauchy sequence $\{y_n\}$ and its subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are also converges to some z in X, i.e.,
$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} ABx_{2n} = \lim_{n\to\infty} T\,x_{2n+1} = z$ and
$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} LQ\,x_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$.

Suppose TX is closed then there exists $w \in X$ such that $z = Tw$. From (3.1.3), since

$$x_{2n} \leq ABx_{2n} \text{ and } ABx_{2n} \to z \text{ as } n \to \infty \Rightarrow x_{2n} \leq z = Tw \leq (AB)Tw \leq w.$$

Using (3.1.2), we have

$$d(ABx_{2n}, LQw) \leq \delta M(x_{2n}, w) + LN(x_{2n}, w) \quad (3.3)$$

where

$$M(x_{2n}, w) = max\{d(Sx_{2n}, Tw), d(ABx_{2n}, Sx_{2n}), d(LQw, Tw), \frac{d(Sx_{2n}, LQw) + d(ABx_{2n}, Tw)}{2}\}$$

$$= max\{d(Sx_{2n}, z), d(ABx_{2n}, Sx_{2n}), d(LQw, z), \frac{d(Sx_{2n}, LQw) + d(ABx_{2n}, z)}{2}\}$$

and

$$N(x_{2n}, w) = \min\{d(ABx_{2n}, Sx_{2n}), d(LQw, Tw), d(Sx_{2n}, LQw), d(ABx_{2n}, Tw)\}$$
$$= \min\{d(ABx_{2n}, Sx_{2n}), d(LQw, z), d(Sx_{2n}, LQw), d(ABx_{2n}, z)\}.$$

Letting $n \to \infty$, we have

$\lim_{n\to\infty} M(x_{2n}, w) = d(LQw, z)$, $\lim_{n\to\infty} N(x_{2n}, w) = 0$.

Therefore from (3.3) as $n \to \infty$, we have

$d(z, LQw) \leq \delta d(LQw, z)$ yields that $LQw = z$. So $LQw = Tw = z$. Now by weakly compatibility of pair (LQ, T), $LQz = (LQ)Tw = T(LQ)w = Tz$.

Using (3.1.2), we have

$$d(z, LQz) = d(ABx_{2n}, LQz) \leq \delta M(x_{2n}, z) + LN(x_{2n}, z) \tag{3.4}$$

where

$$M(x_{2n}, z) = max\{d(Sx_{2n}, Tz), d(ABx_{2n}, Sx_{2n}), d(LQz, Tz), \frac{d(Sx_{2n}, LQz) + d(ABx_{2n}, Tz)}{2}\}$$
$$= max\{d(Sx_{2n}, LQz), d(ABx_{2n}, Sx_{2n}), d(LQz, LQz), \frac{d(Sx_{2n}, LQz) + d(ABx_{2n}, LQz)}{2}\}$$

and

$$N(x_{2n}, z) = \min\{d(ABx_{2n}, Sx_{2n}), d(LQz, Tz), d(Sx_{2n}, LQz), d(ABx_{2n}, Tz)\}$$
$$= \min\{d(ABx_{2n}, Sx_{2n}), d(LQz, Tz), d(Sx_{2n}, LQz), d(ABx_{2n}, Tz)\}.$$

Letting $n \to \infty$, we have

$\lim_{n\to\infty} M(x_{2n}, z) = d(z, LQz)$, $\lim_{n\to\infty} N(x_{2n}, z) = 0$.

Therefore from (3.4) as $n \to \infty$, we have

$d(z, LQz) \leq \delta d(z, LQz)$, yields that $LQz = z$.

Hence $LQz = z$. … (3.5)

Since $LQX \subseteq SX$ then there exists a point $v \in X$ such that $z = LQz = Sv$. From (3.1.3) since $z \leq LQz = Sv \leq (LQ)Sv \leq v$ implies that $z \leq v$.

Using (3.1.2), we have

$$d(ABv, Sv) = d(ABv, LQz) \leq \delta M(v, z) + LN(v, z) \tag{3.6}$$

where

$$M(v, z) = max\{d(Sv, Tz), d(ABv, Sv), d(LQz, Tz), \frac{d(Sv, LQz) + d(ABv, Tz)}{2}\}$$
$$= max\{d(Sv, Tz), d(ABv, Sv), d(LQz, Tz), \frac{d(Sv, LQz) + d(ABv, LQz)}{2}\}$$
$$= max\{0, d(ABv, Sv), 0, \frac{0 + d(ABv, Sv)}{2}\} = d(ABv, Sv),$$

and

$$N(v, z) = \min\{d(ABv, Sv), d(LQz, Tz), d(Sv, LQz), d(ABv, Tz)\}$$
$$= \min\{d(ABv, Sv), 0, d(Sv, LQz), d(ABv, Sv)\} = 0.$$

Therefore from (3.6), we have

$d(ABv, Sv) \leq \delta d(ABv, Sv)$ yields that $ABv = Sv$. Now by weakly compatibility of pair (AB, S), $ABz = (AB)Sv = S(AB)v = Sz$.

Using (3.1.2), we have

$$d(ABz, z) = d(ABz, LQz) \leq \delta M(z, z) + LN(z, z) \tag{3.7}$$

where

$M(z,z) = max\{d(Sz,Tz), d(ABz,Sz), d(LQz,Tz), \frac{d(Sz,LQz)+d(ABz,Tz)}{2}\}$
$= max\{d(ABz,Tz), d(Sz,Sz), d(LQz,Tz), \frac{d(ABz,LQz)+d(ABz,Tz)}{2}\}$
$= max\{d(ABz,z), 0,0, d(ABz,z)\} = d(ABz,z),$

and

$N(z,z) = \min\{d(ABz,Sz), d(LQz,Tz), d(Sz,LQz), d(ABz,Tz)\}$
$= \min\{0,0, d(Sz,z), d(ABz,z)\} = 0.$

Therefore from (3.7), we have

$d(ABz,z) \leq \delta d(ABz,z)$, yields that $ABz = Sz = z$ (3.8)

Hence from (3.5) and (3.8), we have $ABz = LQz = Sz = Tz = z$, i.e. z is the common fixed point of AB, LQ, S and T.

For the uniqueness of z suppose u be another common fixed point of AB, LQ, S and T then from (3.1.2), we have

$$d(z,u) = d(ABz,LQu) \leq \delta M(z,u) + LN(z,u) \tag{3.9}$$

where

$M(z,u) = max\{d(Sz,Tu), d(ABz,Sz), d(LQu,Tu), \frac{d(Sz,LQu)+d(ABz,Tu)}{2}\}$
$= max\{d(z,u), 0,0, d(z,u)\} = d(z,u)$

and

$N(z,u) = \min\{d(ABz,Sz), d(LQu,Tu), d(Sz,LQu), d(ABz,Tu)\}$
$= \min\{0,0, d(z,u), d(z,u)\} = 0.$

Therefore from (3.9), we have

$d(z,u) \leq \delta d(z,u)$, yields that $z = u$, i.e. z is the unique common fixed point of AB, LQ, S and T.

The proof is similar for the cases in which one of ABX, LQX and SX is a closed subspace of X.

From (3.1.6) by commutativity of (A,B), (A,S) and (B,S), we have

$Az = A(ABz) = A(BAz) = AB(Az)$, $Az = A(Sz) = S(Az)$,
$Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz)$, $Bz = B(Sz) = S(Bz)$,

which shows that Az and Bz are the common fixed points of (AB,S). But AB and S have a unique fixed point z, then

$$z = Az = Bz = Sz = ABz \tag{3.10}$$

Again from (3.1.6) by commutativity of (L,Q), (L,T) and (Q,T), we have

$Lz = L(LQz) = L(QLz) = LQ(Lz)$, $Lz = L(Tz) = T(Lz)$,
$Qz = Q(LQz) = QL(Qz) = LQ(Qz)$, $Qz = Q(Tz) = T(Qz)$,

which shows that Lz and Qz are the common fixed points of (LQ,T). But LQ and T have unique fixed point z, then

$$z = Lz = Qz = Tz = LQz \tag{3.11}$$

Using (3.1.2), (3.10) and (3.11), we have

$$d(Az,Lz) = d(AB(Az), LQ(Lz)) \leq \delta M(Az,Lz) + LN(Az,Lz) \tag{3.12}$$

where

$M(Az,Lz) = max\{d(SAz,TLz), d(AB(Az),SAz), d(LQ(Lz),TLz),$
$\frac{d(SAz,LQ(Lz))+d(AB(Az),TLz)}{2}\}$
$= max\{d(z,z), d(z,z), d(z,z), \frac{d(z,z)+d(z,z)}{2}\} = \max\{0,0,0,0\} = 0$

and

$N(Az, Lz) = \min\{d(AB(Az), SAz), d(LQ(Lz), TLz), d(SAz, LQ(Lz)), d(AB(Az), TLz)\}$
$\quad = \min\{d(z,z), d(z,z), d(z,z), d(z,z)\} = 0.$

Therefore from (3.12), we have

$d(Az, Lz) \leq \delta 0 = 0$, yields that $Az = Lz$ (3.13)

Hence from (3.10), (3.11) and (3.13), we have $z = Az = Bz = Lz = Qz = Sz = Tz$, i.e. z is the unique common fixed point of self-mappings A, B, L, Q, S and T.

If we take $S = T$ in Theorem 3.1, we have the following corollary:

Corollary 3.2. Let $(X, \leq, d)$ be an ordered complete metric space. Let $A, B, L, Q, T: X \to X$ satisfying (2.11.3) and

(3.2.1) $ABX \subseteq TX$ and $LQX \subseteq TX$.

(3.2.2) $d(ABx, LQy) \leq \delta M(x,y) + LN(x,y)$, where

$M(x,y) = max\left\{d(Tx,Ty), d(ABx,Tx), d(LQy,Ty), \frac{d(Tx,LQy)+d(ABx,Ty)}{2}\right\},$

$N(x,y) = min\{d(ABx,Tx), d(LQy,Ty), d(Tx,LQy), d(ABx,Ty)\}$

for every comparable elements $x,\ y \in X$, $\delta \in [0,1)$ and some $L \geq 0$.

(3.2.3) (i) the pairs (T, AB) and (T, LQ) are partially weakly increasing.

(ii) AB and LQ are dominating, and weak annihilators of T.

(3.2.4) one of ABX, LQX and TX is a closed subspace of X, then

(i) LQ and T have a coincidence point in X,

(ii) AB and T have a coincidence point in X.

(3.2.5) pairs (AB, T) and (LQ, T) are weakly compatible then

(iii) AB, LQ and T have a unique common fixed point in X.

Furthermore if

(3.2.6) the pairs (A,B), (A,T), (B,T), (L,Q), (L,T) and (Q,T) commute at the common fixed point of AB, LQ and T, then A, B, L, Q and T have a unique common fixed point in X.

Corollary 3.3.

(3.3.1) If we take $B = Q =$ Identity mappings in Theorem 3.1, we can obtain Theorem 2.10 of Aghajani et al. [3].

(3.3.2) If we take $AB = LQ = f$ and $S = T$ in Theorem 3.1, we can obtain Corollary 2.3 of Aghajani et al. [3].

(3.3.3) If we take $AB = LQ = f$ and $S = T =$ Identity mappings in Theorem 3.1, we can obtain Theorem 2.1 of Ćirić et al. [6].

Remark 3.4. Here note that for every self-map T, (T, I) is weakly compatible, I is identity and dominating map so by taking $AB = LQ = I$ in Theorem 3.1, we have the following result.

Corollary 3.5. Let $(X, \leq, d)$ be an ordered complete metric space. Let $S, T: X \to X$ surjective maps such that $Sx \leq x$ and $Tx \leq x$ for all $x \in X$ satisfying (2.10.3) and

$d(x,y) \leq \delta M(x,y) + LN(x,y)$, where

$M(x,y) = max\left\{d(Sx,Ty), d(x,Sx), d(y,Ty), \frac{d(Sx,y)+d(x,Ty)}{2}\right\},$

$N(x,y) = min\{d(x,Sx), d(y,Ty), d(Sx,y), d(x,Ty)\}$

for every comparable elements $x,\ y \in X$, $\delta \in [0,1)$ and some $L \geq 0$, then S and T have a unique common fixed point in X.

Now we illustrate the following example in support of Theorem 3.1.

Example 2.6. Let $X = [0,\infty)$ with relation given by " $\leq$ " and $d(x,y) = |x-y|$, we define a new ordering "$\preccurlyeq$" on X such that $x \preccurlyeq y \Leftrightarrow y \leq x, \forall\, x,y \in X$. Then $(X,\leq,d)$ is an ordered complete metric space. Define $A,B,L,Q,S,T: X \to X$ such that
$Ax = \ln\left(1+\frac{x}{2}\right), Bx = 2x, Lx = \ln\left(1+\frac{x}{8}\right), Qx = 4x,\ Sx = e^{2x}-1$ and $Tx = e^{x}-1$.
Then by routine calculation we can see that all the conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of A,B,L,Q,S and T.

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