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Fuzzy PMS Ideals in PMS Algebras

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Abstract. In this paper, a new notion, namely fuzzification of PMS–algebra, a generalization of BCK/BCI/TM/KUS/PS-algebras is initiated along with fuzzified PMS-ideal and discussed some of its properties in detail.

Keywords: PMS-algebra, fuzzy PMS-subalgebra, fuzzy PMS-ideal, homomorphism, Cartesian product.

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1. Introduction

In 1965, the concept of fuzzy set was introduced by Zadeh [15]. In 1978, Iseki and Tanaka [1] introduced the concept of BCK-algebras and in 1980 Iseki [2] introduced the concept of BCI-algebras. In 2015, Sithar Selvam and Nagalakshmi [4,5] introduced a new algebraic structure, named as PMS algebras, as a generalization of BCK/BCI/TM/KUS /PS-algebras. In this paper we fuzzified PMS-algebras and studied its properties in detail along with fuzzy PMS-ideal, homomorphism and Cartesian products.

2. Preliminaries

In this section, we present the fundamental definitions that will be used in the development of this paper.

Definition 2.1. [1, 14] A BCK- algebra is an algebra $(X,*,0)$ of type $(2,0)$ satisfying the following conditions:

i) $(x * y) * (x * z) \le (z * y)$
ii) $x * (x * y) \le y$
iii) $x \le x$
iv) $x \le y$ and $y \le x \Rightarrow x=y$
v) $0 \le x \Rightarrow x=0$, where $x \le y$ is defined by $x * y = 0$,for all $x, y, z \in X$.

Definition 2.2. [4,5] A nonempty set X with a constant 0 and a binary operation ' $*$ ' is called PMS – algebra if it satisfies the following axioms.

1. $0 * x = x$
2. $(y * x) * (z * x) = z * y$, $\forall$ $x, y, z \in X$.

In X, we define a binary relation $\leq$ by : $x \leq y$ if and only if $x * y = 0$.

Definition 2.3. [4,5] Let X be a PMS - algebra and I be a subset of X, then I is called a PMS - ideal of X if it satisfies the following conditions:

1. $0 \in I$
2. $z * y \in I$ and $z * x \in I \Rightarrow y * x \in I$ for all $x, y, z \in X$.

Example 2.4. Let X = { 0, 1, 2 } be the set with the following table.

*	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Then (X , $*$, 0) is a PMS – algebra.

Example 2.5. Let X = { 0, a, b, c } be the set with the following table.

*	0	a	b	c
0	0	a	b	c
a	b	0	a	b
b	a	b	0	c
c	c	c	a	0

Then (X , $*$, 0) is a PMS – algebra and I = {0,a,b} is a PMS-ideal.

In any PMS - algebra (X, *,0), with $x \leq y$, the following holds good for all $x , y \in X$.

(i) $x * x = 0$
(ii) $(y * x) * x = y$
(iii) $x * (y * x) = y * 0$
(iv) $(y * x) * z = (z * x) * y$
(v) $0 * x = 0 \Rightarrow x = 0$
(vi) $(z * x) * (z * y) \leq x * y$
(vii) $x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z$
(viii) $x * ((y * x) * x) = x * y$
(ix) $(x * y) * 0 = y * x = (0 * y) * (0 * x)$
(x) $0 * (x * y) = (0 * x) * (0 * y)$
(xi) $((y * x) * x) * y = 0$
(xii) $(x * y) * x = y$.

Definition 2.6. [13] Let A be a non-empty subset of an algebra X , then A is called a sub algebra of X if $x * y \in A$, for all $x ,y \in A$.

Definition 2.7. [15,3] Let X be a non-empty set. A fuzzy subset μ of the set X is a mapping $\mu : X \rightarrow [0, 1]$.

Definition 2.8. [10,11] Let μ be a fuzzy set of X. For a fixed $t \in [0, 1]$, the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called the upper level subset of μ. Clearly $\mu^t \cup \mu_t = X$ for $t \in [0,1]$ if $t_1 < t_2$, then $\mu_{t1} \subseteq \mu_{t2}$.

3. Fuzzy PMS-ideal and fuzzy PMS-sub algebra

Definition 3.1. Let X be a PMS-algebra. A fuzzy set μ in X is called a fuzzy PMS-ideal of X if it satisfies the following conditions.
i) $\mu(0) \geq \mu(x)$
ii) $\mu(y * x) \geq \min \{\mu(z * y), \mu(z * x)\}$, for all $x, y, z \in X$

Definition 3.2. A fuzzy set μ in a PMS-algebra X is called a fuzzy PMS- sub algebra of X if $\mu(x * y) \geq \min \{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Definition 3.3. The set of elements that belong to the fuzzy set μ at least to the degree t is called the t-level set. It is represented by $\mu^t = \{ x \in X / \mu(x) \geq t \}$.

Theorem 3.4. Every fuzzy PMS-ideal of a PMS-algebra X is order reversing.
Proof: Let μ be a fuzzy PMS-ideal of a PMS-algebra X.
Let $x, y \in X$ be such that $x \leq y$, then $x * y = 0$

$$
\begin{aligned}
\text{Now} \quad \mu(x) &\geq \min \{\mu(0 * x)\} \\
&= \min \{\mu(z * 0), \mu((x * y)* (z * y))\} \\
&= \min \{\mu(z * 0), \mu(0 * (z * y))\} \\
&= \min \{\mu(z * 0), \mu(z * y)\} \\
&= \min \{\mu(0 * 0), \mu(0 * y)\} \text{ (Taking } z = 0) \\
&= \min \{\mu(0), \mu(y)\} \\
&= \mu(y)
\end{aligned}
$$

$\Rightarrow \mu(x) \geq \mu(y)$. $\therefore$ μ is order reversing.

Theorem 3.5. Every fuzzy PMS-ideal of PMS-algebra is a fuzzy PMS-sub algebra.
Proof: Let μ be a fuzzy PMS-ideal.
To prove: μ is a fuzzy PMS- sub algebra of X.
By definition of fuzzy PMS - ideal, $\mu(y*x) \geq \min \{\mu(z *y), \mu(z*x)\}$, for all $x,y,z \in X$
Now, $\mu(y * x) \geq \min \{\mu(z *y), \mu(z*x)\} = \min \{ \mu(0*y), \mu(0*x)\} = \min \{\mu(y), \mu(x)\}$
$\Rightarrow$ μ is a fuzzy PMS- subalgebra of X.

Theorem 3.6. The intersection of any set of fuzzy PMS - ideals in PMS-algebra is also a fuzzy PMS-ideal.
Proof: Let $\{\mu_i\}$ be a family of fuzzy PMS-ideals of PMS-algebras X. Then for any x, y, $z \in X$.

$$
\begin{aligned}
(\cap \mu_i)(0) &= \text{Inf}(\mu_i(0)) \\
&\geq \text{Inf}(\mu_i(x))
\end{aligned}
$$

$= (\cap \mu_i)(x)$

And $(\cap \mu_i)(y*x) = \text{Inf}(\mu_i(y*x))$

$\geq \text{Inf}\{\min\{\mu_i(z*y), \mu_i(z*x)\}\}$

$= \min\{\text{Inf}(\mu_i(z*y)), \text{Inf}(\mu_i(z*x))\}$

$= \min\{(\cap \mu_i)(z*y), (\cap \mu_i)(z*x)\}$

This completes the proof.

Theorem 3.7. A fuzzy set μ of a PMS - algebra is a fuzzy PMS - sub algebra if and only if the t –level set, μ^t is either empty or a PMS - sub algebra of X, for every $t \in [0, 1]$.

Proof : Assume that μ is a fuzzy PMS - sub algebra of X and $\mu^t \neq \phi$

Then for any $x, y \in \mu^t$, we have $\mu(x) = \mu(y) = t$

$\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$

$= \min\{t, t\} = t$

There fore $x * y \in \mu^t$.

Hence μ^t is a PMS - sub algebra of X.

Conversely, assume that μ^t is a PMS – sub algebra of X.

Let $x, y \in X$. Take $t = \min\{\mu(x), \mu(y)\}$

Then by assumption μ^t is a PMS - sub algebra of X, $x * y \in \mu^t$

$\mu(x*y) \geq t = \min\{\mu(x), \mu(y)\}$

Hence μ is a fuzzy PMS- sub algebra of X.

Theorem 3.8. Any sub algebra of a PMS – algebra X can be realized as a t-level sub algebra of some fuzzy PMS-sub algebra of X.

Proof: Let μ be sub algebra of the given PMS– algebra X.

Let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}$$

where $t \in [0, 1]$ is fixed. It is clear that $\mu^t = A$.

Now we prove such defined μ is a fuzzy PMS- sub algebra of X.

Let $x, y \in X$. If $x, y \in A$, then $x * y \in A$.

Hence, $\mu(x) = \mu(y) = \mu(x*y) = t$ and $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$

If $x, y \notin A$, then $\mu(x) = \mu(y) = 0$ and $\mu(x*y) \geq \min\{\mu(x), \mu(y)\} = 0$.

If at most one of $x, y \in A$, then at least one of $\mu(x)$ and $\mu(y)$ is equal to 0.

Therefore, $\min\{\mu(x), \mu(y)\} = 0$ so that $\mu(x*y) \geq 0$, which completes the proof.

As a generalisation of theorem 3.8, we prove the following theorem.

Theorem 3.9. Let X be a PMS - algebra. Then given any chain of sub algebra $S_0 \subset S_1 \subset S_2 \subset \ldots \subset S_r = X$, there exists a fuzzy PMS-sub algebra μ of X whose t-level sub algebras are exactly the sub algebras of this chain.

Proof : Consider a set of numbers $t_0 > t_1 > t_2 > \ldots\ldots\ldots > t_r$, where each $t_i \in [0,1]$.

Let $\mu : X \to [0,1]$ be a fuzzy set defined by $\mu(s_0) = t_0$ and $\mu(s_i - s_{i-1}) = t_i$, $0 < i \leq r$.

We claim that μ is a fuzzy PMS-sub algebra of X. Let $x, y \in X$. Then we classify it into two cases as follows :

Case (1) : Let $x, y \in s_i - s_{i-1}$. Then by the definition of μ, $\mu(x) = t_i = \mu(y)$.

Since S_i is a sub algebra, it follows that $x*y \in S_i$, and so either $x*y \in S_i - S_{i-1}$ (or) $x * y \in S_{i-1}$
In any case, we conclude that $\mu(x * y) \geq t_i = \min \{\mu(x), \mu(y)\}$.
Case (2) : For $i > j$, Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$.
Then $\mu(x) = t_i$; $\mu(y) = t_j$ and $x * y \in S_i$, since S_i is a subalgebra of X and $S_j \subset S_i$.
Hence $\mu(x * y) \geq t_j = \min \{\mu(x), \mu(y)\}$
Thus μ is a fuzzy PMS-subalgebra of X.
From the definition of μ, it follows that $Im(\mu) = \{t_0, t_1, t_2, \ldots\ldots\ldots t_r\}$.
Hence the t-level subalgebras of μ are given by the chain of subalgebras.
$\mu_{t0} \subset \mu_{t1} \subset \mu_{t2} \subset \ldots\ldots\ldots\ldots \subset \mu_{tr} = X$.
Now $\mu_{t0} = \{x \in X / \mu(x) \geq t_0\} = S_0$.
Finally, we prove that $\mu_{ti} = S_i$ for $0 < i \leq r$.
Clearly $S_i \subseteq \mu_{ti}$.
If $x \in \mu_{ti}$, then $\mu(x) \geq t_i$ which implies that $x \notin S_j$ for $j > i$.
Hence $\mu(x) \in \{t_1, t_2, \ldots\ldots\ldots t_i\}$ and so $x \in S_k$ for some $k \leq i$.
As $S_k \subseteq S_i$, it follows that $x \in S_i$. $\Rightarrow \mu_{ti} = S_i$ for $0 < i \leq r$.
This completes the proof.

Theorem 3.10. Two level sub algebras μ^s, μ^t $(s < t)$ of a fuzzy PMS- algebras are equal if and only if there is no $x \in X$ such that $s \leq \mu(x) < t$.
Proof: Let $\mu^s = \mu^t$ for some $s < t$.
If there exist $x \in X$ such that $s \leq \mu(x) < t$, then μ^t is a proper subset of μ^s, which is a contradiction.
Conversely, assume that there is no $x \in X$ such that $s \leq \mu(x) < t$, since $s < t$, $\mu^t \subseteq \mu^s$.
If $x \in \mu^s$ then $\mu(x) \geq s$ and so $\mu(x) \geq t$, because $\mu(x)$ does not lie between s and t.
Hence $x \in \mu^t$, which gives $\mu^s \subseteq \mu^t$. This completes the proof.

Theorem 3.11. Let μ be a fuzzy set in a PMS-algebra X and let $t \in Im(\mu)$. Then μ is a fuzzy PMS-ideal of X if and only if the t-level subset μ^t is a PMS-ideal of X.
Proof : Assume that μ is a fuzzy PMS-ideal of X.
Clearly $0 \in \mu^t$.
Let $z * x \in \mu^t$ and $z * y \in \mu^t$.
Then $\mu(z * x) \geq t$ and $\mu(z * y) \geq t$
Now $\mu(y * x) \geq \min \{\mu(z * y), \mu(z * x)\} \geq \min \{t, t\} = t$.
Hence the t-level subset μ^t is a PMS-ideal of X.
Conversely assume that, the t-level subset μ^t is a PMS-ideal of X, for any $t \in [0,1]$.
Suppose assume that there exist some $x_0 \in X$ such that $\mu(0) < \mu(x_0)$
Take $s = \frac{1}{2} [\mu(0) + \mu(x_0)]$
$\Rightarrow \mu(0) < s < \mu(x_0)$
$\Rightarrow x_0 \in \mu^s$ and $0 \notin \mu^s$, a contradiction, since μ^s is a PMS-ideal of X.
Therefore, $\mu(0) \geq \mu(x)$ for all $x \in X$.
If possible, assume that $x_0, y_0, z_0 \in X$ such that $\mu(y_0 * x_0) < \min \{\mu(z_0 * y_0), \mu(z_0 * x_0)\}$.
Take $s = \frac{1}{2} [\mu(y_0 * x_0) + \min \{\mu(z_0 * y_0), \mu(z_0 * x_0)\}]$
$\Rightarrow s > \mu(y_0 * x_0)$ and $s < \min \{\mu(z_0 * y_0), \mu(z_0 * x_0)\}$.
$\Rightarrow s > \mu(y_0 * x_0)$, $s < \mu(z_0 * y_0)$ and $s < \mu(z_0 * x_0)$.

$\Rightarrow y_0 * x_0 \notin \mu^s$, a contradiction, since μ^s is a PMS-ideal of X.
Therefore $\mu(y * x) \geq \min \{ \mu(z * y), \mu(z * x)\}$, for any x, y, z $\in$ X.

Theorem 3.12. Let X be a PMS-algebra & μ be a fuzzy PMS-sub algebra of X. If Im(μ) is finite, say $\{t_1, t_2, \dots t_r\}$, then for any $t_i, t_j \in$ Im(μ), $\mu_{t_i} = \mu_{t_j}$, implies $t_i = t_j$.

Proof : Assume that $t_i \neq t_j$ say $t_i < t_j$.
If $x \in \mu_{t_j}$, then $\mu(x) \geq t_j > t_i$, which implies that $x \in \mu_{t_i}$.
Let $x \in X$ be such that $t_i < \mu(x) < t_j$ .Then $x \in \mu_{t_i}$, but $x \notin \mu_{t_j}$.
Hence $\mu_{t_i} \not\subset \mu_{t_j}$ and $\mu_{t_j} = \mu_{t_i}$, a contradiction.

4. Conclusion

In this article, we have been discussed some charecterizations of fuzzy PMS-algebras. It adds an another dimension to the defined PMS--algebras. This concept can further be generalized to soft sets, rough sets and in bi-polar fuzzy for new results in our future work.

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