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Some Relations Related to Centralizers on Semiprime Semiring

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Abstract. In this paper, we generalize the following result. If S is a 2-torsion free semiprime semiring and $T: S \to S$ be an additive mapping such that $2T(xyx) = T(x)yx + xyT(x)$ holds for all $x, y \in S$, then T is a centralizer.

Keywords: Semiring, semiprime semiring, centralizer, jordan centralizer, left (right) centralizer.

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1. Introduction

Semirings has been formally introduced by Vandiver in 1934. Golan [9] discussed the notion of semirings and their applications. In [4], Chandramouleeswaran and Tiruveni worked on the derivations on semirings. Zalar [1] studied centralizers on semiprime rings and proved that Jordan centralizer and centralizers of this rings coincide. In [15], Vukman and Irena proved that if R is a 2-torsion free semiprime ring and $T: R \to R$ is an additive mapping such that $2T(xyx) = T(x)yx + xyT(x)$ holds for all $x, y \in R$, then T is a centralizer. In papers [6,7,8] the authors Hoque and Paul worked on centralizers on semiprime Gamma rings and developed the results of [15] in Gamma rings. Motivated by this Florence and Murugesan [10] studied the notion of semirings and proved that Jordan centralizer of a 2-torsion free semiprime semiring is a centralizer. Here we develop the results of [7,15] in semirings by assuming that S be a 2-torsion free semiprime semiring and $T : S \to S$ be an additive mappingsuch that $2T(xyx) = T(x)yx + xyT(x)$ holds for all $x, y \in S$. Then T is a centralizer.

Now we recall the following definitions and results:

Let S be a non empty set followed with two binary operation '+' and '.' such that

i) $(S, +)$ is a commutative monoid with identity element 0.
ii) $(S, .)$ is a monoid with identity element 1.
iii) Multiplication distributes over addition from either side.
That is, $a.(b + c) = a.b + a.c$,
$(b + c).a = b.a + c.a$. Then S is called a semiring.

A Semiring S is prime if $xSy = 0$ implies $x = 0$ or $y = 0 \forall x, y \in S$, and semiprime if $xSx = 0$ implies $x = 0 \forall x \in S$. A semiring S is 2-torsion free if $2x = 0$, $x \in S \Rightarrow x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$.More over the set $Z(S) = \{ x \in S: xy = yx \ \forall y \in S\}$. we shall use basic commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xz, y] = [x, y]z + x[z, y]$.An additive mapping $T: S \to S$ is called a Left (Right) Centralizer if $T(xy) = T(x)y$ $((T(xy) = xT(y))$ holds for all $x, y \in S$. We call T is a centralizer whichis both left and right centralizer. For a fixed $a \in S$ then $T(x) = ax$ is a left centralizer and $T(x) = xa$ is a right centralizer. An additive mapping $T : S \to S$ is called a left (right) Jordan centralizer if $T(xx) = T(x)x(T(xx) = xT(x))$ holds for all $x \in S$. Every left centralizer is a Jordan left centralizer but the converse is not in general true. An additive mapping $T : S \to S$ is a Jordan centralizer if $T(xy + yx) = T(x)y + yT(x)$ for all $, y \in S$. Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Let $(S, +, .)$ be a semiring. An element a of S is called additively left cacellative if for all $a, b, c \in S, a + b = a + c \Rightarrow b = c$. If every element of a semiring S is additively left cancellative, it is called an additively left cancellative semiring. If every element of a semiring S is additively right cancellative, it is called additively right cancellative semiring. Similarly we can define multiplicatively left and right cancellative semiring.

2. The centralizers of semiprime semiring

Lemma 2.1. Let S be a semiprime semiring. Suppose that the relation
$axb + bxc = 0 \ \forall x \in S$ and some $a, b \in S$. In this case $(a + c)xb = 0$, $\forall x \in S$

Proof: By hypothesis we have $axb + bxc = 0$ (1)

Putting $x = xby$ yields $axbyb + bxbyc = 0$ (2)

On the other hand right multiplying (1) by yb we get

$$axbyb + bxcyb = 0 \ \forall x, y \in S. \quad (3)$$

Subtracting (3) from (2) we obtain $axbyb + bxbyc - axbyb - bxcyb = 0$

$$bxbyc - bxcyb = 0$$

$$bx(byc - cyb) = 0 \quad (4)$$

Putting $x = ycx$ in (4) we get

$$bycx(byc - cyb) = 0 \quad (5)$$

Left multiplying (4) by cy we obtain

$$cybx(byc - cyb) = 0 \quad (6)$$

Subtracting (6) from (5) yields

$$bycx(byc - cyb) - cybx(byc - cyb) = 0$$

$$(byc - cyb)x(byc - cyb) = 0$$

Since S is semiprime, $byc - cyb = 0$

$$\Rightarrow byc = cyb, y \in S$$

Replace $y = x$, in the above relation we get, $bxc = cxb$

So (1) becomes $axb + cxb = 0$

$$(a + c)xb = 0, \forall x \in S$$

Hence the proof is complete.

Lemma 2.2. Let S be a 2-torsion free semiprime semiring and let $T: S \to S$ be an additive mapping such that $2T(xyx) = T(x)yx + xyT(x)$ holds for all $x, y \in S$. Then $2T(xx) = T(x)x + xT(x)$.

Proof:

By the assumption we have $2T(xyx) = T(x)yx + xyT(x)$ (7)

Linearizing the above by putting $x + z$ for x we obtain

$2T((x+z)y(x+z)) = T(x+z)y(x+z) + (x+z)yT(x+z)$

$$2T(xyz + zyx) = T(x)yz + T(z)yx + xyT(z) + zyT(x) \quad (8)$$

Substituting $z = x^2$ the relation (8) yields

$$2T(xyx^2 + x^2yx) = T(x)yx^2 + T(x^2)yx + xyT(x^2) + x^2yT(x) \quad (9)$$

Substitution for $y = xy + yx$ in (7) we arrive at

$$2T(x(xy+yx)x) = T(x)(xy+yx)x + x(xy+yx)T(x)$$

$$2T(x^2yx + xyx^2) = T(x)xyx + T(x)yx^2 + x^2yT(x) + xyxT(x) \quad (10)$$

Subtracting (10) from (9) we get

$$T(x^2)yx + xyT(x^2) - T(x)xyx - xyxT(x) = 0$$

$$(T(x^2) - T(x)x)yx + xy(T(x^2) - xT(x)) = 0$$

From the above relation taking

$$a = T(x^2) - T(x)x, x = y, b = x, c = T(x^2) - xT(x)$$

Now applying lemma 2.1 follows that

$$(T(x^2) - T(x)x + T(x^2) - xT(x))yx = 0$$

$$(2T(x^2) - T(x)x - xT(x))yx = 0$$

Taking $A(x) = 2T(x^2) - T(x)x - xT(x)$, then the above relation becomes,

$$A(x)yx = 0 \quad (11)$$

Applying $y = xyA(x)$ in (11) gives$A(x)\, xyA(x)x = 0$

By the semiprimeness of S, $\quad A(x)x = 0$ (12)

On the other hand left multiplying (11) by$x$ and right multiplying by $A(x)$ we obtain

$$xA(x)yxA(x) = 0$$

Since S is semiprime, $\quad xA(x) = 0$ (13)

Putting $x + y$ for x in (12) we get

$A(x+y)(x+y) = 0$

$A(x)y + A(y)x + B(x,y)x + B(x,y)y + A(x)x + A(y)y = 0$ where

$B(x,y) = 2T(xy + yx) - T(x)y - T(y)x - xT(y) - y(T(x)$

Using (12) the above relation reduces to

$$A(x)y + A(y)x + B(x,y)x + B(x,y)y$$

$$A(x)y + A(y)x + B(x,y)(x+y)$$

Replacing $x + y = x$

$$A(x)y + A(y)x + B(x,y)x.$$

Right multiplication of the above relation by $A(x)$ gives because of (13)

$$A(x)yA(x) + A(y)xA(x) + B(x,y)xA(x)$$

$$A(x)yA(x) = 0 \ \forall\, x, y \in S$$

By the semiprimeness of S, we get $A(x) = 0$.

Thus $2T(x^2) - T(x)x - xT(x) = 0$

$$2T(x^2) = T(x)x + xT(x) \quad (14)$$

This completes the proof.

Lemma 2.3. Let S be a 2-torsion free semiprime semiring and let $T: S \to S$ be an additive mapping, suppose that $2T(xyx) = T(x)yx + xyT(x)$ holds for all pairs $x, y \in S$. Then $[T(x), x] = 0$

Proof: We have $2T(xx) = T(x)x + xT(x)$

Linearizing the above by replacing $x = x + y$ we obtain

$$2T(xy + yx) = T(x)y + T(y)x + xT(y) + yT(x) \quad (15)$$

Replacing $y = 2xyx$ in (15) and using the assumption of the theorem yields

$4T(x^2yx + xyx^2) = 2T(x)xyx + 2T(xyx)x + x2T(xyx) + 2xyxT(x)$

$$= 2T(x)xyx + (T(x)yx + xyT(x))x + x(T(x)yx + xyT(x)) + 2xyxT(x)$$

$$2\big(2T(x^2yx + xyx^2)\big) = 2T(x)xyx + T(x)yx^2 + xyT(x)x + xT(x)yx + x^2yT(x) + 2xyxT(x) \quad (16)$$

Applying (10) in (16) gives

$$2(T(x)xyx + T(x)yx^2 + x^2yT(x) + xyxT(x)) = 2T(x)xyx + T(x)yx^2 + xyT(x)x + xT(x)yx + x^2yT(x) + 2xyxT(x)$$

$$T(x)yx^2 + x^2yT(x) - xyT(x)x - xT(x)yx = 0 \quad (17)$$

Replacing $y = yx$ in (17) we arrive at

$$T(x)yx^3 + x^2yxT(x) - xyxT(x)x - xT(x)yx^2 = 0 \;\; \forall x, y \in S \quad (18)$$

Right multiplication of (17) by x yields,

$$T(x)yx^3 + x^2yT(x)x - xyT(x)x^2 - xT(x)yx^2 = 0 \quad (19)$$

Subtracting (18) from (19) gives

$$x^2yT(x)x - xyT(x)x^2 - x^2yxT(x) + xyxT(x)x = 0$$

$$x^2y[T(x)x - xT(x)] - xy[T(x)x - xT(x)]x = 0$$

$$x^2y[T(x), x] - xy[T(x), x]x = 0 \quad (20)$$

Applying $y = T(x)y$ in (20) leads to

$$x^2T(x)y[T(x), x] - xT(x)y[T(x), x]x = 0 \quad (21)$$

Left multiplication of (20) by $T(x)$ gives

$$T(x)x^2y[T(x), x] - T(x)xy[T(x), x]x = 0 \quad (22)$$

Subtracting (21) from (22) we obtain

$$T(x)x^2y[T(x), x] - T(x)xy[T(x), x]x - x^2T(x)y[T(x), x] + xT(x)y[T(x), x]x = 0$$

$$[T(x)x^2 - x^2T(x)]y[T(x), x] + [xT(x) - T(x)x]y[T(x), x]x = 0$$

$$[T(x), x]xy[T(x), x] + x[T(x), x]y[T(x), x] + [x, T(x)]y[T(x), x]x = 0$$

From (20) we can write $xy[T(x), x] = y[T(x), x]x$ and using $[x, T(x)] = -[T(x), x]$ the above relation becomes

$$[T(x), x]\, y\, [T(x), x]x + x[T(x), x]y[T(x), x] - [T(x), x]y[T(x), x]x = 0$$

$$x[T(x), x]y[T(x), x] = 0 \;\; \forall\, x, y \in S$$

Substituting $y = yx$ in the above relation

$$x[T(x), x]yx[T(x), x] = 0 \;\; \forall\, x, y \in S$$

By the semiprimeness of S, $\quad x[T(x), x] = 0 \quad (23)$

Replacing $y = xy$ in (17) gives

$$T(x)xyx^2 + x^2xyT(x) - xxyT(x)x - xT(x)xyx = 0$$

$$T(x)xyx^2 + x^3yT(x) - x^2yT(x)x - xT(x)xyx = 0 \quad (24)$$

Left multiplication of (17) by x we get

$$xT(x)yx^2 + x^3yT(x) - x^2yT(x)x - x^2T(x)yx = 0 \quad (25)$$

Subtracting (25) from (24) we obtain

$$[T(x)x - xT(x)]yx^2 - x[T(x)x - xT(x)]yx = 0$$

$$[T(x), x]yx^2 - x[T(x), x]yx = 0$$

Using (23) in the above relation yields $\quad [T(x), x]yx^2 = 0 \quad (26)$

Applying $yT(x)$ for y in (26) we obtain $[T(x), x]\, yT(x)x^2 = 0 \quad (27)$

Right multiplication of (26) by $T(x)$ gives $[T(x), x]\, yx^2\, T(x) = 0 \quad (28)$

Subtracting (28) from (27) we get

$$[T(x), x]y(T(x)x^2 - x^2T(x)) = 0$$

$$[T(x), x]y[T(x), x^2] = 0$$

$$[T(x), x]y([T(x), x]x + x[T(x), x]) = 0$$

Using (23) in the above relation reduces to

$$[T(x), x]\, y\, [T(x), x]x = 0$$

Putting $y = xy$ in the above implies $[T(x), x]xy[T(x), x]x = 0$
Since S is semiprime $\quad [T(x), x]x = 0$ (29)
Putting $x = x + y$ in (23) yields
$(x + y)[T(x + y), x + y] = 0$
$$x[T(x), x] + x[T(x), y] + x[T(y), x] + x\,[T(y), y] + y[T(x), x] + y[T(x), y] + y[T(y), x] + y[T(y), y] = 0$$
Using (23) the aboverelation reduces to
$x[T(x), y] + x[T(y), x] + x[T(y), y] + y[T(x), x] + y[T(x), y] + y[T(y), x] = 0$
$(x + y)[T(x), y] + (x + y)[T(y), x] + x[T(y), y] + y[T(x), x] = 0$
Now replacing $x + y = x$ gives
$x[T(x), y] + x[T(y), x] + x[T(y), y] + y[T(x), x] = 0$ (30)
Left multiplication of (30) by $[T(x), x]$ we obtain
$$[T(x), x]x[T(x), y] + [T(x), x]x[T(y), x] + [T(x), x]x[T(y), y] + [T(x), x]y[T(x), x] = 0$$
Using (29) the above relation reduces to $[T(x), x]\,y\,[T(x), x] = 0$
Since S is semiprime $[T(x), x] = 0$.

Theorem 2.1. Let S be a 2-torsion free semiprime semiring. Let $T: S \to S$ be an additive mapping, suppose that $2T(xyx) = T(x)yx + xyT(x)$ holds for all $x, y \in S$. Then T is a centralizer.

Proof: We have $[T(x), x] = 0$
$\Rightarrow T(x)x - xT(x) = 0$
$\Rightarrow T(x)x = xT(x)$
Applying the above results in (14) we obtain $2T(x^2) = 2T(x)x$
Since S is 2-torsion free semiprime semiring $T(x^2) - T(x)x = 0$
$\Rightarrow T(x^2) = T(x)x$
Similarly $T(x^2) = xT(x)$. This means that T is a left and right Jordan Centralizer. By theorem (3.1) in [10] yields that T is a left and right centralizer. Thus the proof is completed.

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