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Size Multipartite Ramsey Numbers for K_4-*e* Versus all Graphs up to 4 Vertices

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Abstract.Let G and H be two simple subgraphs of $K_{j\times s}$. The smallest positive integer s such that any red and blue colouring of $K_{j\times s}$ has a copy of red G or a blue H is called the multipartite Ramsey number of G and H. It is denoted by $m_j(G,H)$. This paper presents exact values for $m_j(B_2,G)$ where G is a isolate vertex free graph up to four vertices.

Keywords: Graph Theory, Ramsey Theory

AMS Mathematics Subject Classification (2010): 05C55, 05D10

1. Introduction

The exact determination of the classical diagonal Ramsey number of $r(s,s)$ (see [6] for a survey) has not made any headway beyond $r(5,5)$ (as of now known to be between 43 and 49) for a few decades. Researches are now trying to approach this problem by using new techniques like fuzzy logic, genetic algorithms to improve the lower bound (see [7], [8] and [9]). One of the main branches of classical Ramsey numbers, introduced by Burger and Vuuren (see [1]) was multipartite Ramsey numbers. It is defined considering $K_{j\times s}$, which consist of j partite sets, each of size s. $V(K_{j\times s})=\{v_{mn} \mid m\in\{1,2,...,j\}$ and $n\in\{1,2,...,s\}\}$ denotes the vertex set of $K_{j\times s}$. The set of vertices in the m^{th} partite set is denoted by $\{v_{mn} \mid n\in\{1,2,...,s\}\}$. The smallest s value for which any two colouring (say red and blue) of $K_{j\times s}$, consist of a red G or a blue H is called the multipartite Ramsey number of G and H and it is denoted by $m_j(G,H)$. Syafrizal and et al (see [10]) investigated a special a case of the multipartite Ramsey number. They found the multipartite Ramsey number $m_j(P_s,G)$ where G is a wheel W_n, a star S_n, a fan F_n and a windmill M_{2n} for $j \geq 2$, $s \in \{2,3\}$ and $n \geq 6$. Jayawardena and et al (see [3,4]) also investigated special cases of the multipartite Ramsey number. They found $m_j(H,G)$ where $H \in \{C_3, C_4\}$ and G is any graph on four vertices and for $j \geq 3$ .This paper is also on a special case of $m_j(B_2,G)$ where G is any isolate vertex free graph up to four vertices where $j \geq 3$. These values are shown in the following table.

$m_j(B_2,G)$ values									
$j =$	3	4	5	6	7	8	9	10	$j \geq 11$
P_2			1	1	1	1	1	1	1
$2K_2$	2	2	1	1	1	1	1	1	1
P_3	3	2	1	1	1	1	1	1	1
P_4	4	2	2	1	1	1	1	1	1
C_3	infinity	infinity	infinity	2	1	1	1	1	1
$K_{1,3}$	4	3	2	2	1	1	1	1	1
C_4	infinity	3	2	2	1	1	1	1	1
$K_{1,3}+x$	infinity	infinity	infinity	2	1	1	1	1	1
B_2	infinity	infinity	infinity	2	2	2	2	1	1
K_4	infinity	infinity	infinity	infinity	infinity	infinity	2	2	1

Table 1: $m_j(B_2,G)$ values

2. Size Ramsey numbers $m_j(B_2,P_2)$

Theorem 1. *If* $j \geq 3$*, then*

$$m_j(B_2,P_2) = \begin{cases} 2 & \text{if } j = 3 \\ 1 & \text{if } j \geq 4 \end{cases}$$

Proof: Consider the $K_{3\times 1}$ where all its edges are colored in red. Then $K_{3\times 1}$ has neither a red B_2 nor a blue P_2. Therefore $m_3(B_2, P_2) \geq 2$.

Next consider any red-blue colouring of $K_{3\times 2}$. If it has a blue P_2, then we are done. Otherwise all the edges of $K_{3\times 2}$ are red and hence it contains a red B_2. Therefore, $m_3(B_2, P_2) = 2$.

Clearly $m_j(B_2, P_2) = 1$ when $j \geq 4$ as $r(B_2, P_2) = 4$ (see [2]).

3.Size Ramsey numbers $m_j(B_2, P_3)$

Theorem 2. *If* $j \geq 3$*, then*

$$m_j(B_2, P_3) = \begin{cases} 3 & \text{if } j = 3 \\ 2 & \text{if } j = 4 \\ 1 & \text{if } j \geq 5 \end{cases}$$

Proof: Consider the coloring $K_{3\times 2} = H_R \oplus H_B$ where $H_B = 3K_2$. Then $K_{3\times 2}$ has neither a red B_2 nor a blue P_3. Therefore $m_3(B_2, P_3) \geq 3$.

Now consider any red-blue colouring of $K_{3\times 3}$. Assume it has no blue P_3 or a red B_2. Then it contains a red C_3, say $v_{11}v_{21}v_{31}v_{11}$ (since $m_3(C_3, P_3) = 2$ by [3]). Due to the absence of a red B_2, at least two of the vertices of $\bigcup_{j=2}^{3}\{v_{ij} : i \in \{1,2,3\}\}$ must be adjacent to each other in blue(say u and v). But then in order to avoid a red B_2, v must be adjacent to some vertex of the red C_3 in blue. This forces a blue P_3. A contradiction. Therefore, $m_3(B_2, P_3) = 3$.

Next consider the red-blue colouring of $K_{4\times 1}$ having $v_{11}v_{21}$ and $v_{31}v_{41}$ as the only blue edges. Then $K_{4\times 1}$ has no red B_2 and has no blue P_3. Therefore, $m_4(B_2, P_3) \geq 2$.

Consider any red-blue colouring of $K_{4\times 2}$ which is blue P_3 free. Then $K_{4\times 2}$ has a red C_3 (as $m_4(C_3, P_3) = 2$ by [3]). Let v be a vertex that does not belong to any of the partite sets which vertices of the red C_3 belong to. Then in order to avoid a blue P_3, v must be adjacent in red to at least two vertices of the red C_3. Thus $K_{4\times 2}$ has a red B_2. Hence $m_4(B_2, P_3) = 2$.

Clearly $m_j(B_2, P_3) = 1$ when $j \geq 5$ as $r(B_2, P_3) = 5$ (see [2]).

4.Size Ramsey numbers $m_j(B_2, 2K_2)$

Theorem 3.*If* $j \geq 3$*, then*

$$m_j(B_2, 2K_2) = \begin{cases} 2 & \text{if } j \in \{3,4\} \\ 1 & \text{if } j \geq 5 \end{cases}$$

Proof: The graph $K_{4\times 1} = H_R \oplus H_B$ where H_R consist only of the red C_3, $v_{11}v_{21}v_{31}v_{11}$ is red B_2 free and blue $2K_2$ free. Therefore $m_3(B_2, 2K_2) \geq m_4(B_2, 2K_2) \geq 2$.

Consider the graph $K_{3\times 2} = H_R \oplus H_B$ which is red B_2 free. Then it contains a blue P_2 (say $v_{11}v_{21}$). If all the edges not incident to v_{11} or v_{21} are red then $K_{3\times 2}$ has a red B_2, a contradiction. Therefore, there is a blue edge not incident to v_{11} or v_{21}. Hence $K_{3\times 2}$ has a blue $2K_2$. Therefore $m_3(B_2, 2K_2) = m_4(B_2, 2K_2) = 2$.

Clearly, $m_j(B_2, 2K_2) = 1$ when $j \geq 5$ as $r(B_2, 2K_2) = 5$ (see [2]).

5.Size Ramsey numbers $m_j(B_2, P_4)$

Theorem 4.*If* $j \geq 3$*, then*

$$m_j(B_2, P_4) = \begin{cases} 4 & \text{if } j = 3 \\ 2 & \text{if } j \in \{4,5\} \\ 1 & \text{if } j \geq 6 \end{cases}$$

Proof: Consider the coloring $K_{3\times 3} = H_R \oplus H_B$ where H_B consist only of the three 3-cycles in $\{v_{1i}v_{2i}v_{3i}v_{1i} : i \in \{1,2,3\}\}$. Then $K_{3\times 3}$ has neither a red B_2 nor a blue P_4. Therefore $m_3(B_2, P_4) \geq 4$.

Now consider any red-blue colouring of $K_{3\times 4}$. Assume it has no red B_2 and no blue P_4. Then each of the subgraphs H_1 and H_2 where $V(H_1) = \cup_{j=1}^{3}\{v_{ij} : i \in \{1,2,3\}\}$ and

$V(H_2)=\cup_{j=2}^{4}\{v_{ij}: i\in\{1,2,3\}\}$ has a blue P_3. By relabeling we can assume them to be $v_{11}v_{21}v_{31}$ and $v_{14}v_{24}v_{33}$. As $K_{3\times 4}$ is blue P_4 free, v_{14} and v_{31} are not incident to any blue edges other than $v_{14}v_{24}$ and $v_{31}v_{21}$ respectively. Then the red edges $\{v_{14}v_{2i}: i\in\{2,3\}\}\cup\{v_{31}v_{2i}: i\in\{2,3\}\}\cup\{v_{14}v_{31}\}$ forces a red B_2, a contradiction. Therefore, $m_3(B_2,P_4)=4$.

Next consider the coloring $K_{5\times 1}=H_R\oplus H_B$. where H_B consist only of the edge $v_{21}v_{31}$ and the 3-cycle $v_{41}v_{51}v_{11}v_{41}$. Then $K_{5\times 1}$ has neither a blue P_4 nor a red B_2. Hence $m_4(B_2,P_4)\geq m_5(B_2,P_4)\geq 2$ .Now consider any red-blue coloring of $K_{4\times 2}$ which is blue P_4 free. Then it has a red C_3, say $v_{11}v_{21}v_{31}v_{11}$ (as $m_4(C_3,P_4)=2$ by [3]). Suppose each of v_{41} and v_{42} are incident to two vertices of $V(C_3)$ in blue. Then $K_{4\times 2}$ has a blue P_4, a contradiction. Therefore at least one of v_{41} or v_{42} (say v_{41}) is incident to two vertices (say v_{11} and v_{21}) of $V(C_3)$ in red. Then the red edges $v_{41}v_{11}, v_{41}v_{21}$ together with the red C_3 forms a red B_2. Therefore, $2\geq m_4(B_2,P_4)$. Thus, we get $m_4(B_2,P_4)=m_5(B_2,P_4)=2$.

Consider any red-blue coloring of $K_{6\times 1}$ with no blue P_4. Then it has a red C_3, say $v_{11}v_{21}v_{31}v_{11}$ (as $m_6(C_3,P_4)=1$ by [3]). Now considering v_{41} and v_{51} and arguing as above, it can be proved that $m_j(B_2,P_4)=1$ when $j\geq 6$.

6.Size Ramsey numbers $m_j(B_2,K_{1,3})$

Theorem 5. *If* $j\geq 3$*, then*

$$m_j(B_2,K_{1,3})=\begin{cases}4 & \text{if } j=3\\ 3 & \text{if } j=4\\ 2 & \text{if } j\in\{5,6\}\\ 1 & \text{if } j\geq 7\end{cases}$$

Proof: Consider a colouring of $K_{3\times 3}=H_R\oplus H_B$ where H_B consists of a blue C_9 as indicated in the following diagram.

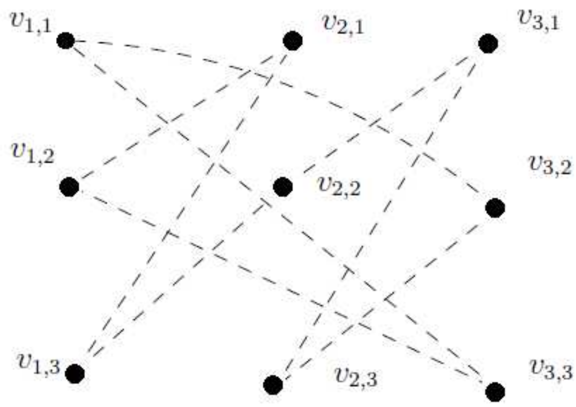

Figure 1: When H_B consists of blue a C_9.

Then $K_{3\times 3}$ has neither a red B_2 nor a blue $K_{1,3}$. Therefore, $m_3(B_2, K_{1,3}) \geq 4$.

Now consider any colouring of $K_{3\times 4} = H_R \oplus H_B$. Assume the graph has no blue $K_{1,3}$. Then $K_{3\times 4}$ has a red C_3, say $v_{11}v_{21}v_{31}v_{11}$ (as $m_3(C_3, K_{1,3}) = 3$ by [3]). Let $W = \{v_{11}, v_{21}, v_{31}\}$. In order to avoid a red B_2 each vertex of $V(K_{3\times 4}) \setminus W$ is adjacent in blue to one vertex in W.

By pigeon hole principle this results in a blue $K_{1,3}$, a contradiction. Therefore, $m_3(B_2, K_{1,3}) \leq 4$.

Hence $m_3(B_2, K_{1,3}) = 4$.

Next as $m_4(C_3, K_{1,3}) = 3$ (see [3]) and C_3 is a subgraph of B_2, we get $m_4(B_2, K_{1,3}) \geq 3.$

Now consider any colouring of $K_{4\times 3} = H_R \oplus H_B$. Assume the graph has no blue $K_{1,3}$. Then $K_{4\times 3}$ has a red C_3, say $v_{11}v_{21}v_{31}v_{11}$ (as $m_4(C_3, K_{1,3}) = 3$ by [3]). In order to avoid a red B_2 each vertex in $\{v_{41}, v_{42}, v_{43}\}$ is adjacent in blue to two vertices in $\{v_{i1} : i \in \{1,2,3\}\}$ and further v_{12} is adjacent in blue to one vertex in $\{v_{i1} : i \in \{2,3\}\}$. By pigeon hole principle this results in a blue $K_{1,3}$, a contradiction. Therefore, $m_4(B_2, K_{1,3}) \leq 3$. Hence $m_4(B_2, K_{1,3}) = 3$.

Next consider any colouring of $K_{6\times 1} = H_R \oplus H_B$ where H_B consist only of the two blue cycles, $v_{11}v_{21}v_{31}v_{11}$ and $v_{41}v_{51}v_{61}v_{41}$. Then $K_{6\times 1}$ has niether a red B_2 nor a blue $K_{1,3}$. Therefore, $m_5(B_2, K_{1,3}) \geq m_6(B_2, K_{1,3}) \geq 2$.

Now consider any red-blue coloring of $K_{5\times 2}$. Assume the graph has no blue $K_{1,3}$. Then there is a red C_3 say $v_{11}v_{21}v_{31}v_{11}$ (since $m_5(C_3, K_{1,3}) = 2$ by [3]). If every vertex in $\{v_{41}, v_{42}, v_{51}, v_{52}\}$ is adjacent in blue to at least two vertices of $V(C_3)$ then the graph has a blue $K_{1,3}$, a contradiction. Therefore there is a vertex in $\{v_{41}, v_{42}, v_{51}, v_{52}\}$ which is adjacent in red to two vertices of $V(C_3)$. This forces a red B_2. Hence $m_5(B_2, K_{1,3}) \leq 2$. Therefore, $m_j(B_2, K_{1,3}) = 2$ when $j \in \{5,6\}$.

Finally, we have $m_j(B_2, K_{1,3}) = 1$ for $j \geq 7$ as $r(B_2, K_{1,3}) = 7$ (see [2]).

7.Size Ramsey numbers $m_j(B_2, G)$ for other graphs G

Theorem 6. *If $j \geq 3$, then*

$$m_j(B_2, K_{1,3}+x) = \begin{cases} \infty & \text{if } j \in \{3,4,5\} \\ 2 & \text{if } j = 6 \\ 1 & \text{if } j \geq 7 \end{cases}$$

Proof: As $m_j(C_3, K_{1,3}+x) = \infty$ when $j \in \{3,4,5\}$ and C_3 is a subgraph of B_2, we have $m_j(B_2, K_{1,3}+x) = \infty$ when $j \in \{3,4,5\}$. As $m_6(C_3, K_{1,3}+x) = 2$ (see [3]) and C_3 is a subgraph of B_2, we have $m_6(B_2, K_{1,3}+x) \geq 2$.

Next consider any red B_2 free colouring of $K_{6\times 2} = H_R \oplus H_B$. Then the graph has a blue C_3, say $v_{11}v_{21}v_{31}v_{11}$ (as $m_6(B_2, C_3) = m_6(C_3, B_2) = 2$ by [3]). If any vertex of $V(C_3)$ is adjacent to any vertex in $\cup_{i=4}^{6}\{v_{i1}, v_{i2}\}$ is blue, then the graph has a blue $K_{1,3}+x$ and hence we are done with the proof. Therefore, assume that every vertex of $V(C_3)$ is adjacent to every vertex in $\cup_{i=4}^{6}\{v_{i1}, v_{i2}\}$ in red. Since the graph has no red B_2 all edges $v_{42}v_{52}, v_{42}v_{61}, v_{52}v_{61}, v_{52}v_{62}$ are blue. This result in a blue $K_{1,3}+x$. Therefore, $m_6(B_2, K_{1,3}+x) \leq 2$. Hence, $m_6(B_2, K_{1,3}+x) = 2$.

$m_j(B_2, K_{1,3}+x) = 1$ as $r(B_2, K_{1,3}+x) = 7$ (see [2]).

Theorem 7. *If* $j \geq 3$, *then*

$$m_j(B_2, C_3) = \begin{cases} \infty & \text{if } j \in \{3,4,5\} \\ 2 & \text{if } j = 6 \\ 1 & \text{if } j \geq 7 \end{cases}$$

Proof: See [3].

Theorem 8. *If* $j \geq 3$, *then*

$$m_j(B_2, C_4) = \begin{cases} 4 & \text{if } j = 3 \\ 3 & \text{if } j = 4 \\ 2 & \text{if } j \in \{5,6\} \\ 1 & \text{if } j \geq 7 \end{cases}$$

Proof: See [4].

Theorem 9. *If* $j \geq 3$, *then*

$$m_j(B_2, B_2) = \begin{cases} \infty & \text{if } j \leq 5 \\ 2 & \text{if } j \in \{6,7,8,9\} \\ 1 & \text{if } j \geq 10 \end{cases}$$

Proof: Consider the coloring of $K_{5\times s}$ where $V(K_{5\times s}) = \{v_{mn} \mid m \in \{1,2,...,5\}$ and $n \in \{1,2,...,s\}\}$, given by $K_{5\times s} = H_R \oplus H_B$ where s represent any positive integer.

Here v_{ij} is connected in blue to $v_{i'j'}$ if $i=1$, $i' \in \{2,5\}$ and $i \in \{2,3,4\}$, $i' = i \pm 1$ and $i=5$, $i' \in \{4,1\}$ for any $j, j' \in \{1,2,...,s\}$.

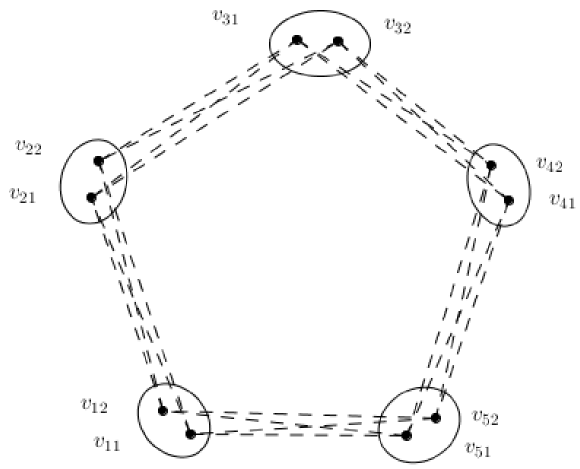

Figure 2: The graph H_B.

Clearly the coloring $K_{5\times s} = H_R \oplus H_B$ doesn't contain a red B_2 nor a blue B_2. Therefore, we get that $m_5(B_2, B_2) \geq s$. Since s is an arbitrary positive integer, we get that $m_5(B_2, B_2) = \infty$. Thus, $m_j(B_2, B_2) = \infty$ for all $j \leq 5$.

Next we will prove that $m_6(B_2, B_2) \leq 2$. To prove this first we will use the following lemma.

Lemma 10. *Suppose that v is any vertex of $K_{6\times 2}$, and that $V = \{v_1, v_2, ..., v_5\}$ is a set of five vertices in $N_B(v)$, belonging to at least 4 partite sets which v doesn't belong.Then, V will induce a red B_2 or V will induce a blue P_3. Thus $K_{6\times 2}$ will contain either a red or a blue B_2.*

Proof: Under the given conditions there are two possible cases. The first namely when V belongs to 5 partite sets. In this case the result follows directly as $r(B_2, P_3) = 5$ (see [2]) . If we assume that no blue P_3 exists then in the other case when Vbelongs to 4 partite sets with v_4, v_5 belonging to the same partite set we get that the only paths the induced subgraph of V in blue can have are P_2. Thus the only subgraphs of induced subgraphs of

V in blue are P_2 or $2P_2$. But then clearly by exhaustive search we see that the induced subgraph of V in red will have a red B_2, as required.

Lemma 11.. *Consider the coloring of* $K_{6\times 2} = H_R \oplus H_B$ *such that* H_R *contains a red* $3K_3$ *where two* K_3*'s will belong to the same partite sets. Then either* H_R *contains a red* B_2 *or* H_B *contains a blue* B_2.

Proof: Suppose that the three disjoint red three cycles in H are generated by the sets $H_1 = \{v_{11}, v_{21}, v_{31}\}$, $H_2 = \{v_{41}, v_{51}, v_{61}\}$ and $H_3 = \{v_{42}, v_{52}, v_{62}\}$. Assume that H_R is B_2-free. As there is no red B_2 containing v_{22}, without loss of generality we may assume that $e = (v_{11}, v_{22})$ will be a blue edge. Again as there is no red B_2, the edge e will have a common neighboring vertex in H_2 and H_3 in blue. Thus H_B contains a blue B_2, as required in the lemma.

To prove that $m_6(B_2, B_2) \leq 2$, consider any red B_2 free and blue B_2 free colouring of $K_{6\times 2} = H_R \oplus H_B$. Since $r(C_3, C_3) = 6$ by symmetry the induced red subgraph H generated by $\{v_{p1} \mid p \in \{1,...,6\}\}$ will have a copy of a red C_3. Without loss of generality let the red C_3 in H be $v_{11}v_{21}v_{31}v_{11}$. Let $H_1 = \{v_{11}, v_{21}, v_{31}\}$ and $H_2 = \{v_{12}, v_{22}, v_{32}\}$. Let $Y = K_{6\times 2} \setminus (H_1 \cup H_2)$. Let B and R denote the blue and red induced subgraphs by Y. Consider any vertex of H_2. As it together with H_1 do not induce a red B_2 we get that without loss of generality one of the possible cases.

Case 1: If one vertex say x (say v_{21}), of H_1 is adjacent to two vertices of H_2 in blue and another vertex of H_1 is adjacent in blue to a the vertex of H_2.

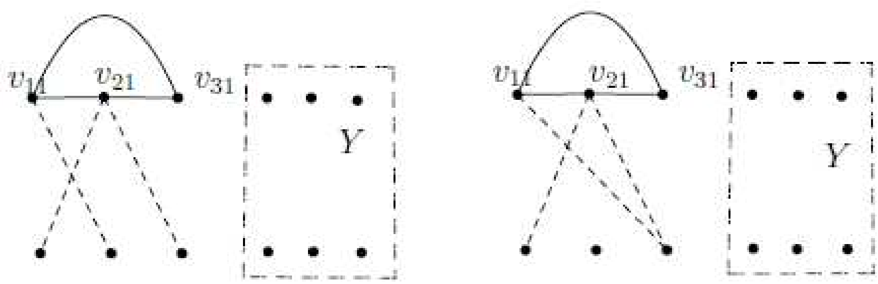

Figure 3: The two possibilities for Case 1

Without loss of generality assume that (v_{11}, v_{22}), (v_{21}, v_{12}) and (v_{21}, v_{32}) are the blue edges between H_1 and H_2. Since each vertex of Y is adjacent in blue to two vertices of

H_1, there are at least 12 blue edges between Y and H_1. But by lemma 10, as v_{11} can be adjacent to at most 4 vertices of Y (note in the case its adjacent to exactly four vertices it must belong to exactly two partite sets of Y) and also v_{21} can be adjacent to at most 2 vertices of Y. Therefore, we get that v_{31} must be adjacent in blue to all the vertices of Y. But then if B has a blue P_3 we are done as it will result in a blue B_2. Therefore, B can have at most one blue edge or two disjoint blue edges or 3 disjoint blue edges. In the first two options clearly R will contain a red B_2, a contradiction. The last option will result in R containing a red $2K_3$ as in lemma 11. This will lead to a contradiction by the lemma.

Case 2: If $(v_{11}, v_{22}), (v_{21}, v_{32})$ and (v_{31}, v_{12}) are the only blue edges between H_1 and H_2.

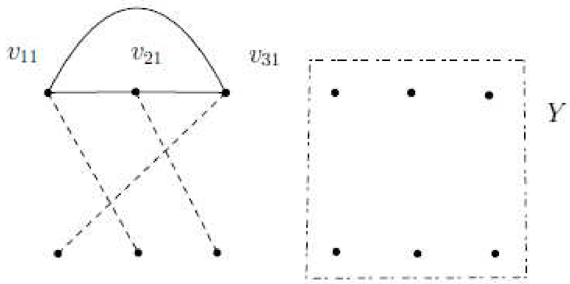

Figure 4: The Case 2.

However, as each vertex of Y has to be adjacent to at least two vertices of H_1 in blue, we get that each of the 3 vertices of H_1 will be adjacent to exactly 4 vertices of Y belonging to exactly 2 partite sets as otherwise H_1 will have a vertex satisfying Lemma 10, resulting in a contradiction. If we investigate the internal structure of B there are two possible subcases.

Subcase 2.1: If B has one blue edge or two disjoint blue edges or 3 disjoint blue edges. If $B \cong K_2 \cup 4K_1$ or $B \cong 2K_2 \cup 2K_1$, clearly we see that R contains a B_2. If $B \cong 3K_2$, then we obtain a contradiction by lemma 11.

Subcase 2.2: If B contains a P_3

There are two possibilities corresponding to this subcase. In the first possibility if B contains a P_3 incident to 2 partite sets. , Without loss of generality say this path is given by $v_{41}v_{51}v_{42}$. But then, by lemma 10, each of the 3 vertices of H_1 will be adjacent to exactly 4 vertices of Y belonging to exactly 2 partite sets. Further, as there is no red B_2, the blue neighborhood in Y of any two vertices belonging to $\{v_{11}, v_{21}, v_{31}\}$ will be distinct. Therefore, $\{v_{i1}, v_{41}, v_{42}, v_{51}\}$ for some $i \in \{1,2,3\}$ will induce a blue B_2, a contradiction.

In the second possibility if B contains a P_3 incident to 3 partite sets and no P_3 incident to 2 partite sets. Without loss of generality say this path is given by $v_{41}v_{52}v_{61}$. Then in order to avoid the first possibility (v_{41}, v_{51}), (v_{51}, v_{61}), (v_{42}, v_{52}) and (v_{52}, v_{62}) will have to be red edges. But then, if (v_{41}, v_{61}) is blue we get that $W = \{v_{41}, v_{52}, v_{61}, v_{i1}\}$ for some $i \in \{1,2,3\}$ will induce a blue B_2, a contradiction. Next suppose that (v_{41}, v_{61}) is red. But then as R contains no red B_2 both v_{42} and v_{62} will have to be adjacent in blue to at least one vertex in $W' = \{v_{41}, v_{51}, v_{61}\}$. This will give rise to a similar situation as in case 1, with H_1 and H_2 replaced by W' and $Y \setminus W'$ respectively, resulting in the required contradiction.

Therefore by the above two cases we can conclude that, $m_6(B_2, B_2) \leq 2$.

Consider the coloring of $K_{9\times1} = H_R \oplus H_B$ where H_B is given in the Figure 5. This coloring $K_{9\times1}$ contains no red B_2 or a blue B_2. Therefore, we obtain that $m_9(B_2, B_2) \geq 2$.

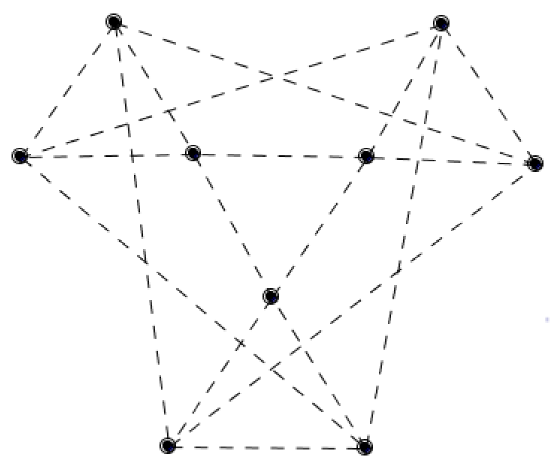

Figure 5: The graph H_B.

Since $m_i(B_2,B_2) \le m_j(B_2,B_2)$ for $i \ge j$, using the inequalities $m_9(B_2,B_2) \ge 2$ and $m_6(B_2,B_2) \le 2$, we can conclude that $m_i(B_2,B_2) = 2$ for $i \in \{6,7,8,9\}$.

Clearly $m_j(B_2,B_2) = 1$ when $j \ge 10$ as $r(B_2,B_2) = 10$ (see [2]). Hence the Theorem.

Theorem 12. *If* $j \ge 3$, *then*

$$m_j(B_2,K_4) = \begin{cases} \infty & \text{if } j \le 8 \\ 2 & \text{if } j \in \{9,10\} \\ 1 & \text{if } j \ge 11 \end{cases}$$

Proof: Let s be a positive integer. Consider the coloring of $K_{8\times s} = H_R \oplus H_B$ where H_R is as follows.

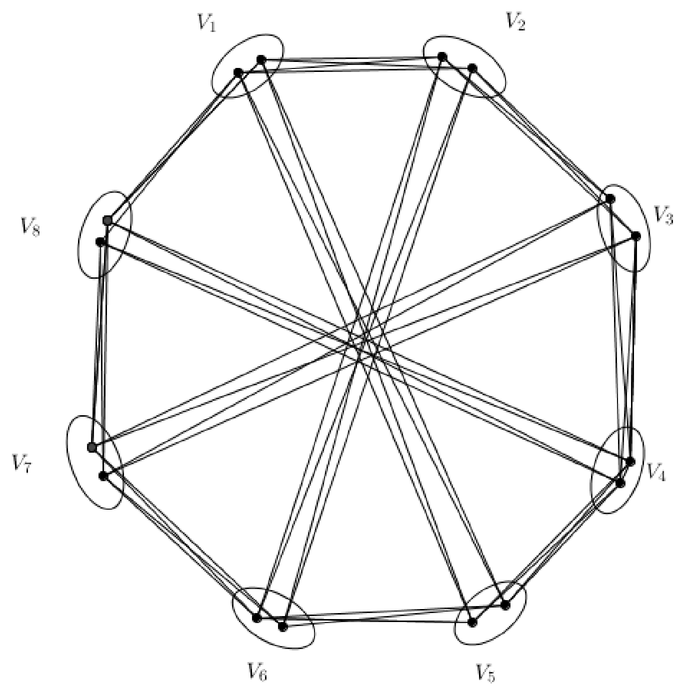

Figure 6: In the case $s = 2$ the partite set V_i consists of the two vertices v_{i1} and v_{i2}

Let the 8 partite sets of $V(K_{8\times s})$ be labelled as $V_1, ..., V_8$. For any $i \in \{2,...,7\}$, the edges between the pair of sets V_{i+1}, V_i and the pair of sets V_{i-1}, V_i are red. For any

$i \in \{1,2,3,4\}$, the edges between the pair of set V_i, V_{i+4} are red. Also, the edges between V_1 and V_8 are red.
Then this coloring $K_{8\times s}$ contains no red B_2 or a blue K_4. Therefore, we obtain that $m_8(B_2, K_4) \geq s$. Since s is arbitrary, we get that $m_8(B_2, K_4) = \infty$. Therefore we could conclude that $m_j(B_2, K_4) = \infty$ for all $j \leq 8$.

Lemma 13. *Suppose that v is any vertex of $K_{9\times 2}$, adjacent in blue to $V = \{v_1, v_2, ..., v_9\}$ belonging to 6 partite sets $V_1, V_2, ..., V_6$ such that $v_7 \in V_1$, $v_8 \in V_2$, $v_9 \in V_3$ and $v_i \in V_i$ for all i where $1 \leq i \leq 6$. Then, $K_{9\times 2}$ will induce a red B_2 or a blue K_4. Also if v is adjacent in blue to seven vertices belonging to 7 partite sets then $K_{9\times 2}$ will induce a red B_2 or a blue K_4.*
Proof: Assume that $K_{9\times 2}$ is red B_2 free.

Remark: Assume that there is a red C_3 in V such that the vertices of the red C_3 belong to the partite sets V_i, V_j and V_k where $3 \leq i, j, k \leq 6$. Then any two vertices of V belonging to distinct partite sets outside of $V_i \cup V_j \cup V_k$ will be adjacent to each other in red. This is clear, as if there is such a blue edge (u, w) belonging to distinct partite sets of V outside of $V_i \cup V_j \cup V_k$, then in order to avoid a red B_2, u and w will have to be adjacent in blue to a common vertex of $V(C_3)$. That is, it will contain a blue C_3 in V. Thus, $K_{9\times 2}$ will induce a blue K_4 (containing v), as required.
As $r(C_3, C_3) = 6$ (see [2]), under the conditions stated in the lemma, there is a red C_3 induced by any 6 vertices of V belonging to 6 partite sets. By the above remark in order to avoid a red B_2, the vertices of $V(C_3)$ must be contained in the first three partite sets. But even in this case this will force a red C_3 contained in $V_4 \cup V_5 \cup V_6$. However, in such a situation, by the repeated use of the remark on this red C_3 will result in blue K_4, as required.
The later part of the lemma follows directly as $r(B_2, C_3) = 7$ (see [2]).

Lemma 14. *Suppose that v is any vertex of $K_{9\times 2}$ adjacent in red to $V = \{v_1, v_2, ..., v_7\}$ belonging to 4 partite sets $V_1, V_2, ..., V_4$ such that $v_4 \in V_4$ and $v_i, v_{i+4} \in V_i$ for all i such that $1 \leq i \leq 3$. Then, $K_{9\times 2}$ will induce a red B_2 or a blue K_4.*

Proof. Assume $K_{9\times 2}$ has no red B_2. Clearly, V cannot have any red P_3 as it would force a red B_2 in $K_{9\times 2}$. Thus the red edges of V must consist of a P_2, $2P_2$ or a $3P_2$. However, exhaustive search shows that in each of these possibilities a vertex not incident to a red edge in V will be contained in a blue induced subgraph of V isomorphic to a K_4. Hence the lemma.

Lemma 15. *Suppose that v is any vertex of $K_{9\times 2}$ adjacent in blue to $V = \{v_1, v_2, ..., v_{11}\}$ belonging to 6 partite sets $V_1, V_2, ..., V_6$ such that $v_6 \in V_6$ and $v_i, v_{i+6} \in V_i$ for all i such that $1 \le i \le 5$. Then, $K_{9\times 2}$ will induce a red B_2 or a blue C_3.*

Proof. Suppose that V doesn't induce a red B_2 nor a blue C_3. As $r(C_3, C_3) = 6$ (see [2]), there is a red C_3 induced by any 6 vertices of V belonging to 3 partite sets. Let the vertices of this red C_3 be $\{x, y, z\}$. In order to avoid a red B_2 each vertex in $V \setminus \{x, y, z\}$, not belonging to the partite sets which x, y and z belong to, must be adjacent to at least 2 vertices of $\{x, y, z\}$ in blue. Also in order to avoid a red B_2 each vertex in $V \setminus \{x, y, z\}$, belonging to the partite sets which x, y and z belong to, must be adjacent to at least 1 vertex of $\{x, y, z\}$ in blue. By pigeon hole principle, without loss of generality we may assume that x is adjacent in blue to 5 vertices of V. Since these 5 vertices belong to at least 3 partite sets and V doesn't contain a red B_2, these 5 vertices must induce a blue edge say (p, q). But then we get the required contradiction as x, p, q will induce a blue C_3. Hence the lemma.

To prove that $m_9(B_2, K_4) \le 2$, consider any red B_2 free and blue K_4 free colouring of $K_{9\times 2} = H_R \oplus H_B$. Since $r(C_3, K_4) = 9$ (see [2]) the induced subgraph H of H_R generated by $\{v_{p2} \mid p \in \{1,...,9\}\}$ will have a copy of a red C_3, say $v_{42}, v_{52}, v_{62}, v_{42}$. Let $W = \{v_{42}, v_{52}, v_{62}\}$. Since $m_6(B_2, C_3) = 2$, we may assume the induced subgraph of H_B generated by $\{v_{p1} \mid p \in \{1,2,3,7,8,9\}\} \cup \{v_{p2} \mid p \in \{1,2,3,7,8,9\}\}$ will have a copy of a blue C_3 with $V(C_3) = \{v_{11}, v_{21}, v_{31}\}$. Let $V = \{v_{11}, v_{21}, v_{31}\}$. Then as there is no red B_2, each vertex in V will be adjacent in blue to two vertices of W. Also as there is no blue K_4 each vertex in W will have to be adjacent in red to at least one vertex of V. Thus we may assume that (v_{11}, v_{42}), (v_{21}, v_{52}) and (v_{31}, v_{62}) are the only red edges between V and W. Let $W_1 = \{v_{41}, v_{51}, v_{61}\}$. Then as there is no red B_2, each vertex in

W_1 will be adjacent in blue to a vertex of W. Thus we get that without loss of generality there are two possible cases.

Remark: Note that we will be using the fact that any vertex outside of $W \cup W_1$ will be adjacent to at least two vertices of W in blue. Also given any six vertices belonging to 3 partite sets outside $W \cup W_1$ will contain at least 4 vertices belonging to exactly 3 partite sets adjacent to some vertex of W in blue or else all three vertices of W will be adjacent in blue to exactly 4 vertices belonging to exactly two partite sets.

Case 1: If (v_{41}, v_{62}), (v_{51}, v_{62}) and (v_{61}, v_{52}) are blue edges between W_1 and W.

Let $U = V(K_{9\times 2})$. In order to avoid a red B_2 each vertex in $U \setminus (W \cup W_1)$, must be adjacent to at least 2 vertices of $\{v_{42}, v_{52}, v_{62}\}$ in blue. Also in order to avoid a red B_2 each vertex in W_1, must be must be adjacent to at least 1 vertex of W in blue. By pigeon hole principleand lemma 13, without loss of generality we may assume that at least one vertex say z of $\{v_{42}, v_{52}, v_{62}\}$ is adjacent in blue to 9or more vertices of $U \setminus (W \cup W_1)$ (by lemma 13, all three vertices cannot be adjacent to exactly 8 vertices of $U \setminus (W \cup W_1)$).Futher, by lemma 13, z can not be incident in blue to 6 or more than 6 partite sets. Therefore, z must be incident to 9 vertices belonging to 5 partite sets. Since v_{62} is incident to v_{41} and v_{51} we get $z \neq v_{62}$. If $z = v_{52}$ then there are two possibilities, namely (v_{41}, v_{52}) is blue or (v_{41}, v_{52}) is red. If (v_{41}, v_{52}) is blue, lemma 13 will give us the required contradiction. If (v_{41}, v_{52}) is red, applying lemma 14 with $v = v_{52}$ or lemma 13 with $v = v_{52}$ will give us a contradiction. Therefore, $z = v_{42}$. Further, as shown above since v_{52} and v_{62} can be adjacent to at most 8 vertices of U in blue, v_{42} will have to be adjacent to 11 vertices of U in blue. Now applying lemma 15 to the 11 vertices v_{42} is adjacent in blue, we get a blue C_3 induced by these 11 vertices unless these 11 vertices belong to at least 7 partite sets. But in this case too, since $r(B_2, C_3) = 7$ (see [2]), we will get a blue C_3 induced by these vertices. Thus, v_{42} together with these 3 vertices will induce a blue K_4, a contradiction.

Case 2: If (v_{41}, v_{62}), (v_{51}, v_{42}) and (v_{61}, v_{52}) are the only blue edges between W_1 and W.

By remark without loss of generality we may assume that v_{62} is adjacent in blue to 4 vertices of $\{v_{mn} \mid m \in \{7,8,9\} \text{ and } n \in \{1,2\}\}$. In this case there are two possible scenarios: In the first if v_{32} is incident to v_{62} in blue. Then applying lemma 13 with

$v = v_{62}$ will give us the required contradiction, unless v_{62} is adjacent to both v_{12} and v_{22} in red and v_{62} is adjacent in blue to at most 2 partite sets out of V_7, V_8 and V_9. But in order to avoid a red B_2, this will force all edges between $\{v_{12}, v_{22}\}$ and $\{v_{42}, v_{52}\}$ to be blue. But then by pigeon hole principle without loss of generality we may assume that v_{52} is adjacent to at least 4 vertices of $\{v_{mn} \mid m \in \{7,8,9\} \text{ and } n \in \{1,2\}\}$ in blue. Irrespective of whether these 4 vertices belong to two or three partite sets of V_7, V_8 and V_9, applying lemma 13 with $v = v_{52}$ will give us the required contradiction.

In the second scenario by symmetry we may assume that (v_{32}, v_{62}), (v_{22}, v_{42}) and (v_{12}, v_{42}) are red. According to whether v_{62} is adjacent to two or three partite sets in blue of $\{v_{mn} \mid m \in \{7,8,9\}$ and $n \in \{1,2\}\}$ we will obtain a required contradiction by applying lemma 14 and lemma 13 respectively to $v = v_{62}$.Therefore we can conclude that, $m_9(B_2, K_4) \leq 2$

Consider the coloring of $K_{10\times 1} = H_R \oplus H_B$ where H_R is given in the Figure 7. This coloring of $K_{10\times 1}$ contains no red B_2 or a blue K_4. Therefore, we obtain that $m_{10}(B_2, K_4) \geq 2$.

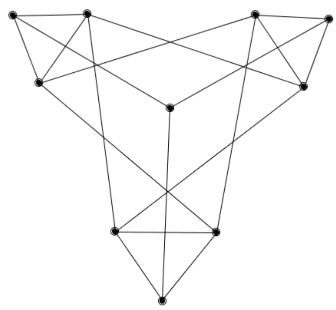

Figure 7: The graph H_R.

Since $m_i(B_2, K_4) \leq m_j(B_2, K_4)$ for $i \geq j$, using $m_{10}(B_2, K_4) \geq 2$ and $m_9(B_2, K_4) \leq 2$, we can conclude that $m_j(B_2, K_4) = 2$ for $j \in \{9,10\}$.

Clearly $m_j(B_2, K_4) = 1$ when $j \geq 11$ as $r(B_2, K_4) = 11$ (see [2]). Hence the Theorem.

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