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On Edge Trimagic Labeling of Umbrella, Dumb Bell and Circular Ladder Graphs

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Abstract. An edge trimagic total labeling of a graph G = (V, E) with p vertices and q edges is a bijection f: V(G) ∪ E(G) → {1, 2, …, p + q} such that for each edge uv ∈ E(G), the value of f(u) + f (uv) + f(v) is either k_1 or k_2 or k_3. In this paper, we prove that the edge trimagic total labeling of Umbrella, Dumb bell and Circular ladder graphs.

Keywords: Graph labeling, Bijective function, Umbrella, Dumb bell, Circular ladder, Edge Trimagic.

AMS Mathematics Subject Classification (2010): 05C72

1. Introduction

Graph labeling was first introduced in the mid sixties. A labeling of a graph is a map of integers to vertices or sometimes edges in a graph based upon certain criteria. In this paper the domain will be the set of all vertices and edges and such labeling are called total labeling [12]. Graph labeling are of many types such as graceful, harmonious, elegant, cordial magic, antimagic, bimagic, etc. This paper is an attempt to study of edge Trimagic total labeling. Harary [4] is referred to know about the notations in graph theory.

Magic labeling was introduced by Sedlacek [11]. Kotzing and Rosa [9], defined edge magic of a graph G with a bijection f: V∪E →{1, 2, …, p+q} such that, for each edge uv ∈ E(G), f(u) + f (uv) + f(v) is a magic constant. In [3] shows the cycle C_n with P_3 chords are edge magic total labeling. Edge bimagic labeling of graphs was introduced by Babujee [2] in 2004, defined by a graph G with a bijection f: V ∪ E → {1, 2, …, p + q} such that for each edge uv∈ E(G), the value of f(u) + f(uv) + f(v) is either k_1 or k_2. Magic and bimagic labeling for disconnected graphs are showed in [1].

In 2013, Jayasekaran et al. [6] introduced the edge trimagic total labeling of graphs. An edge trimagic total labeling of a (p, q) graph G is a bijection f: V ∪ E→{1, 2, …, p+ q} such that for each edge uv ∈ E(G), the value of f(u) + f(uv) + f(v) is equal to any of the distinct constants k_1 or k_2 or k_3. An edge trimagic total labeling is called a super edge trimagic total labeling of a graph G, if the vertices are labeled with the smallest possible integers i.e. 1, 2, …, p. In [7], edge trimagic labeling of digraphs were discussed.

The graph $F_n = P_n+K_1$ is called a fan [10] where $P_n : u_1u_2…u_n$ be a path and $V(K_1) = u$. The umbrella [10] $U_{n,m}$, m > 1 is obtained from a fan F_n by passing the end vertex of the path $P_m : v_1v_2…v_m$ to the vertex of K_1 of the fan F_n. The graph obtained by joining two disjoint cycles $u_1u_2…u_nu_1$ and $v_1v_2...v_nv_1$ with an edge u_1v_1 is called dumbbell [10] graph Db_n. A circular ladder [5] CL(n) is the union of an outer cycle $C_0 : u_1u_2u_3…u_nu_1$ and an inner cycle $C_1 : v_1v_2v_3…v_nv_1$ with additional edges (u_iv_i), i = 1, 2, 3,…, n called spokes.

For more references, we use dynamic survey of graph labeling by Gallian [8]. In this paper, we prove that the graphs such as umbrella, dumb bell and circular ladder are edge trimagic and super edge trimagic labeling.

2. Main results

Theorem 2.1. The Umbrella $U_{n,m}$ is an edge trimagic total labeling for all n.

Proof: Let $V = \{u_i, v_j / 1 \le i \le n, 1 \le j \le m\}$ be the vertex set and $E = \{u_iu_{i+1}, v_jv_{j+1} / 1 \le i \le n-1, 1 \le j \le m-1\} \cup \{v_1u_i / 1 \le i \le n\}$ be the edge set of the graph $U_{n,m}$. Then $U_{n,m}$ has n+m vertices and 2n+m-2 edges.

Case 1. Both n and m are odd

Define a bijection $f: V \cup E \rightarrow \{1,2, …, 3n+2m-2\}$ such that

$$f(u_i) = \begin{cases} m + \dfrac{i+1}{2}, 1 \le i \le n \text{ and } i \text{ is odd} \\ m + \dfrac{n+i+1}{2}, 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_j) = \begin{cases} \dfrac{j+1}{2}, 1 \le j \le m \text{ and } j \text{ is odd} \\ \dfrac{m+j+1}{2}, 1 \le j \le m \text{ and } j \text{ is even} \end{cases}$$

$f(u_iu_{i+1}) = 3n+m-i$, $1 \le i \le n-1$; $f(v_jv_{j+1}) = 3n+2m-j-1$, $1 \le j \le m-1$ and

$$f(v_1u_i) = \begin{cases} 2n + m + 1 - \dfrac{i+1}{2}, \text{ i is odd} \\ n + m + \dfrac{n-i+1}{2}, \text{ i is even} \end{cases}$$

To prove this labeling is an edge trimagic total labeling.

Consider the edges v_1u_i, $1 \le i \le n$.

For odd i, $f(v_1) + f(v_1u_i) + f(u_i) = 1 + 2n + m + 1 - \dfrac{i+1}{2} + m + \dfrac{i+1}{2} = 2n + 2m + 2 = \lambda_1$.

For even i, $f(v_1) + f(v_1u_i) + f(u_i) = 1 + n + m + \dfrac{n-i+1}{2} + m + \dfrac{n+i+1}{2} = 2n + 2m + 2 = \lambda_1$.

Consider the edges u_iu_{i+1}, $1 \le i \le n-1$.

For odd i, $f(u_i)+f(u_iu_{i+1})+f(u_{i+1}) = m+\frac{i+1}{2}+3n+m-i+m+\frac{n+i+2}{2}=\frac{7n+6m+3}{2}=\lambda_2$.

For even i, $f(u_i)+f(u_iu_{i+1})+f(u_{i+1}) = m+\frac{n+i+1}{2}+3n+m-i+m+\frac{i+2}{2}=\frac{7n+6m+3}{2}=\lambda_2$.

Consider the edges v_jv_{j+1}, $1 \le j \le m-1$.

For odd j, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{j+1}{2}+3n+2m-j-1+\frac{m+j+2}{2}=\frac{6n+5m+1}{2}=\lambda_3$.

For even j, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{m+j+1}{2}+3n+2m-j-1+\frac{j+2}{2}=\frac{6n+5m+1}{2}=\lambda_3$.

Hence for edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = 2n+2m+2$, $\lambda_2 = \frac{7n+6m+3}{2}$ and $\lambda_3 = \frac{6n+5m+1}{2}$. Therefore the umbrella $U_{n,m}$ is an edge trimagic for both odd n and m.

Case 2. n is odd and m is even

Define a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, 3n+2m-2\}$ such that

$$f(u_i) = \begin{cases} m+\frac{i+1}{2}, & 1 \le i \le n \text{ and i is odd} \\ m+\frac{n+i+1}{2}, & 1 \le i \le n \text{ and i is even} \end{cases}$$

$$f(v_j) = \begin{cases} \frac{j+1}{2}, & 1 \le j \le m \text{ and j is odd} \\ \frac{m+j}{2}, & 1 \le j \le m \text{ and j is even} \end{cases}$$

$f(u_iu_{i+1}) = 3n+m-i$, $1 \le i \le n-1$; $f(v_jv_{j+1}) = 3n+2m-j-1$, $1 \le j \le m-1$ and

$$f(v_1u_i) = \begin{cases} 2n+m+1-\frac{i+1}{2}, & 1 \le i \le n \text{ and i is odd} \\ n+m+\frac{n-i+1}{2}, & 1 \le i \le n \text{ and i is even} \end{cases}$$

To prove this labeling is an edge trimagic total labeling.

Consider the edges v_1u_i, $1 \le i \le n$.

For odd i, $f(v_1) + f(v_1u_i) + f(u_i) = 1+2n+m+1-\frac{i+1}{2}+m+\frac{i+1}{2}=2n+2m+2=\lambda_1$.

For even i, $f(v_1) + f(v_1u_i) + f(u_i) = 1+n+m+\frac{n-i+1}{2}+m+\frac{n+i+1}{2}=2n+2m+2=\lambda_1$.

Consider the edges u_iu_{i+1}, $1 \le i \le n-1$.

For odd i, $f(u_i)+f(u_iu_{i+1})+f(u_{i+1}) = m+\frac{i+1}{2}+3n+m-i+m+\frac{n+i+2}{2}=\frac{7n+6m+3}{2}=\lambda_2$.

For even i,$f(u_i)+f(u_iu_{i+1})+f(u_{i+1})= m+\frac{n+i+1}{2}+3n+m-i+m+\frac{i+2}{2}=\frac{7n+6m+3}{2}=\lambda_2$.

Consider the edges v_jv_{j+1}, $1 \le j \le m-1$.

For odd j, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{j+1}{2}+3n+2m-j-1+\frac{m+j+1}{2}=\frac{6n+5m}{2}=\lambda_3$.

For even j, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{m+j}{2}+3n+2m-j-1+\frac{j+2}{2}=\frac{6n+5m}{2}=\lambda_3$.

Hence for edge uv $\in$ E, f(u) + f(uv) + f(v) yields any one of the magic constants λ_1 = 2n+2m+2, $\lambda_2 = \frac{7n+6m+3}{2}$ and $\lambda_3 = \frac{6n+5m}{2}$. Therefore the umbrella $U_{n,m}$ is an edge trimagic for n is odd and m is even.

Case 3. n is even and m is odd

Define a bijection f: $V \cup E \rightarrow \{1,2,\ldots,3n+2m-2\}$ such that

$$f(u_i) = \begin{cases} m+\frac{i+1}{2}, 1\le i \le n \text{ and } i \text{ is odd} \\ m+\frac{n+i}{2}, 1\le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_j) = \begin{cases} \frac{j+1}{2}, 1\le j \le m \text{ and } j \text{ is odd} \\ \frac{m+j+1}{2}, 1\le j \le m \text{ and } j \text{ is even} \end{cases}$$

$f(u_iu_{i+1}) = 3n+m-i$, $1 \le i \le n-1$; $f(v_jv_{j+1}) = 3n+2m-j-1$, $1 \le j \le m-1$ and

$$f(v_1u_i) = \begin{cases} 2n+m+1-\frac{i+1}{2}, \text{ i is odd} \\ n+m+\frac{n-i+2}{2}, \text{ i is even} \end{cases}$$

To prove this labeling is an edge trimagic total labeling.

Consider the edges v_1u_i, $1 \le i \le n$.

For odd i, $f(v_1) + f(v_1u_i) + f(u_i) = 1+2n+m+1-\frac{i+1}{2}+m+\frac{i+1}{2}=2n+2m+2=\lambda_1$.

For even i, $f(v_1) + f(v_1u_i) + f(u_i) = 1+n+m+\frac{n-i+2}{2}+m+\frac{n+i}{2}=2n+2m+2=\lambda_1$.

Consider the edges u_iu_{i+1}, $1 \le i \le n-1$.

For odd i, $f(u_i)+f(u_iu_{i+1})+f(u_{i+1})= m+\frac{i+1}{2}+3n+m-i+m+\frac{n+i+1}{2}=\frac{7n+6m+2}{2}=\lambda_2$.

For even i, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = m + \frac{n+i}{2} + 3n + m - i + m + \frac{i+2}{2} = \frac{7n+6m+2}{2} = \lambda_2$.

Consider the edges v_jv_{j+1}, $1 \le j \le m-1$.

For odd j, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{j+1}{2} + 3n + 2m - j - 1 + \frac{m+j+2}{2} = \frac{6n+5m+1}{2} = \lambda_3$.

For even j, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{m+j+1}{2} + 3n + 2m - j - 1 + \frac{j+2}{2} = \frac{6n+5m+1}{2} = \lambda_3$.

Hence for edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = 2n+2m+2$, $\lambda_2 = \frac{7n+6m+2}{2}$ and $\lambda_3 = \frac{6n+5m+1}{2}$. Therefore the umbrella $U_{n,m}$ is an edge trimagic for n is even and m is odd.

Case 4. Both n and m are even

Define a bijection $f : V \cup E \to \{1, 2, \ldots, 3n+2m-2\}$ such that

$$f(u_i) = \begin{cases} m + \frac{i+1}{2}, & 1 \le i \le n \text{ and i is odd} \\ m + \frac{n+i}{2}, & 1 \le i \le n \text{ and i is even} \end{cases}$$

$$f(v_j) = \begin{cases} \frac{j+1}{2}, & 1 \le j \le m \text{ and j is odd} \\ \frac{m+j}{2}, & 1 \le j \le m \text{ and j is even} \end{cases}$$

$f(u_iu_{i+1}) = 3n+m-i$, $1 \le i \le n-1$; $f(v_jv_{j+1}) = 3n+2m-j-1$, $1 \le j \le m-1$ and

$$f(v_1u_i) = \begin{cases} 2n + m + 1 - \frac{i+1}{2}, & 1 \le i \le n \text{ and i is odd} \\ 2n + m + 1 - \frac{n+i}{2}, & 1 \le i \le n \text{ and i is even} \end{cases}$$

To prove this labeling is an edge trimagic total labeling.

Consider the edges v_1u_i, $1 \le i \le n$.

For odd i, $f(v_1) + f(v_1u_i) + f(u_i) = 1 + 2n + m + 1 - \frac{i+1}{2} + m + \frac{i+1}{2} = 2n + 2m + 2 = \lambda_1$.

For even i, $f(v_1) + f(v_1u_i) + f(u_i) = 1 + 2n + m + 1 - \frac{n+i}{2} + m + \frac{n+i}{2} = 2n + 2m + 2 = \lambda_1$.

Consider the edges u_iu_{i+1}, $1 \le i \le n-1$.

For odd i, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = m + \frac{i+1}{2} + 3n + m - i + m + \frac{n+i+1}{2} = \frac{7n+6m+2}{2} = \lambda_2$.

For even i, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = m + \frac{n+i}{2} + 3n + m - i + m + \frac{i+2}{2} = \frac{7n+6m+2}{2} = \lambda_2$.

Consider the edges v_jv_{j+1}, $1 \le j \le m-1$.

For odd j, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{j+1}{2} + 3n + 2m - j - 1 + \frac{m+j+1}{2} = \frac{6n+5m}{2} = \lambda_3$.

For even j, $f(v_j) + f(v_jv_{j+1}) + f(v_{j+1}) = \frac{m+j}{2} + 3n + 2m - j - 1 + \frac{j+2}{2} = \frac{6n+5m}{2} = \lambda_3$.

Hence for edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = 2n+2m+2$, $\lambda_2 = \frac{7n+6m+2}{2}$ and $\lambda_3 = \frac{6n+5m}{2}$. Therefore the umbrella $U_{n,m}$ is an edge trimagic for both even n and m.

Corollary 2.2. The umbrella $U_{n,m}$ is a super edge trimagic total labeling for all n.

Proof: We proved that the umbrella $U_{n,m}$ is an edge trimagic total graph for all n with n+m vertices. The labeling given in Theorem 2.1 is as follows:

For odd n and odd m,

$$f(u_i) = \begin{cases} m + \frac{i+1}{2}, & 1 \le i \le n \text{ and } i \text{ is odd} \\ m + \frac{n+i+1}{2}, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_j) = \begin{cases} \frac{j+1}{2}, & 1 \le j \le m \text{ and } j \text{ is odd} \\ \frac{m+j+1}{2}, & 1 \le j \le m \text{ and } j \text{ is even} \end{cases}$$

For odd n and even m,

$$f(u_i) = \begin{cases} m + \frac{i+1}{2}, & 1 \le i \le n \text{ and } i \text{ is odd} \\ m + \frac{n+i+1}{2}, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_j) = \begin{cases} \frac{j+1}{2}, & 1 \le j \le m \text{ and } j \text{ is odd} \\ \frac{m+j}{2}, & 1 \le j \le m \text{ and } j \text{ is even} \end{cases}$$

For even n and odd m,

$$f(u_i) = \begin{cases} m + \frac{i+1}{2}, & 1 \le i \le n \text{ and } i \text{ is odd} \\ m + \frac{n+i}{2}, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_j) = \begin{cases} \dfrac{j+1}{2}, 1 \le j \le m \text{ and } j \text{ is odd} \\ \dfrac{m+j+1}{2}, 1 \le j \le m \text{ and } j \text{ is even} \end{cases}$$

For even n and m,

$$f(u_i) = \begin{cases} m + \dfrac{i+1}{2}, 1 \le i \le n \text{ and } i \text{ is odd} \\ m + \dfrac{n+i}{2}, 1 \le i \le n \text{ and } i \text{ is even} \end{cases} \qquad f(v_j) = \begin{cases} \dfrac{j+1}{2}, 1 \le j \le m \text{ and } j \text{ is odd} \\ \dfrac{m+j}{2}, 1 \le j \le m \text{ and } j \text{ is even} \end{cases}$$

Hence the n+m vertices get labels 1, 2, …, n+m. Therefore, the umberlla $U_{n,m}$ is a super edge trimagic total labeling graph for all n.

Example 2.3. An edge trimagic total labeling of the Umbrella $U_{5,3}$; $U_{5,4}$; $U_{6,5}$ and $U_{4,6}$ are given in Figure1, Figure 2, Figure 3 and Figure 4 respectively.

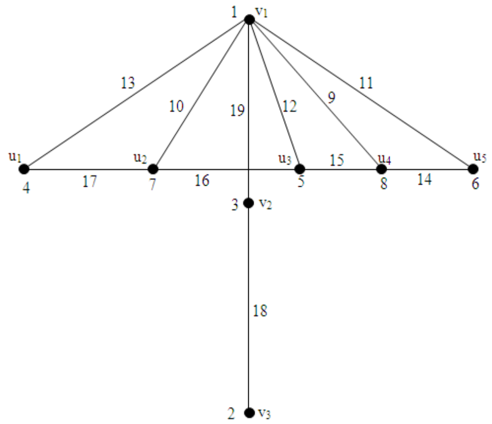

Figure 1: $U_{5,3}$ with $\lambda_1 = 18$, $\lambda_2 = 28$ and $\lambda_3 = 23$

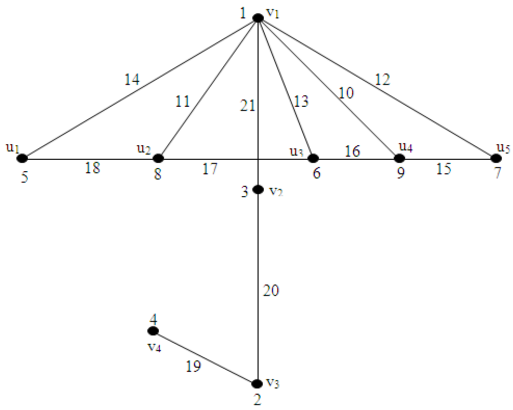

Figure 2: $U_{5,4}$ with $\lambda_1 = 20$, $\lambda_2 = 31$ and $\lambda_3 = 25$

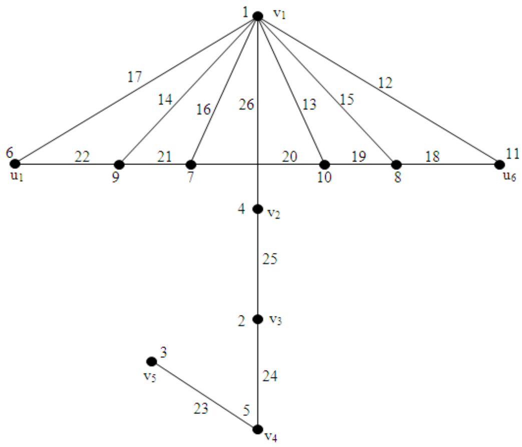

Figure 3: $U_{6,5}$ with $\lambda_1 = 24$, $\lambda_2 = 37$ and $\lambda_3 = 31$

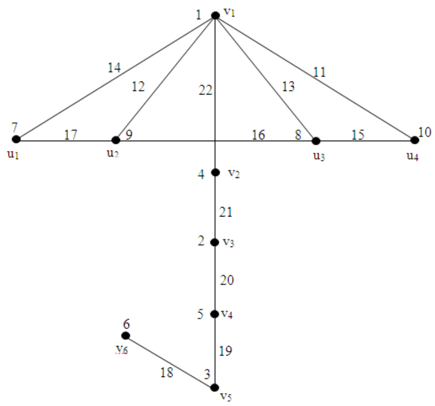

Figure 4. $U_{4,6}$ with $\lambda_1 = 22$, $\lambda_2 = 33$ and $\lambda_3 = 27$

Theorem 2.4. The Dumbbell Db_n is an edge trimagic total labeling for all n.

Proof: Let $V = \{u_i, v_i \,/\, 1 \le i \le n\}$ be the vertex set and $E=\{u_iu_{i+1}, v_iv_{i+1} \,/\, 1 \le i \le n-1\} \cup \{u_1u_n, v_1v_n\} \cup \{u_1v_1\}$ be the edge set of the graph Db_n. Then Db_n has 2n vertices and 2n+1 edges.

Case 1. n is even

Define a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, 4n+1\}$ such that $f(u_i) = 2i-1$, $1 \le i \le n$; $f(v_i) = 2i$, $1 \le i \le n$;

$$f(u_iu_{i+1}) = \begin{cases} 4n - 4i + 3, 1 \le i \le \dfrac{n}{2} \\ 6n - 4i, \dfrac{n}{2} + 1 \le i \le n - 1 \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 4n-4i+1, 1 \le i \le \frac{n}{2} \\ 6n-4i-2, \ \frac{n}{2}+1 \le i \le n-1 \end{cases}$$

$f(u_1v_1) = 4n+1$; $f(u_1u_n) = 4n$; $f(v_1v_n) = 4n-2$.

To prove this labeling is an edge trimagic total labeling.

For the edges u_iu_{i+1}, $1 \le i \le \frac{n}{2}$, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = 2i-1+4n-4i+3+2i+1 = 4n+3 = \lambda_1$.

For the edges v_iv_{i+1}, $1 \le i \le \frac{n}{2}$, $f(v_i) + f(v_iv_{i+1}) + f(v_{i+1}) = 2i+4n-4i+1+2i+2 = 4n+3 = \lambda_1$.

For the edge u_1v_1, $f(u_1) + f(u_1v_1) + f(v_1) = 1+4n+1+2 = 4n+4 = \lambda_2$.

For the edges u_iu_{i+1}, $\frac{n}{2}+1 \le i \le n-1$, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = 2i-1+6n-4i+2i+1 = 6n = \lambda_3$.

For the edges v_iv_{i+1}, $\frac{n}{2}+1 \le i \le n-1$, $f(v_i) + f(v_iv_{i+1}) + f(v_{i+1}) = 2i+6n-4i-2+2i+2 = 6n = \lambda_3$.

For the edge u_1u_n, $f(u_1) + f(u_1u_n) + f(u_n) = 1+4n+2n-1 = 6n = \lambda_3$.

For the edge v_1v_n, $f(v_1) + f(v_1v_n) + f(v_n) = 2+4n-2+2n = 6n = \lambda_3$.

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = 4n+3$; $\lambda_2 = 4n+4$ and $\lambda_3 = 6n$. Therefore the Dumbbell graph Db_n is an edge trimagic for even n.

Case 2. n is odd

Define a bijection $f: V \cup E \to \{1, 2, \ldots, 4n+1\}$ such that

$$f(u_i) = \begin{cases} \frac{i+1}{2}, 1 \le i \le n \text{ and } i \text{ is odd} \\ \frac{n+i+1}{2}, 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} \frac{2n+i+1}{2}, 1 \le i \le n \text{ and } i \text{ is odd} \\ \frac{3n+i+1}{2}, 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$f(u_iu_{i+1}) = 3n-i$, $1 \le i \le n-1$; $f(u_1u_n) = 3n$; $f(v_iv_{i+1}) = 4n-i$, $1 \le i \le n-1$; $f(v_1v_n) = 4n$ and $f(u_1v_1) = 4n+1$.

To prove this labeling is an edge trimagic total labeling.

Consider the edges u_iu_{i+1}, $1 \le i \le n-1$.

For odd i, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = \frac{i+1}{2} + 3n - i + \frac{n+i+2}{2} = \frac{7n+3}{2} = \lambda_1$.

For even i, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = \frac{n+i+1}{2} + 3n - i + \frac{i+2}{2} = \frac{7n+3}{2} = \lambda_1$.

For the edge u_1u_n, $f(u_1) + f(u_1u_n) + f(u_n) = 1+3n+\frac{n+1}{2} = \frac{7n+3}{2} = \lambda_1$.

For the edge u_1v_1, $f(u_1) + f(u_1v_1) + f(v_1) = 1+4n+1+n+1 = 5n+3 = \lambda_2$.

Consider the edges v_iv_{i+1}, $1 \le i \le n-1$.

For odd i, $f(v_i) + f(v_iv_{i+1}) + f(v_{i+1}) = \frac{2n+i+1}{2} + 4n - i + \frac{3n+i+2}{2} = \frac{13n+3}{2} = \lambda_3$.

For even i, $f(v_i) + f(v_iv_{i+1}) + f(v_{i+1}) = \frac{3n+i+1}{2} + 4n - i + n + \frac{i+2}{2} = \frac{13n+3}{2} = \lambda_3$.

For the edge v_1v_n, $f(v_1) + f(v_1v_n) + f(v_n) = n+1+4n+\frac{3n+1}{2} = \frac{13n+3}{2} = \lambda_3$.

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = \frac{7n+3}{2}$, $\lambda_2 = 5n+3$ and $\lambda_3 = \frac{13n+3}{2}$. Therefore the dumbbell graph Db_n is an edge trimagic for odd n. From cases (1) and (2), the Dumbbell Db_n is an edge Trimagic total labeling for all n.

Corollary 2.5. The Dumbbell Db_n is a super edge trimagic total labeling for all n.

Proof: We proved that the Dumbbell Db_n is an edge trimagic total graph for all n with 2n vertices. The labeling given in Theorem 2.4 is as follows:

For even n, $f(u_i) = 2i-1$, $1 \le i \le n$ and $f(v_i) = 2i$, $1 \le i \le n$.

For odd n,

$$f(u_i) = \begin{cases} \frac{i+1}{2}, 1 \le i \le n \text{ and } i \text{ is odd} \\ \frac{n+i+1}{2}, 1 \le i \le n \text{ and } i \text{ is even} \end{cases} \qquad f(v_i) = \begin{cases} \frac{2n+i+1}{2}, 1 \le i \le n \text{ and } i \text{ is odd} \\ \frac{3n+i+1}{2}, 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

Hence the 2n vertices get labels 1, 2, …, 2n. Therefore, the Dumbbell Db_n is a super edge trimagic total labeling graph for all n.

Example 2.6. An edge Trimagic total labeling of the Dumbbell Db_8; and Db_5 are given in figure 5 and figure 6 respectively.

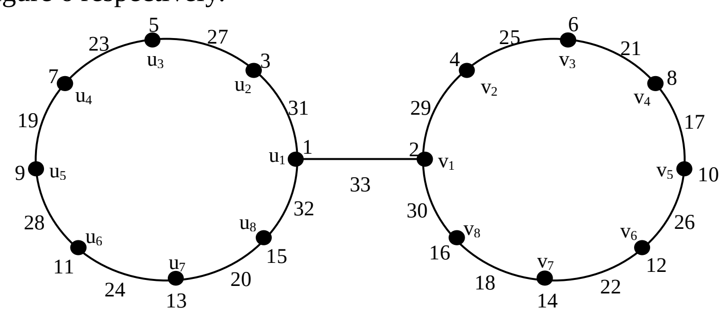

Figure 5: Db_8 with $\lambda_1 = 35$, $\lambda_2 = 36$ and $\lambda_3 = 48$

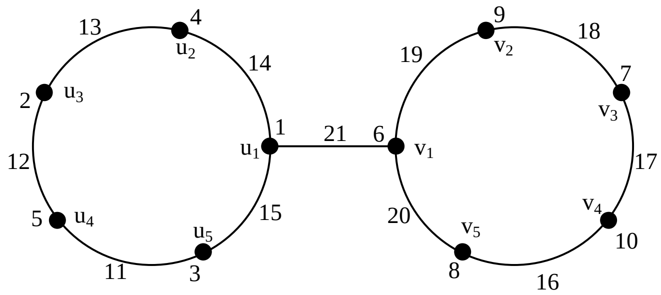

Figure 6: Db_5 with $\lambda_1 = 19$, $\lambda_2 = 28$ and $\lambda_3 = 34$

Theorem 2.7. The circular ladder CL(n) is an edge trimagic total labeling for all n.

Proof: Let $V = \{u_i, v_i \,/\, 1 \le i \le n\}$ be the vertex set and $E = \{u_iu_{i+1}, v_iv_{i+1} \,/\, 1 \le i \le n-1\} \cup \{u_iv_i \,/\, 1 \le i \le n\} \cup \{u_1u_n, v_1v_n\}$ be the edge set of the graph CL(n). Then CL(n) has 2n vertices and 3n edges.

Case 1. n is odd

Define a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, 5n\}$ such that

$$f(u_i) = \begin{cases} n + \dfrac{n+i+2}{2}, & 1 \le i \le n-1 \text{ and } i \text{ is odd} \\ n + \dfrac{i+2}{2}, & 1 \le i \le n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} \dfrac{i+1}{2}, & 1 \le i \le n \text{ and } i \text{ is odd} \\ \dfrac{n+i+1}{2}, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$f(u_n) = n+1$; $f(u_iu_{i+1}) = 4n-i-1$, $1 \le i \le n-2$; $f(u_nu_{n-1}) = 4n$; $f(u_1u_n) = 4n-1$; $f(v_iv_{i+1}) = 5n-i$, $1 \le i \le n-1$; $f(v_1v_n) = 5n$; $f(u_iv_i) = 3n-i$, $1 \le i \le n-1$ and $f(u_nv_n) = 3n$.

To prove this labeling is an edge trimagic total labeling.

Consider the edges u_iu_{i+1}, $1 \le i \le n-2$.

For odd i, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = n + \dfrac{n+i+2}{2} + 4n - i - 1 + n + \dfrac{i+3}{2} = \dfrac{13n+3}{2} = \lambda_1$.

For even i, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = n + \dfrac{i+2}{2} + 4n - i - 1 + n + \dfrac{n+i+3}{2} = \dfrac{13n+3}{2} = \lambda_1$.

For the edge u_1u_n, $f(u_1)+f(u_1u_n)+f(u_n) = n + \dfrac{n+3}{2} + 4n - 1 + n + 1 = \dfrac{13n+3}{2} = \lambda_1$.

For the edge $u_{n-1}u_n$, $f(u_{n-1})+f(u_{n-1}u_n)+f(u_n) = n + \dfrac{n+1}{2} + 4n + n + 1 = \dfrac{13n+3}{2} = \lambda_1$.

Consider the edges u_iv_i, $1 \le i \le n-1$.

For odd i, $f(u_i) + f(u_iv_i) + f(v_i) = n+\frac{n+i+2}{2}+3n-i+\frac{i+1}{2}=\frac{9n+3}{2}=\lambda_2$.

For even i, $f(u_i) + f(u_iv_i) + f(v_i) = n+\frac{i+2}{2}+3n-i+\frac{n+i+1}{2}=\frac{9n+3}{2}=\lambda_2$.

For the edge u_nv_n, $f(u_n) + f(u_nv_n) + f(v_n) = n+1+3n+\frac{n+1}{2}=\frac{9n+3}{2}=\lambda_2$.

Consider the edges v_iv_{i+1}, $1 \le i \le n-1$.

For odd i, $f(v_i) + f(v_iv_{i+1}) + f(v_{i+1}) = \frac{i+1}{2}+5n-i+\frac{n+i+2}{2}=\frac{11n+3}{2}=\lambda_3$.

For even i, $f(v_i) + f(v_iv_{i+1}) + f(v_{i+1}) = \frac{n+i+1}{2}+5n-i+\frac{i+2}{2}=\frac{11n+3}{2}=\lambda_3$.

For the edge v_1u_n, $f(v_1) + f(v_1u_n) + f(u_n) = 1+5n+\frac{n+1}{2}=\frac{11n+3}{2}=\lambda_3$.

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = \frac{13n+3}{2}$, $\lambda_2 = \frac{9n+3}{2}$ and $\lambda_3 = \frac{11n+3}{2}$. Therefore the Circular ladder CL(n) is an edge trimagic for odd n.

Case 2. n is even

Define a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, 5n\}$ such that

$$f(u_i) = \begin{cases} n+\frac{i+1}{2}, 1 \le i \le n \text{ and } i \text{ is odd} \\ n+\frac{n+i}{2}, 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} \frac{i+1}{2}, 1 \le i \le n \text{ and } i \text{ is odd} \\ \frac{n+i}{2}, 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$f(u_iu_{i+1}) = 3n-i+1$, $1 \le i \le n-1$; $f(u_1u_n) = 2n+1$; $f(v_iv_{i+1}) = 5n-i+1$, $1 \le i \le n-1$; $f(v_1v_n) = 4n+1$; and $f(u_iv_i) = 4n-i+1$, $1 \le i \le n$.

To prove this labeling is an edge trimagic total labeling.

For the edge v_1v_n, $f(v_1) + f(v_1v_n) + f(v_n) = 1+4n+1+n = 5n+2 = \lambda_1$.

For the edge u_1u_n, $f(u_1) + f(u_1u_n) + f(u_n) = n+1+2n+1+2n = 5n+2 = \lambda_1$.

Consider the edges u_iv_i, $1 \le i \le n$.

For odd i, $f(u_i) + f(u_iv_i) + f(v_i) = \frac{i+1}{2}+4n-i+1+n+\frac{i+1}{2}=5n+2=\lambda_1$.

For even i, $f(u_i) + f(u_iv_i) + f(v_i) = \frac{n+i}{2}+4n-i+1+n+\frac{n+i}{2}=6n+1=\lambda_2$.

Consider the edges u_iu_{i+1}, $1 \le i \le n-1$.

For odd i, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = n + \frac{i+1}{2} + 3n - i + 1 + n + \frac{n+i+1}{2} = \frac{11n+4}{2} = \lambda_3$.

For even i, $f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = n + \frac{n+i}{2} + 3n - i + 1 + n + \frac{i+2}{2} = \frac{11n+4}{2} = \lambda_3$.

Consider the edges v_iv_{i+1}, $1 \le i \le n-1$.

For odd i, $f(v_i) + f(v_iv_{i+1}) + f(v_{i+1}) = \frac{i+1}{2} + 5n - i + 1 + \frac{n+i+1}{2} = \frac{11n+4}{2} = \lambda_3$.

For even i, $f(v_i) + f(v_iv_{i+1}) + f(v_{i+1}) = \frac{n+i}{2} + 5n - i + 1 + \frac{i+2}{2} = \frac{11n+4}{2} = \lambda_3$.

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = 5n+2$, $\lambda_2 = 6n+1$ and $\lambda_3 = \frac{11n+4}{2}$. Therefore, the Circular ladder CL(n) is an edge trimagic for even n. Hence by case 1 and case 2, the circular ladder CL(n) is an edge Trimagic total labeling for all n.

Corollary 2.8. The Circular ladder CL(n) is a super edge trimagic total labeling for all n.
Proof: We proved that the Circular ladder CL(n) is an edge trimagic total graph for all n with 2n vertices. The labeling given in Theorem 2.7 is as follows:
For odd n,

$$f(u_i) = \begin{cases} n + \frac{n+i+2}{2}, & 1 \le i \le n-1 \text{ and i is odd} \\ n + \frac{i+2}{2}, & 1 \le i \le n-1 \text{ and i is even} \end{cases}$$

$$f(v_i) = \begin{cases} \frac{i+1}{2}, & 1 \le i \le n \text{ and i is odd} \\ \frac{n+i+1}{2}, & 1 \le i \le n \text{ and i is even} \end{cases}$$

For even n,

$$f(u_i) = \begin{cases} n + \frac{i+1}{2}, & 1 \le i \le n \text{ and i is odd} \\ n + \frac{n+i}{2}, & 1 \le i \le n \text{ and i is even} \end{cases}$$

$$f(v_i) = \begin{cases} \frac{i+1}{2}, & 1 \le i \le n \text{ and i is odd} \\ \frac{n+i}{2}, & 1 \le i \le n \text{ and i is even} \end{cases}$$

Hence the 2n vertices get labels 1, 2, …, 2n. Therefore, the Circular ladder CL(n) is a super edge trimagic total labeling graph for all n.

Example 2.9. An edge Trimagic total labeling of the Circular ladder CL(7) and CL(6) are given in figure 7 and figure 8 respectively

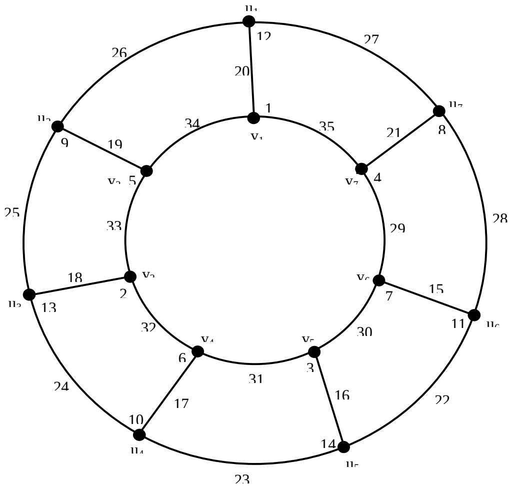

Figure 7: CL(7) with $\lambda_1 = 47$, $\lambda_2 = 33$ and $\lambda_3 = 40$

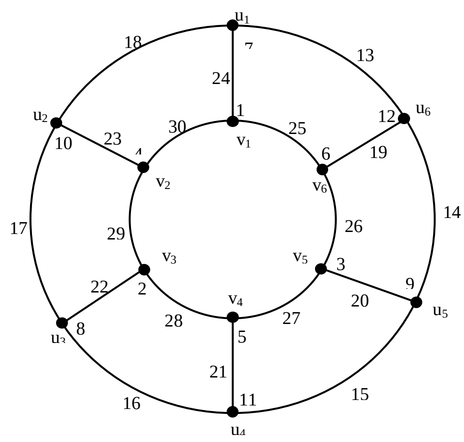

Figure 8: CL(6) with $\lambda_1 = 32$, $\lambda_2 = 37$ and $\lambda_3 = 35$

3. Conclusion

In this paper we have determined the edge trimagic total labeling of the Umbrella, Dumbbell and Circular ladder graphs. Also we have determined the above graphs are super edge Trimagic total labeling.

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REFERENCES

1. S.Babith, A.Amarajothi and J.B.Babujee, Magic and bimagic labeling for disconnected graphs, *Intern. J. Mathematics Trends and Technology*, 3(2) (2012) 86-90.
2. J.B.Babujee, On edge bimagic labeling, *Journal of Combinations Information & System Sciences*, 1-4 (2004) 239-244 .
3. L.Girija and A.Elumalai, Edge magic total labeling of the cycle C_n with P_3 chords, *Annals of Pure and Applied Mathematics*, 8(2) (2014) 175-181.
4. F.Harary, *Graph theory*, Narosa Publishing house, New Delhi (2001).
5. I.Rajasingh, B.Rajan and V.Annamma, Total vertex irregularity strength of circular ladder and windmill graphs, *International Conference on Mathematical Computer Engineering* - ICMCE (2013) 418-423.
6. C.Jayasekaran, M.Regees and C.Davidraj, Edge trimagic labeling of some graphs, *Intern. J. Combinatorial Graph Theory and Applications*, 6(2) (2013) 175-186.
7. C.Jayasekaran and M.Regees, Edge trimagic in digraphs, *Journal of Discrete Mathematical Sciences and Cryptography*, 17(4) (2015) 321-335.
8. J.A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 16 (2013) #DS6.
9. A.Kotzing and A.Rosa, Magic valuations of finite graphs, *Canad. Math. Bull*., 13 (1970) 415-416.
10. R.Ponraj, S.S.Narayanan and A.M.S. Ramasamy, Total mean cordiality of umbrella, butterfly and dumb bell graphs, *Jordan Journal of Mathematics and Statistics,* 8(1) (2015) 59-77.
11. J.Sedlacek, Problem 27: in Theory of Graphs and its Applications, *Proc. Symposium Smolenice*, June (1963) 163-167.
12. S.S.Basha, Reverse super edge-magic labeling on W-trees, *Intern. Journal of Computer Engineering In Research Trends*, 2(11) (2015) 719-721.