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Weak Positive Implicative BRK-Algebras

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Abstract. In this paper we introduce the notion of weak positive implicative BRK-algebra and study its properties via left maps.

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1. Introduction

In 1996, two classes of abstract algebras, BCK-algebras and BCI-algebras,were introduced by Imai and Iseki [1, 2]. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. Since then many researchers introduced and studied different classes of new algebras as a generalization of BCK/BCI-algebras.

In 1983, Hu and Li introduced the notion of BCH-algebras [4] as a generalization of BCI-algebras and studied certain properties of these algebras. In this direction, Jun, Roh and Kim introduced a new class of algebra namely BH-algebras [6] as a generalization of BCH-algebras. Q-Algebras and QS-algebras [5] are further generalizations of BCH algebras. Recently, Ravi Kumar Bandaru introduced the notion of BRK-algebras [3] which is a generalization of BCI/BCI/BCH/Q/QS-algebras and studied various properties of BRK-Algebras. His study was confined to give various characterizations for these BCK/BCI/BCH/Q/QS-algebras with BRK-Algebras.

Ravi kumar defined BRK-algebra as an algebra $X = (X,*,0)$ of type $(2,0)$ which satisfies the axioms (i) $x*0 = x$ and (ii) $(x*y)*x = 0*y$ for any $x, y \in X$. It is known that in any BRK-algebra X the following holds for any $x, y \in X$ (see [3]),

- $x*x = 0$
- $0*(x*y) = (0*x)*(0*y)$
- $x*y = 0$ implies $0*x = 0*y$

He also introduced the notion of positive implicative BRK-algebra as a BRK-algebra which satisfies the condition $((x*y)*y)*(0*y) = x*y$ and gave a necessary and sufficient condition for a BRK-algebra to be positive implicative.

In this paper we establish an isomorphism theorem for quotient BRK-Algebra determined by homomorphism. Furthermore we make use of weak positive implicative BRK-algebras and study their properties via right maps and left maps.

2. Quotient BRK-algebra determined by homomorphism

Definition 2.1. *Let* $X=(X,*,0)$ *and* $Y=(Y,*,0)$ *be BRK-algebras. A mapping* $f:X\to Y$ *is called a homomorphism from* X *into* Y *if* $f(x*y)=f(x)*f(y)$ *for all* $x,y\in X$.

A homomorpism f is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorpism. Two BRK-algebras X and Y are said to be isomorphic, written $X\cong Y$, if there exists an isomorphism $f:X\to Y$. For any homomorphism $f:X\to Y$ the set $\{x\in X: f(x)=0\}$ is called kernel of f, denoted by $Kerf$ and the set $\{f(x):x\in X\}$ is called the image of f, denoted by Imf .

Theorem 2.2. *Let* $f:X\to Y$ *be a homomorphism of BRK-algebras. Define a relation* $\sim$ *on* X *by* $x\sim y$ *if and only if* $f(x)=f(y)$ *for all* $x,y\in X$. *Then* $\sim$ *is a congruence relation on* X *which is called the congruence relation determined by the homomorphism* f .

Proof: Clearly $\sim$ is an equivalence relation on X . Next suppose $x\sim y$ and $u\sim v$. Then $f(x)=f(y)$ and $f(u)=f(v)$. Now since
$f(x*u)=f(x)*f(u)=f(y)*f(v)=f(y*v), x*u\sim y*v$. Thus $\sim$ is a congruence relation on X . ■

We denote the equivalence class of x determined by $\sim$ by $[x]_f$ and the set of all equivalence classes by X/f i.e $[x]_f=\{y\in X: x\sim y\}$ and $X/f=\{[x]_f : x\in X\}$.

Theorem 2.3. *Let* $f:X\to Y$ *be homomorphism on BRK-algebras. Define* $*$ *on* X/f *by* $[x]_f*[y]_f=[x*y]_f$. *Then* $(X/f,*,[0]_f)$ *is a BRK algebra. It is called the quotient BRK-algebra determined by the homomorphism* f .

Proof: Since $\sim$ is a congruence relation on X, * is well defined. Now for any $[x]_f,[y]_f\in X/f$ we have

1. $[x]_f*[0]_f=[x*0]_f=[x]_f$ and
2. $([x]_f*[y]_f)*[x]_f=[x*y]_f*[x]_f=[(x*y)*x]_f=[0*y]_f=[0]_f*[y]_f$.

Thus $(X/f,*,[0]_f)$ is a BRK-algebra. ■

Remark 2.4. *Clearly* $[0]_f=Kerf$.

Theorem 2.5. *Let* $f:X\to Y$ *be homomorphism of BRK-algebras. Then the image of* f *is isomorphic to the quotient BRK-algebra determined by* f *, i.e.* $X/f\cong Imf$.

Proof: Define a mapping $\theta : X/f \to Imf$ by $\theta([x]_f) = f(x)$. Then

1. θ is well defined. Indeed suppose $[x]_f = [y]_f$ but then
   $[x]_f = [y]_f \Rightarrow x \in [y]_f \Rightarrow f(x) = f(y)$. Thus $\theta([x]_f) = \theta([y]_f)$.
2. θ is homomorphism. Indeed for any $[x]_f, [y]_f \in X/f$ we have
   $\theta([x]_f * [y]_f) = \theta([x * y]_f) = f(x * y) = f(x) * f(y) = \theta([x]_f) * \theta([y]_f)$
3. Clearly θ is bijective.

Hence $X/f \cong Imf$. ■

3. Weak implicative BRK-algebra

Here we will define weak implicative BRK-algebra and investigate its properties.

Definition 3.1. *A BRK-algebra $X = (X, *, 0)$ is said to be weak positive implicative if it satisfies $(x * y) * z = (x * z) * (y * z)$ for all x, y and $z \in X$*

Example 3.2. *Let $X = \{0,1,2,3\}$ be a set with the following cayley table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Then $(X, *, 0)$ is a weak positive implicative BRK-algebra.

The next example shows the existence of weak implicative BRK algebra which is not BCK/BCI/BCH-algebra.

Example 3.3. *Let $\mathbb{Z}$ be the set of integers. Define * on $\mathbb{Z}$ by*

$$x * y = \begin{cases} x, & \text{if } y = 0 \\ 0, & \text{if } y \neq 0 \end{cases}$$

Then $(\mathbb{Z}, *, 0)$ is a weak positive implicative BRK-algebra which is not BCK/BCI/BCH-algebra.

Lemma 3.4. *In any weak positive implicative BRK-algebra X, the following hold for all $x, y \in X$.*

1. $0 * x = 0$
2. $(x * y) * x = 0$
3. $x * y = (x * y) * y$
4. $(x * (x * y)) * y = 0$

Proof. Let $x, y \in X$. Then

1. $0*x=(x*x)*x=(x*x)*(x*x)=0*0=0$
2. $(x*y)*x=0*y=0$
3. $x*y=(x*y)*0=(x*y)*(y*y)=(x*y)*y$
4. $(x*(x*y))*y=(x*y)*((x*y)*y)=(x*y)*(x*y)=0$. ■

Theorem 3.5. *Every weak positive implicative BRK-algebra is positive implicative.*
Proof. Let $X=(X,*,0)$ be a weak positive implicative BRK-algebra. For any $x,y\in X$, we have $((x*y)*y)*(0*y)=((x*y)*y)*0=(x*y)*y=x*y$.
Thus X is positive implicative BRK-algebra. ■

Remark 3.6. *The converse of the above theorem is not true.*

Example 3.7. *Let* $X=\{0,1,2\}$ *be a set with Cayley table:*

*	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Then $(X,*,0)$ is a positive implicative BRK-algebra [see 3] which is not a weak positive implicative (as $(1*1)*1=2\neq 0=(1*1)*(1*1)$).

4. R-maps and L-maps in BRK-algebra
In this section we investigate the properties of R-maps and L-maps in weak positive implicative BRK-algebras.

Definition 4.1. *Let* $X=(X,*,0)$ *be a BRK-algebra and* $a\in X$ *be a fixed element. Then the map* $R_a:X\to X$ *given by* $R_a(x)=x*a$ *is called right map of* X *and the map* $L_a:X\to X$ *given by* $L_a(x)=a*x$ *is called left map of* X. *The set of all left maps is denoted by* $\mathrm{L}(X)$.

Definition 4.2. *A right map* R_a *is called idempotent if* $R_a\circ R_a=R_a$ *where* $\circ$ *is the usual composition of maps.*

Remark 4.3. *Clearly for any* a, R_a *is idempotent if and only if* $(x*a)*a=x*a$ *for all* $x\in X$.

Theorem 4.4. *If a BRK-algebra* $X=(X,*,0)$ *is weak positive implicative, then every right map on* X *is idempotent.*
Proof. For any $a\in X$, $R_a(x)=x*a=(x*a)*a=R_a(R_a(x))=(R_a\circ R_a)(x)$ for all

$x \in X$. Hence $R_a \circ R_a = R_a$. ■

Theorem 4.5. *A BRK-algebra* $X = (X,*,0)$ *is a weak positive implicative if and only if every right map is a homomorphism.*

Proof. Suppose X is a weak positive implicative BRK-algebra. Then for each $a \in X$, $R_a(x*y) = (x*y)*a = (x*a)*(y*a) = R_a(x)*R_a(y)$ .Thus R_a is a homomorphism. For the converse suppose every right map is a homomorphism. Now for any $x, y, z \in X$ we have $(x*y)*z = R_z(x*y) = R_z(x)*R_z(y) = (x*z)*(y*z)$. Hence X is weak positive implicative. ■

Theorem 4.6. *In any BRK-algebra* $X = (X,*,0)$ *, if* L_a *is a homomorphism, then* $a = 0$.

Proof. Suppose L_a is a homomorphism. But then

$$a = a*0 = L_a(0) = L_a(0*0) = L_a(0)*L_a(0) = (a*0)*(a*0) = a*a = 0 .$$ ■

For a BRK-algebra $X = (X,*,0)$ we define a binary operation $\otimes$ on $\mathsf{L}(X)$ by $(L_a \otimes L_b)(x) := L_a(x)*L_b(x)$ for any $L_a, L_b \in \mathsf{L}(X)$. We have the following Lemma.

Lemma 4.7. *Let* $X = (X,*,0)$ *be weak positive implicatice BRK-algebra. For any* $L_a, L_b, L_c \in \mathsf{L}(X)$ *, we have*

i. $L_a \otimes L_b = L_{a*b}$ i.e. $L_a \otimes L_b \in \mathsf{L}(X)$.

ii. $(L_a \otimes L_b) \otimes L_c = (L_a \otimes L_c) \otimes (L_a \otimes L_c)$.

Proof. For any $x \in X$ we have

i. $(L_a \otimes L_b)(x) = L_a(x)*L_b(x) = (a*x)*(b*x) = (a*b)*x = L_{a*b}(x)$ and so $L_a \otimes L_b = L_{a*b}$.

ii. $(L_a \otimes L_b) \otimes L_c = L_{a*b} \otimes L_c = L_{(a*b)*c}$

$$= L_{(a*c)*(b*c)} = L_{a*c} \otimes L_{b*c}$$

$$= (L_a \otimes L_c) \otimes (L_a \otimes L_c)$$ ■

Theorem 4.8 *If* $X = (X,*,0)$ *is a weak positive implicative BRK-algebra, then* $\mathsf{L}(X) = (\mathsf{L}(X),\otimes, L_0)$ *is a weak positive implicative BRK-algebra.*

Proof. It is enough to show that $\mathsf{L}(X) = (\mathsf{L}(X),\otimes, L_0)$ is a BRK-algebra. Now for any $L_a, L_b \in \mathsf{L}(X)$ we have

1. $L_a \otimes L_0 = L_{a*0} = L_a$, and
2. $(L_a \otimes L_b) \otimes L_a = L_{(a*b)*a} = L_{0*b} = L_0 \otimes L_b$.

Therefore $\mathsf{L}(X)$ is a BRK-algebra and hence by the above lemma it is weak positive implicative. ■

Corollary 4.9. *Let* $X = (X,*,0)$ *be a weak positive implicative BRK-algebra. Then the map* $f : X \rightarrow \mathrm{L}(X)$ *given by* $f(x) = L_x$ *is an epimorphism and* $X/f \cong \mathrm{L}(X)$ *where* X/f *is the quotient BRK-algebra determined by the homomorphism* f *.*

5. Conclusion

In this paper, we have introduced the notion of weak positive implicative BRK-algebra and showed that the set of all left maps on weak positive implicative BRK-algebra is also weak positive implicative BRK-algebra. We have also investigated the conditions under which right maps and left maps becomes a homomorphsm.

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