Abstract. Molodtsov introduced the concept of soft sets, which can be seen as a new Mathematical tool for dealing with uncertainty. In this paper, we initiate the study of soft intersection ideals of semirings by using the soft set theory. The notions of soft intersection semirings, soft intersection left (right, two-sided) ideals of semiring and soft intersection quasi and bi-ideals of semirings are introduced and several related properties are investigated.

Keywords: Soft set, soft intersection semirings, soft intersection left (right, two-sided) ideals of semiring and soft intersection quasi and bi-ideals of semirings.

AMS Mathematics Subject Classification (2010): 06D72, 16Y60, 16D25

1. Introduction

The notion of a semiring was introduced by Vandiver in 1934. Needless to say, semirings found their full place in Mathematics long before years. The applications of semirings to areas such as optimization theory, graph theory, generalized fuzzy computation, automata theory, formal language theory, coding theory and analysis of computer programs have been extensively studied in the literature (cf. [7, 8]). It is also well-known that ideals usually play a fundamental role in algebra, especially in the study of rings. Nevertheless, ideals in a semiring $S$ do not in general coincide with the usual ring ideals if $S$ is a ring, and so many results in ring theory have no analogues in semirings using only ideals. Consequently, some more restricted concepts of ideals such as $k$-ideals [9] and $h$-ideals [10] have been introduced in the study of the semiring theory. Moreover, the fuzzy set theory initiated by Zadeh [16] has been successfully applied to generalize many basic concepts in algebra. Rosenfeld [14] proposed the concept of group in order to establish the algebraic structure of fuzzy sets. In fact, several researchers have investigated a fuzzy theory in semirings. They introduced the notions of fuzzy semirings, fuzzy (prime) ideals, fuzzy $k$-ideals, fuzzy $h$-ideals and $L$-fuzzy ideals in semirings, and obtained many related results. However, all of these theories have their own difficulties which are pointed out in [13] by Molodtsov who then proposed a completely new approach for modeling vagueness and uncertainty, that is free from the difficulties. This so-called soft set theory has potential applications in many different fields. Maji et al. [11] firstly worked on detailed theoretical study of soft sets. After that, the properties and applications on the soft set theory have been studied by many authors (e.g. [1, 2, 3, 4, 6, 12, 15, 17]), Feng et al. [5] dealt with the algebraic structure of semirings by applying soft set theory and
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defined the notion of a soft semiring and focused on the algebraic properties of soft semirings. In this paper, we make a new approach to the classical semiring theory via soft sets, with the concept of soft intersection semiring and soft intersection ideals of semirings.

2. Preliminaries

Molodtsov [13] defined the notion of a soft set in the following way: Let \( U \) be an initial universe set and \( E \) be a set of parameters. The power set of \( U \) is denoted by \( P(U) \) and \( A \) is a subset of \( U \). A pair \((F, A)\) is called a soft set over \( U \), where \( F : A \rightarrow P(U) \). For \( e \in A, F(e) \) may be considered as the set of \( e \)-approximate elements of the soft set \((F, A)\).

Example 2.1. Let \( U = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\} \) be the set of seven cars and \( E = \{ \text{expensive, fuel efficiency, spacious, maintenance free, ecofriendly, high security measure} \} \) are set of parameters. Let \((F, P)\) be a soft set representing the “suitable cars” given by \( F = \{(c_2, c_3, c_5, c_7)\} \), \( P = \{ \text{expensive cars, fuel efficiency, spacious, maintenance free, ecofriendly, high security measure} \} \). Suppose that Mr X wants to buy a car consisting the parameter fuel efficiency, spacious, eco friendly, high security measure which forms the subset \( P = \{ \text{fuel efficiency, spacious, eco friendly, high security measure} \} \) of the set \( E \). The problem is to select the car which is suitable with the choice parameters set by Mr X.

Definition 2.2. Let \( f_A, f_B \in S(U) \). Then, \( f_A \) is called a soft subset of \( f_B \) and denoted by \( f_A \subseteq f_B \) if \( f_A(x) \subseteq f_B(x) \) for all \( x \in E \).

Definition 2.3. Let \( f_A, f_B \in S(U) \). Then, union of \( f_A \) and \( f_B \) denoted by \( f_A \cup f_B \), is defined as \( f_A \cup f_B = f_{A \cup B} \), where \( f_{A \cup B}(x) = f_A(x) \cup f_B(x) \) for all \( x \in E \).

Definition 2.4. Let \( f_A, f_B \in S(U) \). Then, intersection of \( f_A \) and \( f_B \) denoted by \( f_A \cap f_B \), is defined as \( f_A \cap f_B = f_{A \cap B} \), where \( f_{A \cap B}(x) = f_A(x) \cap f_B(x) \) for all \( x \in E \).

Definition 2.5. Let \( f_A, f_B \in S(U) \). Then, \( \wedge \)-product of \( f_A \) and \( f_B \) denoted by \( f_A \wedge f_B \), is defined as \( f_A \wedge f_B = f_{A \wedge B} \), where \( f_{A \wedge B}(x, y) = f_A(x) \cap f_B(y) \) for all \((x, y) \in E \times E \).

Definition 2.6. Let \( f_A \) and \( f_B \) be soft sets over the common universe \( U \) and \( \Psi \) be a function from \( A \) to \( B \). Then, soft image of \( f_A \) under \( \Psi \), denoted by \( \Psi(f_A) \), is a soft set over \( U \) by
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\[
(\Psi(f_A))(b) = \left\{ \begin{array}{ll}
\bigcup \{ f_A(a) \mid a \in A \text{ and } \Psi(a) = b \}, & \text{if } \Psi^{-1}(b) \neq \emptyset, \\
0, & \text{otherwise}
\end{array} \right.
\]

for all \( b \in B \). And soft pre-image (or soft inverse image) of \( f_B \) under \( \Psi \), denoted by \( \Psi^{-1}(f_B) \), is a soft set over \( U \) by \( \Psi^{-1}(f_B)(a) = f_B(\Psi(a)) \) for all \( a \in A \).

**Definition 2.7.** Let \( f_A \) be a soft set over \( U \) and \( \alpha \subseteq U \). Then, upper \( \alpha \)-inclusion of \( f_A \), denoted by \( U(f_A; \alpha) = \{ x \in A \mid f_A(x) \supseteq \alpha \} \).

### 3. Soft Intersection sum, product and soft characteristic function

In this section, we define soft intersection sum, product and soft characteristic function and study their properties.

**Definition 3.1.** Let \( S_f \) and \( S_g \) be soft sets over the common universe \( U \). Then, soft intersection sum \( S_f + S_g \) is defined by

\[
(f_s + g_s)(x) = \left\{ \begin{array}{ll}
\bigcup_{x = yz} \{ s_f(y) \cap g_s(z) \}, & \text{if there exists } y, z \in S \\
\emptyset, & \text{otherwise}
\end{array} \right.
\]

for all \( x \in S \).

**Definition 3.2.** Let \( S_f \) and \( S_g \) be soft sets over the common universe \( U \). Then, soft intersection product \( S_f \circ g_s \) is defined by

\[
(f_s \circ g_s)(x) = \left\{ \begin{array}{ll}
\bigcup_{x = yz} \{ s_f(y) \cap g_s(z) \}, & \text{if there exists } y, z \in S \\
\emptyset, & \text{otherwise}
\end{array} \right.
\]

for all \( x \in S \).

**Example 3.3.** Consider the semiring \( S = \{0, a, b, c\} \) defined by the following table:

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Let \( U = D_3 = \{< x, y >, x > = y \} = \{ e, x, x^2, y, yx \} = \{ e, x, x^2, y, yx, yx^2 \} \) be the universal set. Let \( f_s \) and \( g_s \) be soft sets over \( U \) such that \( f_s(0) = \{ e, x, y, yx \} \), \( f_s(a) = \{ e, x, y^2 \} \), \( f_s(b) = \{ e, y, yx^2 \} \), \( f_s(c) = \{ e, x, x^2, y \} \) and \( g_s(0) = \{ e, y, y^2 \} \), \( g_s(a) = \{ e, x, yx \} \), \( g_s(b) = \{ e, yx, yx^2 \} \), \( g_s(c) = \{ e, y, yx \} \). \( (f_s \circ g_s)(a) = (f_s(a) \cap g_s(a)) \) = \{ e, x \}. 

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Theorem 3.4. Let \(f_s, g_s, h_s \in S(U)\). Then,

1. \((f_s \circ g_s) \circ h_s = f_s \circ (g_s \circ h_s)\).
2. \(f_s \circ g_s \neq f_s \circ g_s\).
3. \((f_s \circ g_s) \cup (f_s \circ h_s) = (f_s \circ g_s) \cup (f_s \circ h_s)\) and \((f_s \circ g_s) \circ h_s = (f_s \circ h_s) \cup (g_s \circ h_s)\).
4. \((f_s \circ g_s) \cap (f_s \circ h_s) = (f_s \circ h_s) \circ (g_s \circ h_s)\).
5. If \(f_s \subseteq g_s\), then \(f_s \circ h_s \subseteq g_s \circ h_s\) and \(h_s \circ f_s = h_s \circ g_s\).
6. If \(t_s, l_s \in S(U)\) such that \(t_s \subseteq f_s\) and \(l_s \subseteq g_s\), then \(l_s \circ g_s \subseteq f_s \circ g_s\).
7. \((f_s + g_s) + h_s = f_s + (g_s + h_s)\).
8. \(f_s \circ (g_s + h_s) = (f_s \circ g_s) + (f_s \circ h_s)\).

Proof: (1) and (2) follows from Definition 3.1 and Example 3.1.

(3) Let \(a \in S\). If \(a\) is not expressible as \(a = xy\), then 
\[f_s \circ (g_s \cap h_s)(A) = \emptyset.\]

Similarly, \(((f_s \circ g_s) \cap (f_s \circ h_s))(a) = (f_s \circ g_s)(a) \cup (f_s \circ h_s)(a) = \emptyset \cup \emptyset = \emptyset.\]

Now, let there exist \(x, y \in S\) such that \(a = xy\).

\[
(f_s \circ (g_s \cap h_s))(a) = \bigcup_{a \in xy} ((f_s(x) \cap (g_s \cap h_s)(y))
= \bigcup_{a \in xy} ((f_s(x) \cap g_s(y)) \cup (f_s(x) \cap h_s(y)))
= \bigcup_{a \in xy} ((f_s(x) \cap g_s(y))) \cup \bigcup_{a \in xy} ((f_s(x) \cap h_s(y))]
= (f_s \circ g_s)(a) \cup (f_s \circ h_s)(a)
= (f_s \circ g_s \cup f_s \circ h_s)(a).
\]

Thus \((f_s \cup g_s) \circ h_s = (f_s \circ h_s) \cup (g_s \circ h_s).\)

Similarly, we can prove (4) is also clear.

(5) Let \(x \in S\). If \(x\) is not expressible as \(x = yz\), then 
\((f_s \circ h_s)(x) = (g_s \circ h_s)(x) = \emptyset.\) Otherwise,
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\[
(f_s \circ h_s)(x) = \bigcup_{y \in S} (f_s(y) \cap h_s(z))
\]

\[
\subseteq \bigcup_{x \in S} (g_s(y) \cap h_s(z)) \quad \text{(since } f_s(y) \subseteq g_s(y)) = (g_s \circ h_s)(x).
\]

Similarly, one can show that \( h_s \circ f_s \subseteq h_s \circ g_s \).

(6) can be proved similar to (5).

(7) Let \( x \in S \). If \( x \) is not expressible as \( x = y + z \), then

\[
((f_s + g_s) + h_s)(x) = \emptyset. \quad \text{Otherwise},
\]

If \( x = y + z \), then

\[
((f_s + g_s) + h_s)(x) = \bigcup_{x = y + z} \{(f_s + g_s)(y) \cap h_s(z)\}
\]

\[
= \begin{cases} \emptyset, & \text{otherwise} \\ \bigcup_{x = y + z} (f_s(y) \cap g_s(y) \cap h_s(z)) & \text{otherwise} \end{cases}
\]

\[
= \begin{cases} \emptyset, & \text{otherwise} \\ \bigcup_{x = y + z} (f_s(y) \cap g_s(y) \cap h_s(z)) & \text{otherwise} \end{cases}
\]

\[
(f_s + g_s) + h_s = f_s + (g_s + h_s).
\]

(7) Let \( x \in S \). If \( x \) is not expressible as \( x = yz \), then

\[
(f_s \circ (g_s + h_s))(x) = \emptyset. \quad \text{Otherwise,}
\]

If \( x = yz \), then

\[
(f_s \circ (g_s + h_s))(x) = \bigcup_{x = yz} \{(f_s(y) \cap (g_s + h_s)(z))\}
\]

If \( z \neq z_1 + z_2 \), then \((g_s + h_s)(z) = \emptyset\) and so

\[
(f_s \circ (g_s + h_s))(x) = \emptyset. \quad \text{Therefore assume } z = z_1 + z_2,
\]

\[
= \begin{cases} \emptyset, & \text{otherwise} \\ \bigcup_{x = y(z_1 + z_2)} \{(f_s(y) \cap (g_s(z_1) \cap h_s(z_2)))\} & \text{otherwise} \end{cases}
\]

\[
= \begin{cases} \emptyset, & \text{otherwise} \\ \bigcup_{x = y(z_1 + z_2)} \{(f_s(y) \cap g_s(z_1)) \cup (f_s(y) \cap g_s(z_2))\} & \text{otherwise} \end{cases}
\]
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\[
\begin{aligned}
= \left\{ \bigcup_{s \in \mathbb{N}^* + \mathbb{Z}_2^*} \{ (f_s \circ g_s) \cap (f_s \circ h_s)(yz_2) \} \right\} \\
= \emptyset, \text{otherwise}
\end{aligned}
\]

\[
= ((f_s \circ g_s) + (f_s \circ h_s))(x)
\]

\[
\Rightarrow f_s \circ (g_s + h_s) = (f_s \circ g_s) + (f_s \circ h_s).
\]

**Definition 3.5.** Let \( X \) be a subset of \( S \). We denote by \( S_X \) the soft characteristic function of \( X \) and is defined as

\[
S_X(x) = \begin{cases} 
U, & \text{if } x \in X, \\
\emptyset, & \text{if } x \notin X.
\end{cases}
\]

It is obvious that the soft characteristic function is a soft set over \( U \), that is, \( S_X : S \rightarrow \mathcal{P}(U) \).

**Theorem 3.6.** Let \( X \) and \( Y \) be nonempty subsets of a semiring \( S \). Then, the following properties hold:

(i) If \( X \subseteq Y \), then \( S_X \subseteq S_Y \).

(ii) \( S_X \cap S_Y = S_{X \cap Y}, S_X \cup S_Y = S_{X \cup Y} \).

(iii) \( S_X \circ S_Y = S_{XY} \).

(iv) \( S_X + S_Y = S_{X+Y} \).

**Proof:**

(i) is straightforward by Definition 3.3.

(ii) Let \( s \) be any element of \( S \). Suppose \( s \in X \cap Y \). Then, \( s \in X \) and \( s \in Y \). Thus, we have

\[
(S_X \cap S_Y)(s) = S_X(s) \cap S_Y(s) = U \cap U = U = S_{X \cap Y}(s).
\]

Suppose \( s \notin X \cap Y \). Then, \( s \notin X \) or \( s \notin Y \). Hence, we have

\[
(S_X \cap S_Y)(s) = S_X(s) \cap S_Y(s) = \emptyset = S_{X \cap Y}(s).
\]

Let \( s \) be any element of \( S \). Suppose \( s \in X \cup Y \). Then, \( s \in X \) or \( s \in Y \). Thus, we have

\[
(S_X \cup S_Y)(s) = S_X(s) \cup S_Y(s) = U = S_{X \cup Y}(s).
\]

Suppose \( s \notin X \cup Y \). Then, \( s \notin X \) and \( s \notin Y \). Hence, we have

\[
(S_X \cup S_Y)(s) = S_X(s) \cup S_Y(s) = \emptyset = S_{X \cup Y}(s).
\]

(iii) Let \( s \) be any element of \( S \). Suppose \( s \in XY \). Then, \( s = xy \) for some \( x \in X \) and \( y \in Y \). Thus, we have

\[
(S_X \circ S_Y)(s) = \bigcup_{s = xy} (S_X(x) \cap S_Y(y)) \supseteq S_X(x) \cap S_Y(y) = U.
\]

This implies that \( (S_X \circ S_Y)(s) = U \). Since \( s = xy \in XY \), \( S_{XY}(s) = U \). Thus,
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\[ S_X \circ S_Y = S_{XY} \]

In another case, when \( s \notin XY \), we have \( s \neq xy \) for all \( x \in X \) and \( y \in Y \). If \( s = mn \) for some \( m, n \in S \), then we have

\[ (S_X \circ S_Y)(s) = \bigcup_{x=mn} (S_X(m) \cap S_Y(n)) = \emptyset = S_{XY}(s). \]

If \( s \neq mn \) for all \( m, n \in S \), then \( (S_X \circ S_Y)(s) = \emptyset = S_{XY}(s) \). Therefore in all the cases, we have \( S_X \circ S_Y = S_{XY} \).

4. Soft intersection semiring

Definition 4.1. Let \( S \) be a semiring and \( f_s \) be a soft set over \( U \). Then, \( f_s \) is called a soft intersection semiring of \( S \), if

1. \( f_s(x + y) \supseteq f_s(x) \cap f_s(y) \)
2. \( f_s(xy) \supseteq f_s(x) \cap f_s(y) \) for all \( x, y \in S \).

Example 4.2. Consider the semiring \( S = \{0, a, b, c\} \) defined by the following table:

<table>
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<tr>
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<th>0</th>
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Let \( U = D_3 = \{< x, y >: x^3 = y^2 = e, xy = yx^2 \} = \{e, x, x^2, y, yx, yx^2\} \) be the universal set. Let \( f_s \) and \( g_s \) be soft sets over \( U \) such that \( f_s(0) = \{e, x, y, yx\} \), \( f_s(a) = \{e, x, x^2\} \), \( f_s(b) = \{e, y, yx^2\} \), \( f_s(c) = \{e, x, x^2, y\} \).

Clearly \( f_s \) is a SI-semiring over \( U \).

It is easy that if \( f_s = U \) for all \( x \in S \), then \( f_s \) is a SI-semiring over \( U \). We denote that such a kind of SI-semiring by \( \tilde{S} \). It is obvious that \( \tilde{S} = S_s \), that is \( S_X(x) = U \) for all \( x \in S \).

Lemma 4.3. Let \( f_s \) be any SI-semiring over \( U \). Then, we have the following:

1. \( \tilde{S} \subseteq S \)
2. \( f_s \circ \tilde{S} \equiv \tilde{S} \) and \( f_s \circ \tilde{S} \subseteq \tilde{S} \).
3. \( f_s \cup \tilde{S} = \tilde{S} \) and \( f_s \cap \tilde{S} \subseteq \tilde{S} \).
4. \( \tilde{S} + S \subseteq \tilde{S} \).
5. \( \tilde{S} + \tilde{S} \subseteq \tilde{S} \) and \( \tilde{S} + f_s \subseteq \tilde{S} \).

Proof: Obviously (1), (2) and (3) are true.

(4) for any \( x \in S \)

\[ (\tilde{S} + \tilde{S})(x) = U_{x=a+b}(\tilde{S}(a) \cap \tilde{S}(b)) \]

\[ = U_{x=a+b}(U) \subseteq U = \tilde{S}(x) \]
Theorem 4.4. Let $f_S$ be a soft set over $U$. Then, $f_S$ is a SI-semiring over $U$ if and only if
\begin{enumerate}
  \item $f_S + f_S \subseteq f_S$.
  \item $f_S \circ f_S \subseteq f_S$.
\end{enumerate}

Proof: Assume that $f_S$ is a SI-semiring over $U$. Let $a \in S$. If $(f_S + f_S)(a) = \emptyset$, then it is obvious that $(f_S + f_S)(a) \subseteq f_S(a)$, thus $f_S + f_S \subseteq f_S$.

Otherwise, there exist elements $x, y \in S$ such that $a = x + y$. Then, since $f_S$ is a SI-semiring over $U$, we have:
\begin{enumerate}
  \item $(f_S + f_S)(a) = \bigcup_{a = x + y} (f_S(x) \cap f_S(y)) \\
        \subseteq \bigcup_{a = x + y} f_S(x) + f_S(x) = f_S(a)$.
\end{enumerate}

Thus, $f_S + f_S \subseteq f_S$.

Conversely, assume that (1) and (2) are true. Let $a, b \in S$ such that $a = x + y$. Then, we have:
\begin{align*}
  f_S(x + y) = f_S(a) & \supseteq (f_S + f_S)(a) = \bigcup_{a = x + y} f_S(x) \cap f_S(y) \supseteq f_S(x) \cap f_S(y) \\
  f_S(x + y) & \supseteq f_S(x) \cap f_S(y).
\end{align*}

Similarly, $f_S(xy) \supseteq f_S(x) \cap f_S(y)$

Hence, $f_S$ is an SI-semiring over $U$.

Theorem 4.5. Let $X$ be a nonempty subset of a semiring $S$. Then, $X$ is a subsemiring of $S$ if and only if $S_X$ is a SI-semiring of $S$.

Proof: Assume that $X$ is a subsemiring of $S$, that is, $XX \subseteq X$ and $X + X \subseteq X$. Then, we have:
\begin{align*}
  S_X + S_X = S_{X + X} & \subseteq S_X \\
  S_X S_X & \subseteq S_X X_X \\
  \text{by Theorem 3.2(iii) and Theorem 3.2(iv)) and so $S_X$ is a SI-semiring over $U$.}
\end{align*}

Conversely, let $x \in X + X$ and $S_X$ be a SI-semiring of $S$. Then, by Theorem 4.1, $S_X(x) \supseteq (S_X + S_X)(x) = \bigcup_{x = a + b} (S_X(a) \cap S_X(b)) \supseteq S_{X + X}(x) = U$ implying that $S_X(x)$
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\( U \), hence \( X + X \subseteq X \) and let \( x \in X \) and \( S_X \) be a SI-semiring of \( S \). Then, \( S_X(x) \supseteq (S_X \circ S_X)(x) = S_X(x) = U \) implying that \( S_X(x) = U \). Hence \( XX \subseteq X \) and so, \( X \) is a subsemiring of \( S \).

**Proposition 4.6.** Let \( f_S \) and \( f_T \) be a SI-semiring over \( U \). Then, \( f_S \wedge f_T \) is a SI-semiring over \( U \).

**Proof:** Let \( (x_1, y_1), (x_2, y_2) \in S \times T \). Then, \( f_{S,T}((x_1, y_1) + (x_2, y_2)) = f_{S,T}(x_1 + x_2, y_1 + y_2) = f_S(x_1 + x_2) \wedge f_T(y_1 + y_2) \geq [f_S(x_1) \cap f_T(x_2)] \cap [f_T(y_1) \cap f_T(y_2)] = [f_S(x_1) \cap f_T(y_1)] \cap [f_S(x_2) \cap f_T(y_2)] \)

\( f_{S,T}(x_1, y_1) \cap f_{S,T}(x_2, y_2) \)

and \( f_{S,T}(x_1, y_1)(x_2, y_2) = f_{S,T}(x_1x_2, y_1y_2) = f_S(x_1x_2) \cap f_T(y_1y_2) \geq [f_S(x_1) \cap f_S(x_2)] \cap [f_T(y_1) \cap f_T(y_2)] = [f_S(x_1) \cap f_T(y_1)] \cap [f_S(x_2) \cap f_T(y_2)] \)

\( f_{S,T}(x_1, y_1) \cap f_{S,T}(x_2, y_2) \)

Therefore, \( f_S \wedge f_T \) is a SI-semiring over \( U \).

**Definition 4.7.** Let \( f_S \) and \( f_T \) be a SI-semirings over \( U \). Then, the product of soft intersection semirings \( f_S \) and \( f_T \) is defined as \( f_S \times f_T = f_{S,T} \) where \( f_{S,T}(x, y) = f_S(x) \times f_T(y) \) for all \( (x, y) \in S \times T \).

**Proposition 4.8.** If \( f_S \) and \( f_T \) are SI-semiring over \( U \). Then, so is \( f_S \times f_T \) over \( U \times U \).

**Proof:** By Definition 4.2, \( f_S \times f_T = f_{S,T} \) where \( f_{S,T}(x, y) = f_S(x) \times f_T(y) \) for all \( (x, y) \in S \times T \).

Then, for all \( (x_1, y_1), (x_2, y_2) \in S \times T \), \( f_{S,T}((x_1, y_1) + (x_2, y_2)) = f_{S,T}(x_1 + x_2, y_1 + y_2) = f_S(x_1 + x_2) \wedge f_T(y_1 + y_2) \geq [f_S(x_1) \cap f_T(x_2)] \cap [f_T(y_1) \cap f_T(y_2)] = [f_S(x_1) \cap f_T(y_1)] \cap [f_S(x_2) \cap f_T(y_2)] \)

\( f_{S,T}(x_1, y_1)(x_2, y_2) \)

and \( f_{S,T}((x_1, y_1)(x_2, y_2)) = f_{S,T}(x_1x_2, y_1y_2) = f_S(x_1x_2) \times f_T(y_1y_2) \geq [f_S(x_1) \cap f_S(x_2)] \cap [f_T(y_1) \cap f_T(y_2)] = [f_S(x_1) \cap f_T(y_1)] \cap [f_S(x_2) \cap f_T(y_2)] \)

\( f_{S,T}(x_1, y_1) \cap f_{S,T}(x_2, y_2) \)

Therefore, \( f_S \times f_T \) is a SI-semiring over \( U \times U \).

**Proposition 4.9.** If \( f_S \) and \( h_S \) are SI-semiring over \( U \). Then, so is \( f_S \cap h_S \) over \( U \).

**Proof:** Let \( x, y \in S \), then \( (f_S \cap h_S)(x + y) = f_S(x + y) \cap h_S(x + y) \geq (f_S(x) \cap f_S(y)) \cap (h_S(x) \cap h_S(y)) \)

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\[(f_\alpha(x) \cap h_\alpha(x)) \cap (f_\alpha(y) \cap h_\alpha(y)) = (f_\alpha \cap h_\alpha)(x) \cap (f_\alpha \cap h_\alpha)(y)\]

and

\[(f_\alpha \cap h_\alpha)(xy) = f_\alpha(xy) \cap h_\alpha(xy) \supseteq (f_\alpha(x) \cap f_\alpha(y)) \cap (h_\alpha(x) \cap h_\alpha(y))\]

\[= (f_\alpha(x) \cap h_\alpha(x)) \cap (f_\alpha(y) \cap h_\alpha(y)) = (f_\alpha \cap h_\alpha)(x) \cap (f_\alpha \cap h_\alpha)(y)\].

Therefore, \(f_\alpha \cap h_\alpha\) is a SI-semiring over \(U\).

**Proposition 4.10.** Let \(f_\alpha\) be a soft set over \(U\) and \(\alpha\) be a subset of \(U\) such that \(\alpha \in \text{Im}(f_\alpha)\), where \(\text{Im}(f_\alpha) = \{\alpha \subseteq U : f_\alpha(x) = \alpha, \text{for } x \in S\}\). If \(f_\alpha\) is a SI-semiring over \(U\), then \(U(f_\alpha; \alpha)\) is a subsemiring of \(S\).

**Proof:** Let \(f_\alpha(x) = \alpha\) for some \(x \in S\), then \(\emptyset \neq U(f_\alpha; \alpha) \subseteq S\). Let \(x, y \in U(f_\alpha; \alpha)\), then \(f_\alpha(x) \supseteq \alpha\) and \(f_\alpha(y) \supseteq \alpha\). We need to show that \(xy \in U(f_\alpha; \alpha)\) and \(x + y \in U(f_\alpha, \alpha)\). Since \(f_\alpha\) is a SI-semiring over \(U\), it follows that \(f_\alpha(x) \supseteq f_\alpha(x) \cap f_\alpha(y) \supseteq \alpha \cap \alpha = \alpha \Rightarrow xy \in U(f_\alpha; \alpha)\).

\[f_\alpha(x + y) \supseteq f_\alpha(x) \cap f_\alpha(y) \supseteq \alpha \cap \alpha = \alpha\]

This shows \(x + y \in U(f_\alpha; \alpha)\).

**Definition 4.11.** Let \(f_\alpha\) be a SI-semiring over \(U\). Then, the subsemiring \(U(f_\alpha; \alpha)\) are called upper \(\alpha\)-subsemiring of \(f_\alpha\).

**Proposition 4.12.** Let \(f_\alpha\) be a soft set over \(U\), \(U(f_\alpha; \alpha)\) be upper \(\alpha\)-subsemiring of \(f_\alpha\) for each \(\alpha \subseteq U\) and \(\text{Im}(f_\alpha)\) be an ordered set by inclusion. Then, \(f_\alpha\) is a SI-semiring over \(U\).

**Proof:** Let \(x, y \in S\) and \(f_\alpha(x) = \alpha_1\) and \(f_\alpha(y) = \alpha_2\). Suppose that \(\alpha_1 \subseteq \alpha_2\). It is obvious that \(x \in U(f_\alpha; \alpha_1)\) and \(y \in U(f_\alpha; \alpha_1)\). Since \(\alpha_1 \subseteq \alpha_2\), \(x, y \in U(f_\alpha; \alpha_1)\) and \(U(f_\alpha; \alpha)\) is a subsemiring of \(S\) for all \(\alpha \subseteq U\), it follows that \(xy \in U(f_\alpha; \alpha_1)\). Hence, \(f_\alpha(x + y) \supseteq \alpha_1 = \alpha_1 \cap \alpha_2 = f_\alpha(x) \cap f_\alpha(y)\).

This shows \(x + y \in U(f_\alpha; \alpha)\).

**Proposition 4.13.** Let \(f_\alpha\) and \(f_\beta\) be soft sets over \(U\) and \(\phi\) be a semiring isomorphism from \(S\) to \(T\). If \(f_\alpha\) is a SI-semiring over \(U\), then so is \(\phi(f_\alpha)\).

**Proof:** Let \(t_1, t_2 \in T\). Since \(\phi\) is surjective, then there exists \(s_1, s_2 \in S\) such that \(\phi(s_1) = t_1\) and \(\phi(s_2) = t_2\). Then,

\[
(\phi(f_\alpha))(t_1t_2) = \bigcup \{f_\alpha(s) : s \in S, \phi(s) = t_1t_2\}
\]

\[= \bigcup \{f_\alpha(s) : s \in S, s = \phi^{-1}(t_1t_2)\} = \bigcup \{f_\alpha(s) : s \in S, s = \phi^{-1}(\phi(s_1s_2))\} = s_1s_2\]
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\[
\begin{align*}
&= \bigcup \{ f_S(s_i, s_j) : s_i \in S, \phi(s_i) = t_i, i = 1, 2 \} \\
\supseteq & \bigcup \{ f_S(s_i) \cap f_S(s_j) : s_i \in S, \phi(s_i) = t_i, i = 1, 2 \} \\
&= (\bigcup \{ f_S(s_i) : s_i \in S, \phi(s_i) = t_i \}) \cap (\bigcup \{ f_S(s_j) : s_j \in S, \phi(s_j) = t_j \}) \\
&= (\phi(f_S))(t_i) \cap (\phi(f_S))(t_j) \quad \text{and} \\
&= (\phi(f_S))(t_i + t_j) = \bigcup \{ f_S(s) : s \in S, \phi(s) = t_i + t_j \} = \bigcup \{ f_S(s) : s \in S, s = \phi^{-1}(t_i + t_j) \} \\
&= \bigcup \{ f_S(s_i + s_j) : s_i \in S, \phi(s_i) = t_i, i = 1, 2 \} \\
&= \bigcup \{ f_S(s_i) \cap f_S(s_j) : s_i \in S, \phi(s_i) = t_i, i = 1, 2 \} \\
&= (\bigcup \{ f_S(s_i) : s_i \in S, \phi(s_i) = t_i \}) \cap (\bigcup \{ f_S(s_j) : s_j \in S, \phi(s_j) = t_j \}) \\
&= \phi(f_S)(t_i) \cap \phi(f_S)(t_j) \\
\end{align*}
\]

Hence \( \phi(f_S) \) is a SI-semiring over \( U \).

**Proposition 4.14.** Let \( f_S \) and \( f_T \) be soft sets over \( U \) and \( \phi \) be a semiring homomorphism from \( S \) to \( T \). If \( f_T \) is a SI-semiring over \( U \), then so is \( \phi^{-1}(f_T) \).

**Proof:** Let \( s_1, s_2 \in S \). Then,

\[
\begin{align*}
\phi^{-1}(f_T)(s_1, s_2) &= \phi_T(\phi(s_1), \phi(s_2)) = f_T(\phi(s_1), \phi(s_2)) \\
\phi^{-1}(f_T)(s_1, s_2) &= (\phi^{-1}(f_T))(s_1) \cap (\phi^{-1}(f_T))(s_2)
\end{align*}
\]

Let \( s_1, s_2 \in S \). Then,

\[
\begin{align*}
\phi^{-1}(f_T)(s_1 + s_2) &= f_T(\phi(s_1 + s_2)) = f_T((\phi(s_1)) + (\phi(s_2))) \\
\phi^{-1}(f_T)(s_1 + s_2) &\supseteq (\phi^{-1}(f_T))(s_1) \cap (\phi^{-1}(f_T))(s_2)
\end{align*}
\]

Hence, \( \phi^{-1}(f_T) \) is an SI-semiring over \( U \).

**5. Soft intersection left (right, two-sided) ideals of semiring**

**Definition 5.1.** A soft set \( f_S \) over \( U \) is called a soft intersection left (right, two-sided) ideals of \( S \) over \( U \) if

\[
\begin{align*}
(1) & \quad f_S(x + y) \supseteq f_S(x) \cap f_S(y) \\
(2) & \quad f_S(xy) \supseteq f_S(x)(f_S(y) \supseteq f_S(y))
\end{align*}
\]

for all \( x, y \in S \). A soft set over \( U \) is called a soft intersection two-sided ideal (soft intersection ideal) of \( S \) if it is both soft intersection left and soft intersection right ideal of \( S \) over \( U \).

**Example 5.2.** Consider the semiring \( S=\{0,x,1\} \) defined by the following table:
Let $f_S$ be soft set over $S$ such that $f_S(0) = \{0, x, 1\}$, $f_S(x) = \{0, x\}$, $f_S(1) = \{x\}$. Then, one can easily show that $f_S$ is a SI-ideal of $S$ over $U$.

**Theorem 5.3.** Let $f_S$ be a SI-semiring over $U$. Then, $f_S$ is a SI-left ideal of $S$ over $U$ if and only if

1. $f_S + f_S \subseteq f_S$,
2. $S \circ f_S \subseteq f_S$.

**Proof:** Assume that $f_S$ is a SI-left ideal of $S$ over $U$. Let $a \in S$. If $(f_S + f_S)(a) = \emptyset$, then it is obvious that $(f_S + f_S)(a) \subseteq f_S(a)$, thus $f_S + f_S \subseteq f_S$.

Otherwise, there exist elements $x, y \in S$ such that $a = x + y$. Then, since $f_S$ is a SI-left ideal of $S$ over $U$, we have:

$$(f_S + f_S)(a) = \bigcup_{a = x + y} (f_S(x) \cap f_S(y)) \subseteq \bigcup_{a = x + y} f_S(x + y) = \bigcup_{a = x + y} f_S(a) = f_S(a)$$

Thus, $f_S + f_S \subseteq f_S$.

(2) If $(S \circ f_S)(a) = \emptyset$, then it is obvious that $(S \circ f_S)(a) \subseteq f_S(a)$, thus $S \circ f_S \subseteq f_S$.

Otherwise, there exist elements $x, y \in S$ such that $a = xy$. Then, since $f_S$ is a SI-left ideal of $S$ over $U$, we have:

$$(S \circ f_S)(a) = \bigcup_{a = xy} (S(x) \cap f_S(y)) \subseteq \bigcup_{a = xy} (U \cap f_S(xy)) = \bigcup_{a = xy} (U \cap f_S(a)) = f_S(a)$$

Thus, $S \circ f_S \subseteq f_S$.

Conversely, assume that (1) and (2) are true. Let $x, y \in S$ and $a = x + y$. Then, we have:

$$f_S(x + y) = f_S(a) \supseteq (f_S + f_S)(a) = \bigcup_{a = x + y} (f_S(x) \cap f_S(y)) \supseteq f_S(x) \cap f_S(y).$$

Similarly, $f_S(xy) \supseteq f_S(y)$

Hence, $f_S$ is an SI-left ideal over $U$. This competes the proof.

**Theorem 5.4.** Let $f_S$ be a SI-semiring over $U$. Then, $f_S$ is a SI-right ideal of $S$ over $U$ if and only if

1. $f_S + f_S \subseteq f_S$,
2. $S \circ f_S \subseteq f_S$.

Let $f_S$ be soft set over $S$ such that $f_S(0) = \{0, x, 1\}$, $f_S(x) = \{0, x\}$, $f_S(1) = \{x\}$.
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(2) \( f_s \circ S \subseteq f_s \).

Proof: Similar to the proof of Theorem 5.1.

**Corollary 5.5.** \( \tilde{S} \) is both SI-right and SI-left ideal of \( S \).

Proof: Follows from Lemma 4.1-(1).

**Theorem 5.6.** Let \( X \) be a nonempty subset of a semiring \( S \). Then, \( X \) is a left (right, two-sided) ideal of \( S \) if and only if \( S_X \) is an SI-left (right, two-sided) ideal of \( S \) over \( U \).

Proof: We give the proof for the SI-left ideals. Assume that \( X \) is a left ideal of \( S \), that is, \( X \subseteq X \subseteq X \) and \( S \subseteq X \). Then, we have \( S_X \subseteq S_X \subseteq S_X \) and \( S \circ S_X = S \circ S_X = S \subseteq S_X \). Thus, \( S_X \) is an SI-left ideal of \( S \) over \( U \) by Theorem 5.1.

Conversely, let \( \tilde{S}X \) and \( S_X \) be an SI-left ideal of \( S \) over \( U \). Then, \( S_X(x) \supseteq (S \circ S_X)(x) = (S \circ S_X)(x) = S_{SX}(x) = U \) implying that \( S_{SX}(x) = U \), hence \( x \in X \). Similarly \( S_X(x) \supseteq (S_X + S_X)(x+X) \). Thus, \( S_X \subseteq X \) and \( X \) is a left ideal of \( S \).

**Theorem 5.7.** Let \( f_s \) and \( g_s \) be SI-left (right) ideals of a semiring \( S \). Then \( f_s + g_s \) is a SI-left (right) ideal of \( S \).

Proof: Suppose \( f_s \) and \( g_s \) are SI-left ideals of a semiring \( S \) and \( x, y \in S \). If \( (f_s + g_s)(x) = \emptyset \) or \( (f_s + g_s)(y) = \emptyset \) then, \( (f_s + g_s)(x) \cap (f_s + g_s)(y) = \emptyset \subseteq (f_s + g_s)(x+y) \). If \( (f_s + g_s)(x) \neq \emptyset \) and \( (f_s + g_s)(y) \neq \emptyset \) then, \( (f_s + g_s)(y) = \bigcup_{x=a+b} \{f_s(c) \cap g_s(d)\} \). Thus,

\[
(f_s + g_s)(x) \cap (f_s + g_s)(y) = \left( \bigcup_{x=a+b} \{f_s(a) \cap g_s(b)\} \right) \cap \left( \bigcup_{y=c+d} \{f_s(c) \cap g_s(d)\} \right)
\]

\[
= \bigcup_{x=a+b} \bigcup_{y=c+d} \{f_s(a) \cap g_s(b)\} \cap \{f_s(c) \cap g_s(d)\}
\]

\[
= \bigcup_{x=a+b} \bigcup_{y=c+d} \{f_s(a) \cap f_s(c)\} \cap \{g_s(b) \cap g_s(d)\}
\]

\[
\subseteq \bigcup_{x=a+b} \bigcup_{y=c+d} \{f_s(a+c) \cap g_s(b+d)\}
\]

\[
\subseteq (f_s + g_s)(x+y).
\]

Again, if \( (f_s + g_s)(x) = \emptyset \) then \( (f_s + g_s)(x) \subseteq (f_s + g_s)(yx) \). If \( (f_s + g_s)(x) \neq \emptyset \), then
(\(f_s + g_s\))(x) = \bigcup_{x=a+b} \{ f_s(a) \cap g_s(b) \} \subseteq \bigcup_{x=a+y} \{ f_s(ya) \cap g_s(yb) \}
\subseteq \bigcup_{y=x+c+d} \{ f_s(c) \cap g_s(d) \} = (f_s + g_s)(yx).

Hence \(f_s + g_s\) is a soft fuzzy left ideal of \(S\).

**Theorem 5.8.** If \(f_s, g_s\) are SI-left (right) ideals of a semiring \(S\), then \(f_s \circ g_s\) is a SI-left (right) ideal of \(S\).

**Proof:** Suppose \(f_s, g_s\) are SI-left ideals of a semiring \(S\) and \(x, y \in S\). If
\((f_s \circ g_s)(x) = \emptyset\) or \((f_s \circ g_s)(y) = \emptyset\),
then \((f_s \circ g_s)(x) \cap (f_s \circ g_s)(y) = \emptyset \subseteq (f_s \circ g_s)(x + y)\).

If \((f_s \circ g_s)(x) \neq \emptyset\) and \((f_s \circ g_s)(y) \neq \emptyset\), then
\((f_s \circ g_s)(x) = \bigcup_{x=a+b} \{(f_s(a) \cap g_s(b))\}\)
\((f_s \circ g_s)(y) = \bigcup_{y=x+c+d} \{(f_s(c) \cap g_s(d))\}\)
\((f_s \circ g_s)(x) \cap (f_s \circ g_s)(y) = \bigcup_{x=a+b} \{(f_s(a) \cap g_s(b))\} \cap \bigcup_{y=x+c+d} \{(f_s(c) \cap g_s(d))\} \subseteq \bigcup_{x+y+e+f} \{(f_s(e) \cap g_s(f))\}
= (f_s \circ g_s)(x + y).

Again, if \((f_s \circ g_s)(x) = \emptyset\) then \((f_s \circ g_s)(x) \subseteq (f_s \circ g_s)(yx)\).

If \((f_s \circ g_s)(x) \neq \emptyset\), then
\((f_s \circ g_s)(x) = \bigcup_{x=a+b} \{(f_s(a) \cap g_s(b))\}\)
\subseteq \bigcup_{x=a+b} \{(f_s(ya) \cap g_s(yb))\}
\subseteq \bigcup_{x+y=a+b} \{(f_s(c) \cap g_s(d))\} = (f_s \circ g_s)(yx).

Hence \(f_s \circ g_s\) is a SI- left ideal of \(S\).

**Theorem 5.9.** Let \(f_s\) be a soft set over \(U\). Then, if \(f_s\) is a SI-left (right, two sided) ideal of \(S\) over \(U\), \(f_s\) is a SI-semiring over \(U\).

**Proof:** We give the proof for SI-ideals. Let \(f_s\) be a SI-left ideal of \(S\) over \(U\). Then,
\(f_s(x + y) \supseteq f_s(x) \cap f_s(y)\) and \(f_s(xy) \supseteq f_s(y)\) for all \(x, y \in S\). Thus \(f_s(xy) \supseteq f_s(y) \supseteq f_s(x) \cap f_s(y)\), so \(f_s\) is a SI-semiring over \(U\).

**Theorem 5.10.** Let \(f_s\) be a SI-right ideal and \(g_s\) a soft intersection left ideal of a semiring \(S\). Then \(f_s \circ g_s \subseteq f_s \cap g_s\).
Proof: Let \( f_S \) and \( g_S \) be a SI-right ideal of \( S \). Then, since \( f_S, g_S \subseteq S \) always holds, we have \( f_S \circ g_S \subseteq f_S \circ S \subseteq f_S \) and \( f_S \circ g_S \subseteq S \circ g_S \subseteq g_S \). Hence \( f_S \circ g_S \subseteq f_S \cap g_S \).

Proposition 5.11. Let \( f_S \) be a soft set over \( U \) and \( \alpha \) be a subset of \( U \) such that \( \alpha \in \text{Im}(f_S) \). If \( f_S \) is an SI-left (right) ideal of \( S \) over \( U \), the \( U(f_S;\alpha) \) is a left (right) ideal of \( S \) over \( U \).

6. Soft intersection quasi-ideals of semiring

Definition 6.1. A SI-semiring \( f_S \) over \( U \) is called a soft intersection quasi-ideal of \( S \) over \( U \).

(i) \( f_S(x + y) \supseteq f_S(x) \cap f_S(y) \) for all \( x, y \in S \)

(ii) \( (S \circ f_S) \cap (f_S \circ S) \subseteq f_S \).

Theorem 6.2. A soft set \( f_S \) of a semiring \( S \) is a soft intersection quasi-ideal of \( S \) if and only if each nonempty level subset \( U(f_S;\alpha) \) of \( f_S \) is a quasi-ideal of \( S \).

Proof: Suppose \( f_S \) is a fuzzy quasi-ideal of \( S \). Let \( a, b \in U(f_S;\alpha) \). Then \( f_S(a) \supseteq \alpha \) and \( f_S(b) \supseteq \alpha \). As \( f_S(a + b) \supseteq f_S(a) \cap f_S(b) \), so \( f_S(a + b) \supseteq \alpha \). Hence \( a + b \in U(f_S;\alpha) \).

Let \( x = \sum_{i=1}^{n} u_i r_i \) and \( x = \sum_{k=1}^{p} s_k v_k \) for some \( u_i, v_k \in U(f_S;\alpha) \) and \( r_i, s_k \in S \).

Now,
\[
f_S(x) \supseteq \bigcup_{x = \sum_{i=1}^{n} u_i r_i} \left[ S(u_i) \cap f_S(r_i) \right] \cup \bigcup_{x = \sum_{k=1}^{p} s_k v_k} \left[ f_S(s_k) \cap S(v_k) \right] \supseteq \alpha \cap \alpha = \alpha.
\]

So, \( f_S(x) \supseteq \alpha \). Thus, \( x \in U(f_S;\alpha) \). Hence \( U(f_S;\alpha) \cap SU(f_S;\alpha) \subseteq U(f_S;\alpha) \).

Conversely, assume that each nonempty subset \( U(f_S;\alpha) \) of \( S \) is a quasi-ideal of \( S \). Let \( a, b \in S \) be such that \( f_S(a + b) \subseteq f_S(a) \cap f_S(b) \). Take \( \alpha \subseteq U \) such that \( f_S(a + b) \subseteq \alpha \subseteq f_S(a) \cap f_S(b) \). Then \( a, b \in U(f_S;\alpha) \) but \( a + b \not\in U(f_S;\alpha) \), a contradiction. Hence \( f_S(a + b) \supseteq f_S(a) \cap f_S(b) \).

Let \( x \in S \). If possible let \( f_S(x) \subset [(f_S \circ S) \cap (S \circ f_S)](x) \). Take \( \alpha \subset U \) such that \( f_S(x) \subset \alpha \subseteq [(f_S \circ S) \cap (S \circ f_S)](x) \). If \( [(f_S \circ S) \cap (S \circ f_S)](x) \supseteq \alpha \), then...
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\[
[(\tilde{S} \circ f_S) \cap (f_S \circ \tilde{S})](x) = \bigcup_{x = \sum_{i, k} v_i} \tilde{S}(s_k) \cap f_S(v_i) \bigcap \bigcup_{x = \sum_{i, k} u_i} f_S(u_i) \cap \tilde{S}(r) .
\]

Hence, \[ \bigcup_{x = \sum_{i, k} v_i} \tilde{S}(s_k) \cap f_S(v_i) \supseteq \alpha \] and \[ \bigcup_{x = \sum_{i, k} u_i} f_S(u_i) \cap \tilde{S}(r) \supseteq \alpha .\]

so, \( f_S(u_i) \supseteq \alpha, f_S(v_k) \supseteq \alpha \), that is, \( u_i, v_k \in U(f_S ; \alpha) \) for all \( i, k \). Thus

\[
\sum_{i=1}^n u_i r_i \in U(f_S ; \alpha) S \quad \text{and} \quad \sum_{k=1}^p s_k v_k \in SU(f_S ; \alpha).
\]

This implies \( x \in U(f_S ; \alpha) S \bigcap SU(f_S ; \alpha) \subseteq U(f_S ; \alpha) \), and hence \( x \in U(f_S ; \alpha) \), that is \( f_S(x) \supseteq \alpha \), a contradiction.

Hence \( f_S \circ \tilde{S} \cap (\tilde{S} \circ f_S) \subseteq f_S \). Thus \( f_S \) is a soft intersection quasi-ideal of \( S \).

**Corollary 6.3.** Let \( Q \) be a nonempty subset of a semiring \( S \). Then \( Q \) is a quasi-ideal of \( S \) if and only if the characteristic function \( \tilde{S} \) of \( Q \) is a fuzzy quasi-ideal of \( S \).

**Proposition 6.4.** The intersection of any two SI quasi-ideals of a semiring \( S \) is a SI quasi-ideal of \( S \).

**Proof:** Let \( f_S, g_S \) be SI-semiring quasi-ideals of a semiring \( S \) and \( x, y \in S \). Then

\[
(f_S \cap g_S)(x + y) = f_S(x + y) \cap g_S(x + y) \supseteq [f_S(x) \cap f_S(y) \cap [g_S(x) \cap g_S(y)]
\]

\[= [f_S(x) \cap g_S(x)] \cap [f_S(y) \cap g_S(y)] = (f_S \cap g_S)(x) \cap (f_S \cap g_S)(y) .
\]

Also,

\[
((f_S \cap g_S) \circ \tilde{S}) \cap (\tilde{S} \circ (f_S \cap g_S)) \subseteq (f_S \circ \tilde{S}) \cap (\tilde{S} \circ f_S) \subseteq f_S .
\]

\[
((f_S \cap g_S) \circ \tilde{S}) \cap (\tilde{S} \circ (f_S \cap g_S)) \subseteq (g_S \circ \tilde{S}) \cap (\tilde{S} \circ g_S) \subseteq g_S .
\]

Thus \( ((f_S \cap g_S) \circ \tilde{S}) \cap (\tilde{S} \circ (f_S \cap g_S)) \subseteq f_S \cap g_S .\)

**Corollary 6.5.** Let \( f_S \) and \( g_S \) be SI- right and SI- left ideals of a semiring \( S \), respectively. Then \( f_S \cap g_S \) is a SI- quasi-ideal of \( S \).

**7. Soft intersection bi-ideals of semiring**

**Definition 7.1.** A soft set over \( U \) is called a SI- bi-ideals of \( S \) over \( U \) if

1. \( f_S(x + y) \supseteq f_S(x) \cap f_S(y) \)
2. \( f_S(xy) \supseteq f_S(x) \cap f_S(y) \)
3. \( f_S(xyz) \supseteq f_S(x) \cap f_S(z) \) for all \( x, y, z \in S \).

**Example 7.2.** Consider the semiring \( S = \{0, a, b, c\} \) defined by the following table:
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Define the soft set \( S_f \) over \( U = \mathbb{Z}_4 \) such that
\[ S_f(0) = \{0, 1, 2\}, \quad S_f(a) = \{0, 1\}, \quad S_f(b) = \{0\}, \quad S_f(c) = \{1, 2\}. \]
Clearly, \( S_f \) is a SI-Bi-ideal of \( S \) over \( U \).

**Theorem 7.3.** Let \( f_S \) be a soft set over \( U \). Then, \( f_S \) is a SI-bi-ideal of \( S \) over \( U \) if and only if

(i) \( f_S + f_S \subseteq f_S \)

(ii) \( f_S \circ f_S \subseteq f_S \)

(iii) \( \tilde{f}_S \circ \tilde{f}_S \subseteq f_S \).

**Proof:** Let \( f_S \) be a SI-bi-ideal of \( S \) and \( x \in S \). Then

(1) \( (f_S + f_S)(x) = \bigcup_{x+y=z} \{f_S(y) \cap f_S(z)\} \subseteq \bigcup_{x+y=z} f_S(y+z) = f_S(x). \)

Therefore, \( f_S + f_S \subseteq f_S \).

(2) \( f_S \circ f_S \subseteq f_S \) is evident.

(3) Let \( s \in S \). If \( (f_S \circ \tilde{S} \circ f_S)(s) = \emptyset \), then \( f_S \circ \tilde{S} \circ f_S \subseteq f_S \). Otherwise, there exist elements \( x, y, p, q \in S \) such that \( s = xy \) and \( x = pq \).

Then, since \( f_S \) is an SI-bi-ideal of \( S \) over \( U \), we have

\[ (f_S \circ \tilde{S} \circ f_S)(s) = [(f_S \circ \tilde{S} \circ f_S)(s)] = \bigcup_{x=py} (f_S(p) \cap f_S(y)) \subseteq \bigcup_{x=py} f_S(pq) = f_S(xy) = f_S(s) \]

Hence, \( f_S \circ \tilde{S} \circ f_S \subseteq f_S \). Here, note that if \( x \neq pq \), then \( (f_S \circ \tilde{S} \circ f_S \subseteq f_S)(a) = \emptyset \), and so \( (f_S \circ \tilde{S} \circ f_S = \emptyset \subseteq f_S(s) \).

Conversely, assume (1) and (2). By theorem 4.1, \( f_S \) is a SI-semiring of \( S \). Let \( x, y, z \in S \) and \( s = xyz \). Then, since \( f_S \circ \tilde{S} \circ f_S \subseteq f_S \), we have

\[ f_S(xyz) = f_S(s) \supseteq (f_S \circ \tilde{S} \circ f_S)(s) = [(f_S \circ \tilde{S} \circ f_S)(s)](s) \]
Thus, $f_S$ is a SI-bi-ideal of $S$ over $U$.

**Theorem 7.4.** Let $X$ be a nonempty subset of a semiring $S$. Then, $X$ is a bi-ideal of $S$ if and only if $S_X = X$ is a SI-bi-ideal of $S$ over $U$.

**Proof:** Assume that $X$ is a bi-ideal of $S$, that is, $X X X \subseteq X X$ and $X X X \subseteq X$. Then, we have

$$S_X + S_X = S_{XX} \subseteq S_X \text{ and } S_X \circ S_X = S_{XX} (\text{since } xx \subseteq X).$$

Moreover $S_X \circ S_X = S_X \circ S_X = S_{XX} \subseteq S_X (\text{since } X X \subseteq X)$.

This means that $S_X$ is a bi-ideal of $S$.

Conversely, let $S_X$ be an SI-bi-ideal of $S$. It means that $S_X$ is a SI-semiring. Let $x \in X + X$. Then, $S_X (x) \subseteq (S_X + S_X)(x) = S_{XX}(x) = U \Rightarrow x \in X \Rightarrow X X \subseteq X$ and let $x \in XX$. Then, $S_X (x) \subseteq (S_X \circ S_X)(x) = S_{XX}(x) = U \Rightarrow x \in X \Rightarrow XX \subseteq X$.

Therefore $X$ is a subsemiring of $S$. Next, let $y \in X X$. Thus $S_X (y) \subseteq (S_X \circ S_X)(y) = (S_X \circ S_X)(y) = S_{XX}(y) = U$ and so $y \in X$. Thus $X X \subseteq X$ and $X$ is a bi-ideal of $S$.

**Theorem 7.5.** Every SI-left (right, two sided) ideal of a semiring $S$ over $U$ is a SI-bi-ideal of $S$ over $U$.

**Proof:** Let $f_S$ be a SI-left (right, two sided) ideal of a semiring $S$ over $U$ and $x, y, z \in S$. Then $f_S$ is as SI-semiring by Theorem (5.6.). Moreover, $f_S(xyz) = f_S((xy)z) \supseteq f_S(z) \supseteq f_S(x) \cap f_S(z)$. Thus $f_S$ is a SI-bi-ideal of $S$.

**Theorem 7.6.** Let $f_S$ be any SI-ideal of a semiring $S$ and $g_S$ any SI-bi-ideal of $S$. $f_S \circ g_S$ and $g_S \circ f_S$ are SI-bi-ideals of $S$.

**Proof:** To show that $f_S \circ g_S$ is a SI-bi-ideal of $S$, first we need to show that $f_S \circ g_S$ is a SI-semiring. Thus

$$(f_S \circ g_S) \circ (f_S \circ g_S) = f_S \circ (g_S \circ (f_S \circ g_S))$$

$$(f_S \circ g_S) \subseteq f_S \circ (g_S \circ (f_S \circ g_S)) (\text{since } f_S \subseteq S) = f_S \circ (g_S \circ (S \circ g_S))$$

$$(f_S \circ g_S) \subseteq (f_S \circ g_S) (\text{since } g_S \circ (S \circ g_S) \subseteq g_S, g_S \text{ being SI} \text{ bi-ideal})$$

Hence by Theorem 4.1, $f_S \circ g_S$ is a SI-semiring over $U$. Moreover we have
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\[(f_s \circ g_s) \circ (f_s \circ g_s) = f_s \circ (g_s \circ (S \circ f_s) \circ g_s)\]
\[\subseteq f_s \circ (g_s \circ (S \circ g_s))(\text{since } S \circ f_s \subseteq S) \subseteq f_s \circ g_s.\]

Thus, it follows that \(f_s \circ g_s\) is a SI-bi-ideal of \(S\). It can be proved in a similar way that \(g_s \circ f_s\) is a SI-bi-ideal of \(S\) over \(U\).

**Theorem 7.7.** Intersection of a non-empty collection of SI-bi-ideal of \(S\) over \(U\) is also a SI-bi-ideal of \(S\) over \(U\).

**Proof:** The proof follows by routine verifications.

**Theorem 7.8.** Let \(\{f_i : i \in I\}\) be a family of SI-bi-ideal of \(S\) over \(U\) such that \(f_i \subseteq f_j\) or \(f_j \subseteq f_i\) for \(i, j \in I\). Then \(\bigcup_i f_i\) is a SI-bi-ideal of \(S\) over \(U\).

**Proof:** The proof follows by routine verifications.

**Theorem 7.9.** In a semiring every SI-quasi ideals are SI-bi-ideals.

**8. Conclusion**

Through this paper, SI-semiring, SI-left (right, two-sided) ideals of semiring, SI-quasi ideal of semiring and SI-bi-ideal of semirings are studied and properties pertaining to them elicited.

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