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A note on the Diophantine Equation $p + q + r = M^2$ and the Goldbach Conjectures

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Abstract. We discuss the title equation with primes p, q, r, and the connection to the binary and ternary Goldbach Conjectures. The binary conjecture states the every even integer $N \geq 4$ is the sum of two primes, whereas the ternary conjecture states that every odd integer $N \geq 7$ is the sum of three primes. Under the assumption that the binary conjecture is true, it is established for each and every fixed prime $p \geq 2$ and also for each and every $M \geq 3$, that there exist primes q, r so that the equation has at least one solution. The ternary conjecture is also verified in the case of this equation.

Keywords: Diophantine equations, Goldbach conjectures

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In this paper, we consider the branch of additive prime number theory. Our main focus is the diophantine equation

$$x_1 + x_2 + x_3 = K$$

where x_i and K are positive integers.

Goldbach in his correspondence to Euler in 1742 conjectured that every even number greater than 2 can be written as the sum of two primes. This problem is one of the most famous unsolved problems in mathematics today. It is very easy to state, but very difficult to prove. Goldbach's second conjecture states that every odd number greater than 5 can be written as the sum of three primes. The literature contains numerous articles on both these problems, and only a minute abstract [6, 7, 8,11] is provided here. In 1937 Vinogradov [11] showed that every sufficiently large odd positive integer can be written as the sum of three primes.

The two conjectures are stated as follows:

Goldbach Conjectures. *Every even integer $N \geq 4$ is the sum of two primes, and every odd integer $N \geq 7$ is the sum of three primes.*

The two parts of this conjecture are known as the binary Goldbach problem and the ternary Goldbach problem, respectively. Clearly, the binary conjecture is the stronger one, and also the more difficult.

The equation

$$p^x + q^y = z^2 \qquad (1)$$

where p, q are odd positive integers, and x, y, z are positive integers has interested various authors, among them [1, 2, 3, 4, 5, 9, 10]. The objective of this paper is to extend equation (1) to the following equation

$$p + q + r = M^2 \qquad M \geq 3 \qquad p, q, r \text{ are primes.} \qquad (2)$$

If the binary Goldbach Conjecture is true, it was shown [5] for a particular case that equation (1) has infinitely many solutions. Here, it will be shown for each and every fixed prime $p \geq 2$ and each and every value $M \geq 3$, that equation (2) has at least one solution in which q and r are primes.

2. The main result

In (2), without loss of generality, we assume hereafter that $p \leq q \leq r$. We shall distinguish two cases, namely: the case $p = 2$, and the case of each and every prime $p > 2$. The two cases are considered in the following lemmas.

The case $p = 2$ is discussed in Lemmas 2.1 and 2.2.

Lemma 2.1. If in (2)

(a) $2 = p = q = r$, then (2) has no solutions,

or

(b) $2 = p = q$, then (2) has exactly one solution.

Proof: (a) Suppose $2 = p = q = r$. Then

$$p + q + r = 6 \neq M^2.$$

Thus, equation (2) has no solutions.

(b) Suppose $2 = p = q$. From equation (2) we have

$$2 + 2 + r = M^2.$$

Hence, $r = M^2 - 4 = (M - 2)(M + 2)$. Since r is prime, it follows that $M - 2 = 1$ and $M + 2 = r$. The values M and r are then $M = 3$ and $r = 5$, the only possible solution of (2).

The proof of Lemma 2.1 is complete. □

In Lemma 2.2 we consider all even values $M \geq 4$.

Lemma 2.2. Suppose that the binary Goldbach Conjecture is indeed true. If in equation (2) $p = 2$ is fixed, then for each and every even value $M \geq 4$, there exists at least one pair of primes (q, r) which satisfies equation (2).

Proof: Consider the equation

$$2 + A + B = M^2. \qquad (3)$$

In (3), for every even fixed value $M \geq 4$, certainly there exist two odd positive integers $A \leq B$ not necessarily both primes which satisfy the equation. Hence $A + B$ is even. For instance:

$A = 5$, $B = 9$, $M = 4$, $A = 13$, $B = 21$, $M = 6$, $A = 27$, $B = 35$, $M = 8$.

Our supposition now implies that the even value $A + B$ is the sum of two primes, say q, r. Then, in (3) respectively substitute A and B by q and r. Equation (2) is now satisfied. Therefore, the equation

$$2 + q + r = M^2 \qquad \text{for each and every even } M \geq 4 \qquad (4)$$

has at least one solution.

This concludes our proof. □

Lemma 2.2 yields the following Corollary 2.1.

Corollary 2.1. Equation (4) has infinitely many solutions.

Remark 2.1. For every even fixed value $M \geq 4$, one can clearly see that equality (3) i.e., $A + B = M^2 - 2$ yields more than one solution for A and B. Hence, the statement which follows (4): has at least one solution - is justified.

In the next Lemma 2.3, all odd values M are considered.

Lemma 2.3. Suppose that the binary Goldbach Conjecture is indeed true. If in equation (2) $p \geq 3$ is any fixed prime, then for each and every odd value $M \geq 3$, there exists at least one pair of primes (q, r) such that equation (2) is satisfied.

Proof: If $p = q = r$, then from (2) we have $3p = M^2$, and the only possible solution is then $p = 3$ and $M = 3$. When $p = q$, we demonstrate in Table 1, the first six values p and their respective values r, M forming solutions of equation (2).

Table 1.

p	q	r	M	M^2
3	3	19	5	25
3	3	43	7	49
5	5	71	9	81
7	7	11	5	25
7	7	67	9	81
11	11	59	9	81
13	13	23	7	49
17	17	47	9	81

From Table 1 it is quite evident, that for the same prime p various values M exist. On the other hand, for the same values M, distinct primes p exist.

We now consider the equation

$$p + A + B = M^2 \tag{5}$$

where $p \geq 3$ is any fixed prime, and $A \leq B$ are odd positive integers. Thus $A + B$ is even. Certainly, (5) exists when both A and B are not primes. As for example: $p = 7$, $A = 35$, $B = 39$, $M = 9$, and $p = 11$, $A = 35$, $B = 35$, $M = 9$ satisfy (5), but A and B are not both primes.

For every fixed odd $M \geq 3$ which satisfies (5), our supposition then implies that $A + B$ is the sum of two primes, say q, r. In (5), set $A = q$ and $B = r$. Thus (2) is satisfied. Therefore, for any prime $p \geq 3$, the equation

$$p + q + r = M^2 \qquad \text{for each and every odd } M \geq 3 \tag{6}$$

has at least one solution as asserted. □

The following Corollary 2.2 is a consequence of Lemma 2.3.

Corollary 2.2. Equation (6) has infinitely many solutions.

Final remark. Under the assumption that the binary Goldbach Conjecture is true, it follows from Lemma 2.3 that each and every odd square M^2 is the sum of three primes. Hence, the ternary Goldbach Conjecture has been verified in the particular case of equation (2).

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