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On a Ramsey Problem Involving Quadrilaterals

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Abstract. Let $j \geq 3$. Given any two coloring (consisting of say red and blue colors) of the edges of a complete graph $K_{j\times s}$, we say that $K_{j\times s} \rightarrow (C_4,G)$, if there exists a copy of a red C_4 or a copy of blue G in it. Let $m_j(C_4,G)$ denote the smallest positive integer s such that $K_{j\times s} \rightarrow (C_4,G)$. This paper deals with finding the exact values $m_j(C_4,G)$ for all possible proper subgraphs G of K_4.

Keywords: Ramsey theory, multipartite Ramsey numbers

AMS Mathematics Subject Classification (2010): 05C55, 05D10

1. Introduction

All graphs listed in this paper are graphs without loops or parallel edges. Let G, H and K represent three graphs. Given any two coloring (consisting of say red and blue colors) of the edges of a graph G, we say that $G \rightarrow (H,K)$ if there exists a red copy of H in G or a blue copy of K in G. The Ramsey number $r(H,K)$ is defined as the smallest positive integer t such that $K_t \rightarrow (H,K)$. The diagonal classical Ramsey number $r(n,n)$ is defined as the smallest positive integer t such that $K_t \rightarrow (K_n,K_n)$. In the last four decades most of the Ramsey numbers $R(H,K)$ have been studied in detail for $|V(H)| <7$ and $|V(K)| <7$ (see [6]). The size Ramsey multipartite number $m_j(H,K)$ is defined as the smallest natural number t such that $K_{j\times t} \rightarrow (H,K)$ (see [1,3,5,7]). In this paper we concentrate on determining multipartite Ramsey numbers $m_j(C_4,G)$ for all possible proper subgraphs G of K_4.

2. Notation

The vertices of $K_{j\times s}$ are labeled as $\{v_{k,i}|\ 1 \leq i \leq s,\ 1 \leq k \leq j\}$, with the m^{th} partite set consisting of $\{v_{m,i}|\ 1 \leq i \leq s\}$. It is worth noting that all values of $m_j(C_4,P_4)$ and $m_j(C_4,C_3)$ (see [4])are currently known. Also all values of $m_j(C_4,K_{1,3}+x)$ are known as $m_j(C_4,C_3) = m_j(C_4,K_{1,3}+x)$ for any integer j.

3. Some useful lemmas on connected proper subgraphs of K_4

Theorem 1. *If $j \geq 3$, then*

$$m_j(C_4,C_4) = \begin{cases} 1 & j \geq 6 \\ 2 & j = \{4,5\} \\ 3 & j = 3 \end{cases}$$

Proof. If $j \geq 6$ (see [2]), since $r(C_4,C_4) = 6$ we get $m_j(C_4,C_4) = 1$. So we are left with the cases $j = 3$, $j = 4$ and $j = 5$. If $j = 4$ or $j = 5$, consider the coloring of $K_{j\times 1} = H_R \oplus H_B$, generated by $H_R = C_5$ and $H_R = C_3$ respectively. Then, $K_{j\times 1}$ has no red C_4 or a blue C_4. Therefore, we obtain that $m_4(C_4,C_4) \geq 2$ and $m_5(C_4,C_4) \geq 2$.

In order to show, $m_5(C_4,C_4) \leq 2$ and $m_4(C_4,C_4) \leq 2$, first note that it suffices only to show that $m_4(C_4,C_4) \leq 2$. Consider any red/blue coloring given by $K_{4\times 2} = H_R \oplus H_B$, such that H_R contains no red C_4 and H_B contains no blue C_4.

In the first possibility that H_R is a regular graph of order 3, we get from the above remark that H_R must contain a red C_3. Without loss of generality assume that this red 3 cycle is incident to the first three partite sets and consists of say $v_{1,1}$, $v_{2,1}$, $v_{3,1}$. Then both $v_{4,1}$ and $v_{4,2}$ have to be adjacent to two verticesof $v_{1,1}$, $v_{2,1}$, $v_{3,1}$ in blue in order to avoid a red C_4. Without loss of generality assume $v_{4,1}$ and $v_{4,2}$ are adjacent $v_{3,1}$, $v_{2,1}$ and $v_{3,1}$, $v_{1,1}$ respectively. But then as $deg_R(v_{1,1}) = 3$, $(v_{1,1},v_{2,2})$ and $(v_{1,1},v_{3,2})$ will have to be blue edges. This is illustrated in the following figure.

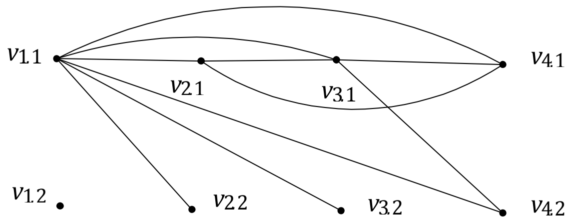

Figure 1: In the first possibility, the derived red/blue graphs

As there is no blue C_4, $(v_{2,2},v_{3,1})$ and $(v_{4,2},v_{2,1})$ have to be red edges. Next as there is no red C_4, $(v_{2,2},v_{4,1})$ has to be a blue edge. But then if we consider the edge $(v_{2,2},v_{4,2})$ we see that it can be neither a red edge or a blue edge as it will give rise to a red C_4 or a blue C_4, a contradiction.In the second possibility that H_R is not a regular graph of order 3, by symmetry we can assume that red degree of a vertex (say $v_{1,2}$) is greater than or equal to 4 and without loss of generality $v_{1,2}$ is connected in red to $v_{2,1}$, $v_{2,2}$, $v_{3,1}$ and $v_{4,1}$ as illustrated in the following figure (note if $v_{1,2}$ is adjacent to $v_{2,1}$, $v_{2,2}$, $v_{3,1}$ and $v_{3,2}$, it would clearly force a monochromic.

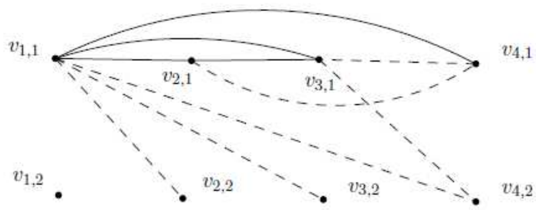

Figure 2: In the first possibility, the derived red/blue graphs

If $v_{1,1}$ is connected in red to two vertices of $N_R(v_{1,2})$ will result in a red C_4. Therefore, without loss of generality get either $v_{1,1}$ to connected in blue to $v_{2,1}$, $v_{3,1}$, $v_{4,1}$ or else $v_{1,1}$ to connected in blue to $v_{2,1}$, $v_{2,2}$, $v_{3,1}$. In the first scenario, as the induced subgraph of $v_{2,1}$, $v_{3,1}$, $v_{4,1}$ will contains 2 red edges or 2 blue edges and these two edges will be contained in a C_4 of that same color, a contradiction. In the second scenario, $v_{4,2}$ will be adjacent to two vertices of $v_{2,1}$, $v_{2,2}$, $v_{3,1}$ in some color and then this would force these two vertices to be contained in a monochromatic C_4 in that same color, a contradiction.

If j = 3, consider the coloring of $K_{j\times 2} = H_R \oplus H_B$, generated by $H_R = 2K_3$. Then, $K_{j\times 2}$ has no red C_4 or a blue C_4. Therefore, we obtain that $m_3(C_4,C_4) \geq 3$.

To show, $m_3(C_4,C_4) \leq 3$. Consider any red/blue coloring given by $K_{3\times 3} = H_R \oplus H_B$, such that H_R contains no red C_4 and H_B contains no blue C_4. By handshaking lemma all vertices cannot have red degree 3.

Therefore, without loss of generality, using symmetry, we may assume that $v_{1,1}$ is adjacent to at least 4 vertices in red. Let V be any subset of size 4 of $N_R(v_{1,1})$. In order to avoid a red C_4, both $v_{1,2}$ and $v_{1,3}$ must be adjacent to at least three vertices of V in blue. This will result in a blue C_4 containing $v_{1,2}$ and $v_{1,3}$ contrary to our assumption. Therefore, we could conclude that $m_3(C_4,C_4) = 3$.

Theorem 2.

$$m_j(C_4,B_2) = \begin{cases} 1 & j \geq 7 \\ 2 & j = \{5,6\} \\ 3 & j = 4 \\ 4 & j = 3 \end{cases}$$

Proof. Clearly $m_j(C_4,B_2) = 1$ if $j \geq 7$, since $r(C_4,B_2) = 7$ (see [2]).
Let $j \in \{5,6\}$. Consider the coloring of $K_{j\times 1} = H_R \oplus H_B$, generated by $H_R = C_5$ and $H_R = 2K_3$ when j = 5 and j = 6 respectively. Then, $K_{j\times 1}$ has no red C_4 or a blue B_2. Therefore, we obtain that $m_5(C_4,B_2) \geq 2$ and $m_6(C_4,B_2) \geq 2$. Next we have to show $m_5(C_4,B_2) \leq 2$. For this consider any coloring consisting of (red, blue) given by $K_{5\times 2} = H_R \oplus H_B$, such that H_R contains no red C_4 and H_B contains no red B_2. Then since $m_5(C_4,C_3) = 2$, without loss of generality we may assume that $(v_{1,1},v_{2,1},v_{3,1})$ is a blue cycle. Define$T = \{v_{4,1},v_{4,2},v_{5,1},v_{5,2}\}$ and $S = \{v_{1,1},v_{2,1},v_{3,1}\}$. Then, if any vertex of T is adjacent to two vertices of S in blue, it will result blue B_2, contrary to our assumption. Therefore, we will be left with the option every vertex of T is adjacent to at least two vertices of S in red. But then as $|S| = 3$ there will be twovertices of T adjacent in red to the same pair of vertices in S. This will result in a red C_4, a contradiction. From this we can conclude that $m_j(C_4,B_2) \leq 2$ if $j \in \{5,6\}$. That is, $m_j(C_4,B_2) = 2$ if $j \in \{5,6\}$.We are left with the following two cases, namely j = 4(case 1) and j = 3(case 2).

Case 1: j = 4
Consider the coloring of $K_{4\times 2} = H_R \oplus H_B$, generated by H_R illustrated in the following figure.

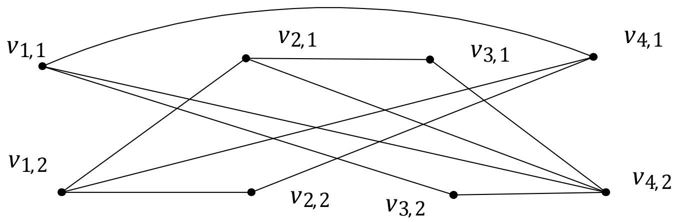

Figure 3: Case 1

Then H_R will contain three disjoint triangles except for two triangles containing a common vertex. Thus the red-blue coloring generated by the figure will be such that H_R contains no red C_4 and H_B contains no blue B_2. Therefore, we will get $m_4(C_4,B_2) \geq 3$. To show that $m_3(C_4,B_2) \leq 3$ consider any coloring consisting of (red , blue) given by $K_{3\times 4} = H_R \oplus H_B$, such that H_R contains no red C_4 and H_B contains no blue B_2. Since $m_4(C_4,C_3) = 2$ without loss of generality, we get that, $v_{1,1}v_{2,1}v_{3,1}v_{1,1}$ is a blue cycle. Define $S = \{v_{1,1},v_{2,1},v_{3,1}\}$ and $T = \{v_{4,1},v_{4,2},v_{4,3}\}$. If any vertex of T has adjacent in blue to 2 vertices of S we will get a blue B_2 and if any two vertex of T has adjacent in red to the same 2 vertices of S we will get a red C_4. Therefore, we may assume that $v_{4,1}$ is adjacent in blue to $v_{1,1}$ and in red to $v_{2,1}$ and $v_{3,1}$; $v_{4,2}$ is adjacent in blue to $v_{2,1}$ and in red to $v_{1,1}$ and $v_{3,1}$; $v_{4,3}$ is adjacent in blue to $v_{3,1}$ and in red to $v_{1,1}$ and $v_{2,1}$.

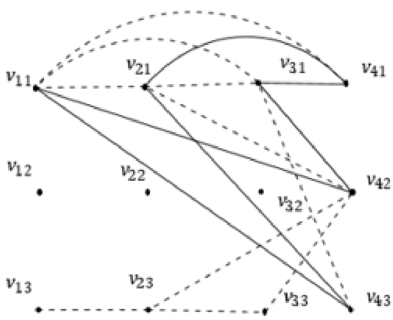

Figure 4: The first scenario

Also in the remaining 6 vertices (in $S^c \cap T^c$) must contain a blue P_3 as $m_3(C_4,P_3) = 2$. Thus, without loss of generality we may assume that $v_{1,3}v_{2,3}v_{3,3}$ is a blue P_3. In order to avoid a red C_4, all three vertices of $\{v_{1,3},v_{2,3},v_{3,3}\}$ must be adjacent in blue to at least two vertices of T. Thus, without loss of generality this gives rise to two possible scenarios illustrated in Figure 5 and Figure 6 respectively.

In the first scenario, in order to avoid a blue B_2 both $(v_{1,3},v_{4,2})$ and $(v_{1,3},v_{3,3})$ must be a red edges. Then in order to avoid a red C_4, $(v_{1,1},v_{3,3})$ must be a blue edge; in order to avoid a blue B_2, $(v_{3,3},v_{2,1})$ and $(v_{1,1},v_{2,3})$ must be red edges; in order to avoid a red C_4,$(v_{3,1},v_{2,3})$, $(v_{1,3},v_{4,3})$ and $(v_{1,3},v_{4,1})$ must be a blue edges.But then in order to avoid a blue B_2, $(v_{2,3},v_{4,3})$ must be red. In order to avoid a red C_4, $(v_{2,3},v_{4,1})$ must be a blue edge.

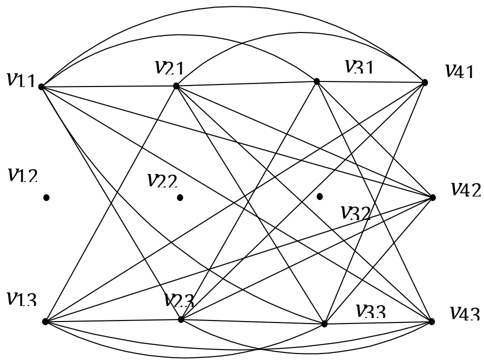

Figure 5: The final graph generated by the first scenario

In order to avoid a blue B_2, $(v_{3,3},v_{4,1})$ must be a red edge and in order to avoid a red C_4, $(v_{1,3},v_{2,1})$ must be a blue edge. The red/blue graph generated is illustrated in the above figure.

But then if $(v_{1,3},v_{3,1})$ is red we get a red C_4 and if is blue we get a blue we get a B_2, a contradiction.

In the second scenario, we may assume that $v_{4,1}$ is adjacent in blue to $v_{2,3}$ and $v_{3,3}$; $v_{4,2}$ is adjacent in blue to $v_{1,3}$ and $v_{3,3}$; $v_{4,3}$ is adjacent in blue to $v_{1,3}$ and $v_{3,3}$. Next in order to avoid a blue B_2, $(v_{3,3},v_{4,3})$, $(v_{2,3},v_{4,2})$, $(v_{1,3},v_{4,1})$ and $(v_{2,3},v_{3,1})$ must be a red edges.Then in order to avoid a red C_4, $(v_{1,1},v_{2,3})$ must be a blue edge; in order to avoid a blue B_2, $(v_{3,3},v_{1,1})$ must be a red edge.In order to avoid a red C_4, $(v_{2,1},v_{3,3})$ must be a blue edge; in order to avoid a blue B_2, $(v_{1,3},v_{2,1})$ must be a red edge; and in order to avoid a red C_4, $(v_{1,3},v_{3,1})$ must be a blue edge. But then the vertices in $\{v_{1,3},v_{3,1},v_{4,3},v_{2,3}\}$ will induce a blue B_2, a contradiction.

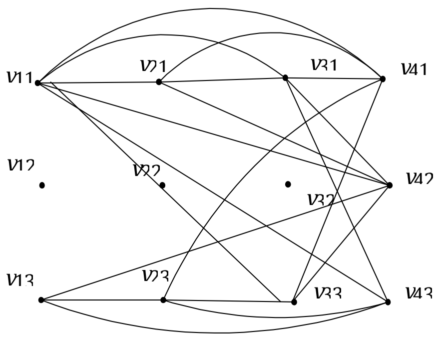

Figure 6: The second scenario

Case 2: j = 3

Consider the coloring of $K_{3\times 4} = H_R \oplus H_B$, generated by H_R and H_B illustrated in the following figure. The red blue coloring generated by the following figure will be such that H_R contains no red C_4 and H_B contains no red B_2. Therefore, we will get $m_3(C_4,B_2) \geq 4$.

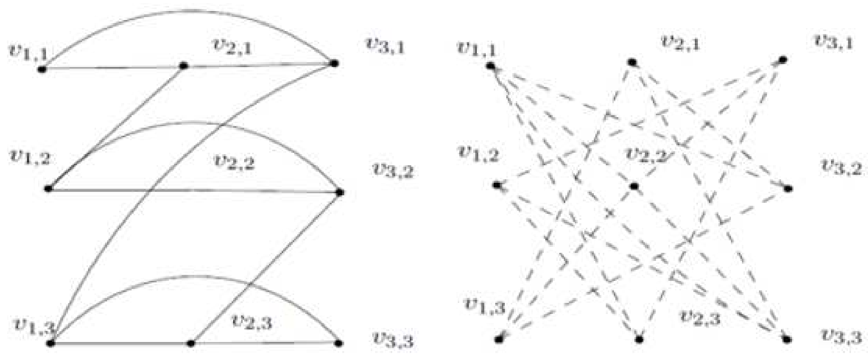

Figure 7: Case 2

To show that $m_3(C_4,B_2) \leq 4$, consider any coloring consisting of (red,blue) given by $K_{3\times4} = H_R \oplus H_B$ such that there is no red C_4 in H_R or a red B_2 in H_B. Subject to these conditions will first show the following three claims.

Notation: Let $1 \leq i, j \leq 4$. A vertex $v \in K_{3\times4}$ having blue degree $i + j$ is said to consists of a blue (i, j) split if v is adjacent in blue to i vertices of one particle set and j vertices of the other partite set.

Claim 1: All vertices of $K_{3\times4}$ have blue degree at most five.
Proof. Suppose $v_{1,4}$ has blue degree at least 6. Without loss of generality, there are two possibilities. Then one of the following two scenarios must be true. The first $v_{1,4}$ is adjacent in blue to all vertices of $S = \{v_{2,1},v_{2,2},v_{2,3},v_{2,4},v_{3,3},v_{3,4}\}$ and the second $v_{1,1}$ is adjacent in blue to all vertices of $S = \{v_{2,1},v_{2,2},v_{2,3},v_{3,1},v_{3,2},v_{3,3}\}$.

In the first scenario to avoid a blue B_2, both $v_{3,3}$ and $v_{3,4}$ will have to be adjacent in red to at least three vertices of the second partite set. This will result in a red C_4 containing $v_{3,3}$ and $v_{3,4}$, a contradiction. Hence the claim follows.

In the second scenario, In order to avoid a blue B_2 all edges between $\{v_{2,1},v_{2,2},v_{2,3}\}$ and $\{v_{3,1},v_{3,2},v_{3,3}\}$. Next, applying $m_3(C_4,C_3) = 3$ to $K_{3\times3}$ consisting of the first three elements of the three partite sets, we obtain a blue B_2 containing $v_{1,4}$. Hence the claim follows.

Claim 2: $K_{3\times4}$ has at least one vertex of blue degree five.
Proof. Applying $m_3(C_4,C_3) = 3$ to $K_{3\times3}$ consisting of the first three elements of the three partite sets, without loss of generality we obtain a blue C_3 containing $S = \{v_{1,1},v_{2,1},v_{3,1}\}$. Next, as there is no blue B_2, each vertex outside S will have to be adjacent in red to at least one vertex of S. Thus by pigeon-hole principle at least one vertex must have degree greater than 5. Thus by Claim 1, we can conclude that S has at least one vertex of blue degree five as required.

Claim 3: $K_{3\times4}$ has at least one vertex of blue (3,2) split.
Proof. Suppose that the claim is false. Let $v_{1,4}$ be a vertex having a blue (4,1) split. In particular, suppose that $v_{1,4}$ is adjacent in blue to all vertices of

$S = \{v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}, v_{3,4}\}$ and adjacent in red to all vertices of $T = \{v_{3,1}, v_{3,2}, v_{3,3}\}$. Since there is no blue B_2 without loss of generality we may assume that $v_{3,4}$ adjacent in red to all vertices of $T = \{v_{2,1}, v_{2,2}, v_{2,3}\}$. Next we deal with 2 possible cases.

Case a: $v_{3,4}$ is adjacent in blue to at least one vertex of $\{v_{1,1}, v_{1,2}, v_{1,3}\}$ (say $v_{1,3}$). Then, since there is no red C_4 and by Claim 1, we would get that $v_{1,3}$ will be a blue (3,2) split.

Case b: $v_{3,4}$ is adjacent in red to all three vertices of $\{v_{1,1}, v_{1,2}, v_{1,3}\}$. Then, since there is no red C_4 and by Claim 1, we would get that $v_{2,4}$ will be a blue (3,2) split.

Now let us try to complete the proof of $j = 3$ case. According to lemma 3, $v_{1,4}$ be a vertex having a blue (3,2) split.
In particular, suppose that $v_{1,4}$ is adjacent in blue to all vertices of $S = \{v_{2,2}, v_{2,3}, v_{2,4}, v_{3,3}, v_{3,4}\}$ and adjacent in red to all vertices of $T = \{v_{2,1}, v_{3,1}, v_{3,2}\}$.

Since there is no blue B_2 or a red C_4,without loss of generality we may assume that $v_{3,4}$ adjacent in red to $v_{2,3}, v_{2,4}$ and that $v_{3,3}$ adjacent in red to $v_{2,2}, v_{2,3}$. Next as there is no red C_4, we would get that $(v_{2,4}, v_{3,3})$ and $(v_{2,2}, v_{3,4})$ are blue edges. This is illustrated in the following figure.

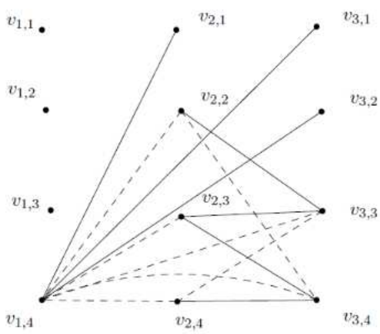

Figure 8: The generated graph for both cases 1 and 2, when $j = 3$

Next we get that as in claim 3, there are two possible cases to consider.

Case 2.1: $(v_{2,3}, v_{1,3})$ is red. First in order to avoid a red C_4 both $(v_{1,3}, v_{2,2})$ and $(v_{1,3}, v_{2,4})$ will have to be blue. Next in order to avoid a blue B_2 both $(v_{1,3}, v_{3,3})$ and $(v_{1,3}, v_{3,4})$ will have to be red. But this would give us $v_{1,3}v_{3,4}v_{2,3}v_{3,3}v_{1,3}$ is a red C_4.

Case 2.2: $v_{2,3}$ is adjacent in blue to all three vertices of $\{v_{1,1}, v_{1,2}, v_{1,3}\}$. Next, as there is no red C_4, $v_{2,3}$ will be adjacent to at least one vertex of $\{v_{3,1}, v_{3,2}\}$ in blue. Without loss of generality assume $(v_{2,3}, v_{3,2})$is blue. Then, by Claim 1, $(v_{2,3}, v_{3,1})$ will be red.Next, as there is no red C_4, $(v_{2,2}, v_{3,1})$ and $(v_{2,4}, v_{3,1})$ will have to be blue.

In order to avoid a blue B_2 the vertex $v_{3,2}$ must be adjacent to two vertices of $\{v_{1,1}, v_{1,2}, v_{1,3}\}$ in red. Without loss of generality assume that $(v_{3,2}, v_{1,2})$ and $(v_{3,2}, v_{1,3})$ are red. But then in order to avoid a red C_4, $(v_{3,1}, v_{1,2})$ and $(v_{3,1}, v_{1,3})$ are blue. Consider four vertices, $v_{3,1}$ is adjacent to in blue, given by $W = \{v_{1,2}, v_{1,3}, v_{2,2}, v_{2,4}\}$. In order to avoid a blue B_2 there can be at most one blue edge among them. That is there are three red edges in the subgraph induced by W. Exhaustive search will show that in each of the possibilities either $v_{1,2}v_{3,2}v_{1,3}v_{2,2}v_{1,2}$ or $v_{1,2}v_{3,2}v_{1,3}v_{2,4}\,v_{1,2}$will be a red C_4, a contradiction.

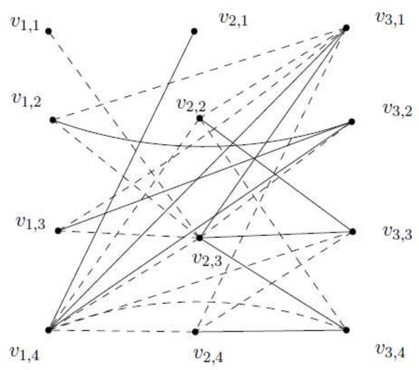

Figure 9: The final graph of case 2, when j = 3

4. Disconnected graphs up to 4 vertices

It is worth noting that if G and H are two graphs with at most 4 vertices satisfying $G = H \cup K_1$. Then clearly, $m_j(C_4,G) = m_j(C_4,H)$ for $j \geq 4$. In the case of j = 3, as $m_3(C_4,H) > 1$ for all connected graphs H up to 3 vertices we $m_3(C_4,G) = m_3(C_4,H)$. Therefore, by this remark we are left are left only to consider $m_j(C_4,2K_2)$. However, this follows directly from $m_j(C_4,P_4)$ as $m_j(C_4,2K_2) \leq m_j(C_4,P_4)$ for any integer j.

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