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Interval-valued Q-Fuzzy Ideals Generated by an Interval-valued Q-Fuzzy Subset in Ordered Semi-groups

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Abstract. This paper expose a study on interval-valued Q-fuzzy ideals generated by an interval-valued Q-fuzzy subset on ordered semi-groups. Some characterizations of such generated interval-valued Q-fuzzy ideals are also discussed.

Keywords: Ordered semi-group, interval-valued Q-fuzzy sub semi-group, interval-valued Q-fuzzy left(right, two-sided) ideal, interval-valued Q-fuzzy interior ideal, interval-valued Q-fuzzy bi-ideal.

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1. Introduction

The important concept of a fuzzy set introduced by Zadeh in 1965 (see [18]) has opened up keen insights and applications in wide range of scientific fields. Rosenfeld introduced the concept of fuzzy groups [15]. Among others, fuzzy semi-groups were introduced by Kuroki [10]. A theory of fuzzy sets on ordered semi-groups has been recently developed [4,5,6,8,9]. Fuzzy sets in ordered semi-groups were first studied by Kehayopulu and Tsingelis in [5], then they defined fuzzy analogies for several notations, which have proved useful in the theory of ordered semi-group. In [7], they have discussed fuzzy bi-ideals in ordered semi-groups and they discuss fuzzy interior ideals in ordered semi-group in [9]. Fuzzy semi-groups were generalized in two folds: fuzzy ordered semi-groups and fuzzy ternary semi-groups. Since ordered semi-groups are useful for computer science, especially in theory of automata and formal language, fuzzy ordered semi-group has been extensively studied (see [1,2,3,5,6]). Interval-valued fuzzy subsets were proposed thirty years ago as a natural extension of fuzzy sets by Zadeh [18]. In [18], Zadeh also constructed a method of approximate inference using his interval-valued fuzzy subsets. In [13], Narayanan and Manikantan introduced the notions of interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in semi-groups. Shabir and Khan [16] have studied about interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in ordered semi-groups. Thillaigovindan and Chinnadurai [17] initiated some study on Interval-valued fuzzy generalized bi-ideals. Some interesting studies are carried out in

[11,12,14] using interval-valued fuzzy set. This paper characterize the ordered semi-groups in terms of interval-valued Q-fuzzy left (right, interior and bi-)ideals.

2. Preliminaries

In [8], Kehayopulu and Tsingelis used the operation $\wedge$ on the unit interval [0,1] to define an operation '$\circ$' on $\mathcal{F}(S)$ as follows:

$$(f\circ g)(x) = \begin{cases} \bigvee_{y,z\in S_x} f(y)\wedge g(z) \ if\ S_x \neq \emptyset,\ for\ all\ x\in S, \\ \qquad 0 \qquad\qquad otherwise, \end{cases}$$

where $S_x = \{(y,z)\in S\times S : x \leq yz\}$

An ordered semi-group is an ordered set S at the same time a semi-group such that a $\leq b \Rightarrow ax \leq bx$ and $xa \leq xb$ for all $\forall\, a,b,x\in S.$.

A non-empty subset A of an ordered semi-group S is called a **sub semi-group** of S if $AA \subseteq A$.

A non-empty subset A of an ordered semi-group S is called a **right (resp. left) ideal** of S if

1. $AS \subseteq A$ (resp. $SA \subseteq A$)
2. $a\in A, S \ni b \leq a \Rightarrow b\in A$.

A is called an **ideal** of S if it is both a right and a left ideal of S.

A sub semi-group A of an ordered semi-group S is called an **interior ideal** of S if

1. $SAS \subseteq A$.
2. $a\in A, S \ni b \leq a \Rightarrow b\in A.$

A semi-group A of an ordered semi-group S is called a **bi-ideal** of S if

1. $ASA \subseteq A$.
2. $a\in A, S \ni b \leq a \Rightarrow b\in A$.

A fuzzy subset f of an ordered semi-group S is a function from S to the unit interval [0,1] [5]. Let S be an ordered semi-group. A fuzzy subset f of S is called a **fuzzy sub semi-group** of S if $f(xy) \geq min\{f(x), f(y)\}$ $\forall\, x,y\in S$.

A fuzzy subset f of an ordered semi-group S is called a fuzzy left (resp. right) ideal of S if

1) $x \leq y \Rightarrow f(x) \geq f(y)$
2) $f(xy) \geq f(y)$ $(resp. f(xy) \geq f(x))$.

If f is both a fuzzy left ideal and a fuzzy right ideal of S, then f is called a fuzzy ideal of S or a fuzzy two sided ideal of S. Equivalently, f is called a **fuzzy ideal** of S if

1) $x \leq y \Rightarrow f(x) \geq f(y)$
2) $f(xy) \geq \sup\{f(x), f(y)\}$.

A fuzzy sub semi-group f of an ordered semi-group S is called a **fuzzy interior ideal** of S if

1) $x \le y \Rightarrow f(x) \ge f(y)$
2) $f(xyz) \ge f(y)$.

A fuzzy sub semi-group f of an ordered semi-group S is called a **fuzzy bi-ideal** of S if

1) $x \le y \Rightarrow f(x) \ge f(y)$
2) $f(xyz) \ge min\{f(x), f(y)\}$.

3. Interval-valued Q-fuzzy ideals generated by an interval-valued Q-fuzzy subset in ordered semi-groups

We now give the interval-valued Q-fuzzy concepts.

An interval number on [0,1], say $\bar{a}$ is a closed sub-interval of [0,1], that is $\bar{a} = [a^-, a^+]$ where $0 \le a^- \le a^+ \le 1$. Let D[0,1] denote the family of all closed sub-intervals of [0,1], $\bar{0}$ =[0,0] and $\bar{1}$ =[1,1]. For any two elements $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$ in D[0,1]. We define

(i) $\bar{a} \le \bar{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$
(ii) $\bar{a} = \bar{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$,
(iii) $Min^i \ \{\bar{a}, \bar{b}\} = [min\{a^-, b^-\}, min\{a^+, b^+\}]$.
(iv) $Max^i \ \{\bar{a}, \bar{b}\} = [max\{a^-, b^-\}, max\{a^+, b^+\}]$.

Similarly we can define inf^i and sup^i in case of family of elements in $D[0,1]$.

A mapping $\bar{A}: X \times Q \to D[0,1]$ is called an interval-valued Q-fuzzy subset of X. where $\bar{A}(x,q) = [A^-(x,q), A^+(x,q)]$. $\forall x \in X$ and $q \in Q$.

Let S be an ordered semi-group with identity element 1 and IQF(S) denotes the set of all i-v Q-fuzzy subsets of S.

Definition 3.1. *Let* $\bar{A}, \bar{B}$ *be i-v Q-fuzzy subsets of* S. Then we have the following:

(i) $\bar{A} \le \bar{B}$ if and only if $\bar{A}(x,q) \le \bar{B}(x,q)$.
(ii) $\bar{A} = \bar{B}$ if and only if $\bar{A}(x,q) = \bar{B}(x,q)$.
(iii) $(\bar{A} \cup \bar{B})(x,q) = max^i\{\bar{A}(x,q), \bar{B}(x,q)\}$.
(iv) $(\bar{A} \cap \bar{B})(x,q) = min^i\{\bar{A}(x,q), \bar{B}(x,q)\}$, for all $x \in S$ and $q \in Q$.

Definition 3.2. *Let* "$\circ$" *be a binary composition in* S. *The product* $\bar{A} \circ \bar{B}$ *of any two i-v Q-fuzzy subsets* $\bar{A}, \bar{B}$ *of* S *is defined by*

$$(\bar{A} \circ \bar{B})(x,q) = \begin{cases} \vee_{x=ab}^{i} \left\{ Min^i\{\bar{A}(a,q), \bar{B}(b,q)\} \right\} & if\ x\ is\ expressible\ as\ x = ab \\ \bar{0} & otherwise \end{cases}$$

Since semi-group S is associative, The operation "$\circ$" is associative.

We denote xy instead of $x \cdot y$ and $\bar{A}\bar{B}$ for $\bar{A} \circ \bar{B}$. Let B be a subset of a set S.

Define a function (characteristic function) $\overline{\chi}_B : S\times Q \to$ D[0,1] by

$$\overline{\chi}_B(x,q) = \begin{cases} \overline{1}\ if\ x \in B & \forall x \in S\ and\ q \in Q. \\ \overline{0} & otherwise \end{cases}$$

clearly $\overline{\chi}_B$ is an $i-v$ Q-fuzzy subset of S. Throughout this paper $\overline{\chi}_s$ is denoted by $\overline{S}$ and S will denote a semi-group unless mentioned.

Definition 3.3. *Let A be a Q-fuzzy subset of an ordered semi-group S with identity element 1. Then the smallest Q-fuzzy left (right, two sided, interior, bi-)ideal of S containing A is called a Q-fuzzy left (right, twosided,interior, bi-)ideal of S generated by A denoted by $< A >_L$ $(< A >_R, < A >, < A >_I, < A >_B)$ respectively.*

Definition 3.4. *An i-v Q-fuzzy subset $\overline{A}$ of an ordered semi-group S is called an i-v Q-fuzzy sub semi-group of S if for all $x, y \in S$ and $q \in Q$,*

$$\overline{A}(xy,q) \geq Min^i\{\overline{A}(x,q), \overline{A}(y,q)\}.$$

Definition 3.5. *An i-v Q-fuzzy subset $\overline{A}$ of an ordered semi-group S is called an i-v Q-fuzzy left (resp. right) ideal of S if for all $x, y \in S$ and $q \in Q$*

i) $x \leq y \Rightarrow \overline{A}(x,q) \geq \overline{A}(y,q)$ and $q \in Q$

ii) $\overline{A}(xy,q) \geq \overline{A}(y,q)$ (resp. $\overline{A}(xy,q) \geq \overline{A}(x,q)$).

An i-v Q-fuzzy subset $\overline{A}$ in S is called an interval valued fuzzy two sided ideal of S if it is an i-v Q-fuzzy left ideal and an i-v Q-fuzzy right ideal of S.

Definition 3.6. *An i-v Q-fuzzy sub semi-group $\overline{A}$ of an ordered semi-group S is called an i-v Q-fuzzy interior ideal of S if for all $x, y, z \in S$ and $q \in Q$*

i) $x \leq y \Rightarrow \overline{A}(x,q) \geq \overline{A}(y,q)$

ii) $\overline{A}(xyz,q) \geq \overline{y}$.

Definition 3.7. *An i-v Q-fuzzy sub semi-group $\overline{A}$ of an ordered semi-group S is called an i-v Q-fuzzy bi-ideal of S if for all $x, y, z \in S$ and $q \in Q$*

i) $x \leq y \Rightarrow \overline{A}(x,q) \geq \overline{A}(y,q)$

ii) $\overline{A}(xyz,q) \geq Min^i\{\overline{A}(x,q), \overline{A}(z,q)\}$.

Theorem 3.8. *An interval-valued Q-fuzzy subset $\overline{A}$ of an ordered semi-group S is an i-v Q-fuzzy sub semi-group (left ideal, right ideal, two sided ideal, interior ideal, bi-ideal) of S iff A^- and A^+ are Q-fuzzy sub semi-group (left ideals, right ideals,interior ideals, bi-ideals) of S.*

Proof: Let $\overline{A}$ be an i-v Q-fuzzy sub semi-group of S.

Then for all $x, y \in S$, and $q \in Q$ $\overline{A}(xy,q) \geq Min^i\{\overline{A}(x,q), \overline{A}(y,q)\}$, where

$Min^i\{\overline{A}(x,q), \overline{A}(y,q)\} = [min\{A^-(x,q), A^-(y,q)\}, min\{A^+(x,q), A^+(y,q)\}]$.

Thus $\overline{A}(xy,q) \geq [min\{A^-(x,q), A^-(y,q)\}, min\{A^+(x,q), A^+(y,q)\}]$.

Hence $[A^-(xy,q), A^+(xy,q) \geq [min\{A^-(x,q), A^-(y,q)\}, min\{A^+(x,q), A^+(y,q)\}]$.

Thus $A^-(xy,q) \geq min\{A^-(x,q), A^-(y,q)\}$ and $A^+(xy,q) \geq min\{A^+(x,q), A^+(y,q)\}]$.

Hence A^-, A^+ are Q-fuzzy sub semi-groups of S. The converse is straightforward.

Now suppose $\overline{A}$ is an i-v Q-fuzzy left ideal of S.

Then for all $x, y \in S$, and $q \in Q$ $x \leq y \Rightarrow \overline{A}(x,q) \geq \overline{A}(y,q)$ and $\overline{A}(xy,q) \geq \overline{A}(y,q)$

Now $\overline{A}(x,q) \geq \overline{A}(y,q) \Rightarrow [A^-(x,q), A^+(x,q) \geq [A^-(y,q), A^+(y,q)]$.

That is $A^-(x,q) \geq A^-(y,q)$ and $A^+(x,q) \geq A^+(y,q)$.

$\overline{A}(xy,q) \geq \overline{A}(y,q) \Rightarrow [A^-(xy,q)], A^+(xy,q) \geq [A^-(y,q), A^+(y,q)]$.

Thus $A^-(xy,q) \geq A^-(y,q)$ and $A^+(xy,q) \geq A^+(y,q)$.

Hence, A^- and A^+ are Q-fuzzy left ideals of S.
The converse is straightforward. Similarly we can prove for other cases.

Definition 3.9. *Let $\overline{A}$ be an i-v Q-fuzzy set. Then the smallest i-v Q-fuzzy left (right, two sided) ideal of S containing $\overline{A}$ is called an* ***i-v Q-fuzzy left (right, two sided) ideal*** *of S generated by $\overline{A}$, denoted by $\langle\overline{A}\rangle_L (\langle\overline{A}_R\rangle, \langle\overline{A}\rangle)$ respectively.*

Theorem 3.10. *Let $\overline{A}$ be an i-v Q-fuzzy set. Then $\langle\overline{A}\rangle_L = \overline{J}$,*

where $\overline{J} = [J^-, J^+]$ such that

$J^-(x,q) = \sup\limits_{\substack{x \leq x_1x_2 \\ x_1, x_2 \in S}} A^-(x_2,q)$ and $J^+(x,q) = \sup\limits_{\substack{x \leq x_1x_2 \\ x_1, x_2 \in S}} A^+(x_2,q)$ for all $x \in S$ and $q \in Q$.

Proof: For all $a \in S$ and $q \in Q$,

$J^-(a,q) = \sup\limits_{a \leq x_1x_2} A^-(x_2,q) \geq A^-(a,q)$, since $a = 1.a \Rightarrow A^-(a,q) \leq J^-(a,q)$.

Similarly $A^+(a,q) \leq J^+(a,q) \Rightarrow \overline{A}(a,q) = [A^-(a,q), A^+(a,q)] \leq [J^-(a,q), J^+(a,q)]$.

Thus $\overline{A} \subseteq \overline{J}$. Now we show that $\overline{J}$ is an interval-valued Q-fuzzy left ideal of S, for this we have to show $(x,q) \leq (y,q) \Rightarrow \overline{J}(x,q) \geq \overline{J}(y,q)$ and $\overline{J}(xy,q) \geq \overline{J}(y,q)$ for all $x, y \in S$ and $q \in Q$. Let $x, y \in S$, such that $x \leq y$. If $y \leq x_3x_4$, then $x \leq x_3x_4$.

Hence $J^-(y,q) = \sup\limits_{y \leq x_3x_4} A^-(x_4,q) \leq \sup\limits_{x \leq x_1x_2} A^-(x_2,q) = J^-(x,q)$.

Similarly $J^+(x,q) \geq J^+(y,q) \Rightarrow \overline{J}(x,q) \geq \overline{J}(y,q)$.

Now if $y \leq x_1x_2$, then $(xy,q) \leq ((xx_1)x_2, q)$.

Hence $J^-(y,q) = \sup_{y\le x_1x_2} A^-(x_2,q) \le \sup_{xy\le x_3x_4} A^-(x_4,q) = J^-(xy,q)$.

Similarly $J^+(xy,q) \ge J^+(y,q)$. Hence $\overline{J}(xy,q) \ge \overline{J}(y,q)$.

Let $\overline{B}$ be any i-v Q-fuzzy left ideal of S such that $\overline{B} \supseteq \overline{A}$.

Then for all $a \in S$ and $q \in Q$, $B^-(a,q) \ge A^-(a,q)$ and $B^+(a,q) \ge A^+(a,q)$. Now

$J^-(a,q) = \sup_{a\le a_1a_2} A^-(a_2,q) \le \sup_{a\le a_1a_2} B^-(a_2,q) \le \sup_{a\le a_1a_2} B^-(a_1a_2,q) \le B^-(a,q).$

Similarly, $B^+(a,q) \ge J^+(a,q)$ for all $a \in S$ and $q \in Q$ $\Rightarrow \overline{J} \subseteq \overline{B}$.

Hence, $\langle \overline{A} \rangle_L = \overline{J}$.

Theorem 3.11. *Let* $\overline{A}$ *be an i-v Q-fuzzy set, Then* $\langle \overline{A} \rangle_R = \overline{J}$,

where $\overline{J} = [J^-, J^+]$ such that

$J^-(x,q) = \sup_{\substack{x\le x_1x_2 \\ x_1,x_2\in S}} A^-(x_1,q)$ and $J^+(x,q) = \sup_{\substack{x\le x_1x_2 \\ x_1,x_2\in S}} A^+(x_1,q)$ for all $x \in S$.

Proof: The proof is similar to the proof of Theorem 3.10.

Theorem 3.12. *Let* $\overline{A}$ *be an i-v Q-fuzzy set. An interval-valued Q-fuzzy subset* $\overline{J}$ *is an i-v Q-fuzzy left (right) ideal of S generated by* $\overline{A}$ *if and only if* J^- *and* J^+ *are Q-fuzzy left (right) ideals of* S *generated by* A^- *and* A^+ *respectively.*

Proof: Suppose $\overline{J}$ is an i-v Q-fuzzy left ideal of S generated by $\overline{A}$.

Then by Theorem 3.8, J^+ and J^- are Q-fuzzy left ideals of S.

Since $\overline{A} \subseteq \overline{J}$, we have $A^- \subseteq J^-$ and $A^+ \subseteq J^+$. If B is a Q-fuzzy left ideal of S containing A^-, then define $\overline{B}: S \times Q \to$ D[0,1] by $\overline{B}(x,q) = [B^-(x,q), B^-(x,q)]$, where $B^-(x,q) = B(x,q)$ for all $x \in S$ and $B^+(x,q)$ =1 for all $x \in S$ and $q \in Q$.

Since B^- and B^+ are Q-fuzzy left ideals of S, by Theorem 3.8, $\overline{B}$ is an i-v Q-fuzzy left ideal of S. Clearly $\overline{A} \subseteq \overline{B}$ so $\overline{J} \subseteq \overline{B} \Rightarrow J^- \subseteq B^- = B \Rightarrow J^-$ is Q-fuzzy left ideal of S generated by A^-. Similarly we can show that J^+ is Q-fuzzy left ideal of S generated by A^+.

Conversely, assume that J^- and J^+ are Q-fuzzy left ideals of S generated by A^- and A^+ respectively.Then by Theorem 3.8, $\overline{J}$ is an i-v Q-fuzzy left ideals of S containing $\overline{A}$. If $\overline{B}$ is an i-v Q-fuzzy left ideal of S containing $\overline{A}$, then $A^- \subseteq B^-$ and $A^+ \subseteq B^+$ Since B^- and B^+ are Q-fuzzy left ideals of S, we have $J^- \subseteq B^-$ and $J^+ \subseteq B^+$.

Hence $\overline{J} \subseteq \overline{B}$. Thus $\overline{J}$ is an i-v Q-fuzzy left ideal of S generated by $\overline{A}$.

Theorem 3.13. *Let* $\overline{A}$ *be an i-v Q-fuzzy set, then* $\langle \langle \overline{A} \rangle_L \rangle_R = \langle \overline{A} \rangle = \langle \langle \overline{A} \rangle_R \rangle_L$.

Proof: By Theorem 3.11, $\langle\langle \overline{A}\rangle_L\rangle_R$ is an i-v Q-fuzzy right ideal of S.

Clearly, $\langle\langle \overline{A}\rangle_L\rangle_R = [\langle\langle A^-\rangle_L\rangle_R, \langle\langle A^+\rangle_L\rangle_R]$. For all $x, y \in S$, $q \in Q$

$\langle\langle A^-\rangle_L\rangle_R(xy,q) = \sup\limits_{xy \le a_1a_2} \langle A^-\rangle_L(a_1,q) = \sup\limits_{xy\le a_1a_2}\sup\limits_{a_1\le z_1z_2} A^-(z_2,q)$ and

$\langle\langle A^-\rangle_L\rangle_R(xy,q) = \sup\limits_{y \le y_1y_2} \langle A^-\rangle_L(y_1,q) = \sup\limits_{y\le y_1y_2}\sup\limits_{y_1\le w_1w_2} A^-(w_2,q)$

Obviously, $\langle\langle A^-\rangle_L\rangle_R(xy,q) \ge \langle\langle A^-\rangle_L\rangle_R(y,q)$.

Similarly we have $\langle\langle A^+\rangle_L\rangle_R(xy,q) \ge \langle\langle A^+\rangle_L\rangle_R(y,q)$.

It follows that,

$$\begin{aligned}\langle\langle \overline{A}\rangle_L\rangle_R(xy,q) &= [\langle\langle A^-\rangle_L\rangle_R(xy,q), \langle\langle A^+\rangle_L\rangle_R(xy,q)]\\ &\ge [\langle\langle A^-\rangle_L\rangle_R(y,q), \langle\langle A^+\rangle_L\rangle_R(y,q)] = \langle\langle \overline{A}\rangle_L\rangle_R(y,q)\end{aligned}$$

Hence $\langle\langle A^-\rangle_L\rangle$ is an i-v Q-fuzzy left ideal of S.

So $\langle\langle A^-\rangle_L\rangle_R$ is an i-v Q-fuzzy ideal of S.

Since $\overline{A} \subseteq \langle \overline{A}\rangle_L \subseteq \langle\langle \overline{A}\rangle_L\rangle_R$, we have $\langle\langle \overline{A}\rangle_L\rangle_R \supseteq \overline{A}$.

Suppose $\overline{B}$ is any i-v Q-fuzzy ideal of S such that $\overline{B} \supseteq \overline{A}$.

Since $\langle \overline{A}\rangle_L$ is the smallest i-v Q-fuzzy left ideal of S containing $\overline{A}$,

we have $\overline{B} \supseteq \langle \overline{A}\rangle_L$. Also $\overline{B} \supseteq \langle\langle A^-\rangle_L\rangle_R$,

since $\langle\langle \overline{A}\rangle_L\rangle_R$ is the smallest i-v Q-fuzzy left ideal of S containing $\langle \overline{A}\rangle_L$.

This show that $\langle\langle \overline{A}\rangle_L\rangle_R$ is a smallest i-v Q-fuzzy left ideal of S containing $\overline{A}$.

Therefore $\langle\langle \overline{A}\rangle_L\rangle_R = \langle \overline{A}\rangle$.

Similarly we can prove that $\langle\langle \overline{A}\rangle_R\rangle_L = \langle \overline{A}\rangle$.

Hence $\langle\langle \overline{A}\rangle_L\rangle_R = \langle \overline{A}\rangle = \langle\langle \overline{A}\rangle_R\rangle_L$.

Definition 3.14. *Let $\overline{A}$ be an i-v Q-fuzzy set. Then the smallest i-v Q-fuzzy interior ideal of S containing $\overline{A}$ is called an **i-v Q-fuzzy interior ideal** of S generated by $\overline{A}$, denoted by $\langle \overline{A}\rangle_I$.*

Theorem 3.15. *Let $\overline{A}$ be an i-v Q-fuzzy set, then $\langle \overline{A}\rangle_I = \overline{J}$, where*

$\overline{J} = [J^-, J^+]$ such that $J^-(x,q) = \sup\limits_{\substack{x\le x_1x_2x_3\\ x_1,x_2,x_3\in S}} A^-(x_2,q)$

$J^+(x,q) = \sup\limits_{\substack{x\le x_1x_2x_3\\ x_1,x_2,x_3\in S}} A^+(x_2,q)$ for all $x \in S$, $q \in Q$.

Proof: For all $a \in S$ and $q \in Q$, we have

$J^-(a,q) = \sup_{a \leq x_1 x_2 x_3} A^-(x_2, q) \geq A^-(a,q)$ because $a \leq 1a1$.

Similarly $J^+(a,q) \geq A^+(a,q)$.

Therefore $\overline{J}(a,q) = [J^-(a,q), J^+(a,q)] \geq [A^-(a,q), A^+(a,q)] = \overline{A}(a,q)$.

Let $x, y \in S$, and $q \in Q$ such that $(x,q) \leq (y,q)$.

If $(y,q) \leq (x_1 x_2 x_3, q)$ then $(x,q) \leq (x_1 x_2 x_3, q)$.

Hence $J^-(y,q) = \sup_{y \leq x_1 x_2 x_3} A^-(x_2, q) \leq \sup_{x \leq x_4 x_5 x_6} A^-(x_5, q) = J^-(x,q)$.

Similarly $J^+(y,q) \leq J^+(x,q)$.

Hence $\overline{J}(x,q) = [J^-(x,q), J^+(x,q)] \geq [J^-(y,q), J^+(y,q)] = \overline{J}(y,q)$.

Also for all $x, y, z \in S$ and $q \in Q$,

if $(y,q) \leq (a_1 a_2 a_3, q)$ then $(xyz, q) \leq \left((xa_1)a_2(a_3 z), q\right)$.

Hence $J^-(y,q) = \sup_{y \leq a_1 a_2 a_3} A^-(a_2) \leq \sup_{xyz \leq b_1 b_2 b_3} A^-(b_2, q) = J^-(xyz, q)$.

Similarly $J^+(y,q) \geq J^+(xyz,q)$.

Thus $\overline{J}(y,q) = [J^-(y,q), J^+(y,q)] \leq [J^-(xyz,q), J^+(xyz,q)] = \overline{J}(xyz,q)$.

This shows that $\overline{J}$ is an i-v Q-fuzzy interior ideal of S such that $\overline{B} \supseteq \overline{A}$.

Then for all $a \in S$ and $q \in Q$,

$$J^-(a,q) = \sup_{a \leq a_1 a_2 a_3} A^-(a_2, q) \leq \sup_{a \leq a_1 a_2 a_3} B^-(a_2, q)$$

$$\leq \sup_{a \leq a_1 a_2 a_3} B^-(a_1 a_2 a_3, q) \leq B^-(a,q).$$

Similarly $J^+(a,q) \leq B^+(a,q)$.

Hence $\overline{J}(a,q) \leq \overline{B}(a,q)$.

This shows that $\overline{J}$ is the smallest i-v Q-fuzzy interior ideal of S containing $\overline{A}$, that is $\langle \overline{A} \rangle_I = \overline{J}$.

Theorem 3.16. *Let $\overline{A}$ be an i-v Q-fuzzy set. An interval-valued Q-fuzzy subset $\overline{J}$ is an i-v Q-fuzzy interior ideal of S generated by $\overline{A}$ iff J^- and J^+ are fuzzy interior ideals of S generated by A^- and A^+ respectively.*

Proof: The proof is similar to the proof of Theorem 3.12.

Definition 3.17. *An i-v Q-fuzzy sub semi-group $\overline{A}$ of S is called an i-v Q-fuzzy submonoid of S if $\overline{A}(1,q) \geq \overline{A}(x,q)$ for all $x \in S$ and $q \in Q$.*

Theorem 3.18. *Let $\overline{A}$ be an i-v Q-fuzzy submonoid of S then $\langle \overline{A} \rangle_B = \overline{J}$ where*

$J^{-}(x,q) \ = \sup\limits_{x\le x_1x_2x_3} min\{A^{-}(x_1,q), A^{-}(x_3,q)\}$ and

$J^{+}(x,q) \ = \sup\limits_{x\le x_1x_2x_3} min\{A^{+}(x_1,q), A^{+}(x_3,q)\}$ for all $x\in S$ and $q\in Q$.

Proof: For all $x\in S$ and $q\in Q$

$$\begin{aligned} J^{-}(x,q) &= \sup_{x\le x_1x_2x_3} min\{A^{-}(x_1,q), A^{-}(x_3,q)\} \\ &\ge min\{A^{-}(1), A^{-}(x,q)\} = A^{-}(x,q) \end{aligned}$$

because $x\le 1.1.x$ and $\overline{A}(1,q)\ge \overline{A}(x,q)$ for all $x\in S$.

Similarly, we have $J^{+}(x,q)\ge A^{+}(x,q)$.

Therefore $\overline{J}(x,q)\ge \overline{A}(x,q)$ and so $\overline{J}\supseteq \overline{A}$.

Let $x,y\in S$ such that $x\le y$.

If $y\le x_1x_2x_3$ then $x\le x_1x_2x_3$. Hence

$$\begin{aligned} J^{-}(y,q) &= \sup_{y\le x_1x_2x_3} min\{A^{-}(x_1,q), A^{-}(x_3,q)\} \\ &\le \sup_{x\le a_1a_2a_3} min\{A^{-}(a_1,q), A^{-}(a_3,q)\} = J^{-}(x,q). \end{aligned}$$

Similarly, we have $J^{+}(x,q)\ge J^{+}(y,q)$.

Thus $\overline{J}(x,q)\ge \overline{J}(y,q)$.

Also for all $x,y,z\in S$.

If $(x,q)\le (x_1x_2x_3,q)$ and $(z,q)\le (z_1z_2z_3,q)$,

then $(xyz,q)\le ((x_1x_2x_3)y(z_1z_2z_3),q)$.

Hence

$$\begin{aligned} J^{-}(xyz,q) &= \sup_{xyz\le a_1a_2a_3} min\{A^{-}(a_1,q), A^{-}(a_3,q)\} \\ &\ge \sup_{\substack{xyz\le x_1(x_2x_3yz_1z_2)z_3 \\ x\le x_1x_2x_3,\, z\le z_1z_2z_3}} min\{A^{-}(x_1,q), A^{-}(z_3,q)\}. \end{aligned}$$

We can write $A^{-}(x_1,q)\ge min\{A^{-}(x_1), A^{-}(x_3,q)\}$,

$A^{-}(z_3,q)\ge min\{A^{-}(z_1,q), A^{-}(z_3,q)\}$.

It follows that

$$\begin{aligned} J^{-}(xyz,q) &\ge \sup_{\substack{xyz\le x_1(x_2x_3yz_1z_2)z_3 \\ x\le x_1x_2x_3,\, z\le z_1z_2z_3}} min\{min\{A^{-}(x_1,q), A^{-}(x_3,q)\}, min\{A^{-}(z_1,q), A^{-}(z_3,q)\}\} \\ &= min\{\sup_{x\le x_1x_2x_3} min\{A^{-}(x_1), A^{-}(x_3)\}, \sup_{z\le z_1z_2z_3} min\{A^{-}(z_1,q), A^{-}(z_3,q)\}\} \\ &= min\{J^{-}(x,q), J^{-}(z,q)\}. \end{aligned}$$

Similarly we have $J^{+}(xyz,q) \geq min\{J^{+}(x,q), J^{+}(z,q)\}$.
Therefore

$$\begin{aligned}\overline{J}(xyz,q) &= [J^{-}(xyz,q), J^{+}(xyz,q)] \\ &\geq [min\{J^{-}(x,q), J^{-}(z,q)\}, min\{J^{+}(x,q), J^{+}(z,q)\}] \\ &= Min^{i}\{\overline{J}(x,q), \overline{J}(z,q)\}\end{aligned}$$

and so, $\overline{J}(xyz,q) \geq Min^{i}\{\overline{J}(x,q), \overline{J}(z,q)\}$.
Taking y = 1, we have $\overline{J}(xz,q) \geq Min^{i}\{\overline{J}(x,q), \overline{J}(z,q)\}$.
This shows that $\overline{J}$ is an i-v Q-fuzzy bi-ideal of S.
Let $\overline{B}$ be an i-v Q-fuzzy bi-ideal of S such that $\overline{B} \supseteq \overline{A}$.

Then for all $a \in S$, we have

$$\begin{aligned}J^{-}(a,q) &= \sup_{a \leq a_1 a_2 a_3} min\left\{A^{-}(a_1,q), A^{-}(a_3,q)\right\} \\ &\leq \sup_{a \leq a_1 a_2 a_3} min\left\{B^{-}(a_1,q), B^{-}(a_3,q)\right\} \leq \sup_{a \leq a_1 a_2 a_3} B^{-}(a_1 a_2 a_3, q) \leq B^{-}(a,q).\end{aligned}$$

Similarly, we have $J^{+}(a,q) \leq B^{+}(a,q)$. Thus $\overline{J} \subseteq \overline{B}$.
Hence $\overline{J}$ is a smallest i-v Q-fuzzy bi-ideal of S containing $\overline{A}$.
That is $\langle \overline{A} \rangle_B = \overline{J}$.

Theorem 3.19. *Let $\overline{A}$ be an i-v Q-fuzzy submonoid of S. Then an i-v Q-fuzzy subset $\overline{J}$ is an i-v Q-fuzzy bi-ideal of S generated by $\overline{A}$ if and only if J^{-} and J^{+} are Q-fuzzy bi-ideals of S generated by A^{-} and A^{+} respectively.*
Proof: The proof is similar to the proof of Theorem 3.18.

Theorem 3.20. *Let S be regular ordered semi-group and $\overline{A}$ be an i-v Q-fuzzy set, then $\langle \overline{A} \rangle_B = \overline{J}$ where*

$$J^{-}(x,q) = \sup_{x \leq x_1 x_2 x_3} \min \{A^{-}(x_1,q), A^{-}(x_3,q)\} \text{ and}$$

$$J^{+}(x,q) = \sup_{x \leq x_1 x_2 x_3} \min \{A^{+}(x_1,q), A^{+}(x_3,q)\} \text{ for all } x \in S,\ q \in Q.$$

Proof: From the proof of Theorem 3.18, it enough to prove that $\overline{J} \supseteq \overline{A}$.
For all $x \in S$ and $q \in Q$, we have

$$\begin{aligned} J^{-}(x,q) &= \sup_{x \le x_1 x_2 x_3} min\{A^{-}(x_1,q), A^{-}(x_3,q)\} \\ &\ge \sup_{x \le xax} min\{A^{-}(x,q), A^{-}(x,q)\} \\ &= A^{-}(x,q) \end{aligned}$$

Similarly, we have $J^{+}(x,q) \ge A^{+}(x,q)$.

Therefore, $\overline{J}(x,q) \ge \overline{A}(x,q)$ and so $\overline{J} \supseteq \overline{A}$.

Remark 3.21. *Theorem 3.19 is also true for an i-v Q-fuzzy subset* $\overline{A}$ *of a regular ordered semi-group* S.

4. Conclusion

Interval-valued Q-fuzzy ideals generated by an interval-valued Q-fuzzy subset on ordered semi-groups has been studied and some characterization are establisted.

REFERENCES

1. Y.B.Jun, J.Meng and X.L.Xin, On ordered filters of implicative semi-groups, *Semi-group Forum*, 54 (1997) 75-82.
2. N.Kehayopulu, On prime, weakly prime ideals in ordered semi-groups, *Semi-group Forum*, 44(3) (1992) 341-346.
3. N.Kehayopulu and M.Tsingelis, On intra-regular ordered semi-groups, *Semi-group Forum*, 46 (1993) 271-278.
4. N.Kehayopulu and M.Tsingelis, A note on fuzzy sets in semi-groups, *Sci. Math*., 5(3) (1999) 411-413.
5. N.Kehayopulu and M.Tsingelis, Fuzzy sets in ordered groupoids. *Semi-group Forum*, 65(1) (2002) 128-132.
6. N.Kehayopulu and M.Tsingelis, The embedding of an ordered groupid into a poe-groupid in terms of fuzzy sets, *Information Sciences,* 152 (2003) 231-236.
7. N.Kehayopulu and M.Tsingelis, Fuzzy bi-ideals in ordered semi-groups, *Information Sciences*, 171 (2005) 13-28.
8. N.Kehayopulu and M.Tsingelis, Regular ordered semi-groups in terms of fuzzy sets, *Information Sciences,* 171 (2005) 13-28.
9. N.Kehayopulu and M.Tsingelis, Fuzzy interior ideals in ordered semi-groups, *Lovachevskli Journal of Mathematics*, 21 (2006) 65-71.
10. N.Kuroki, On fuzzy semi-group, *Information Sciences*, 53(3) (1991) 203-236.
11. M.Pal and H.Rashmanlou, Irregular Interval-valued Fuzzy graphs, *Annals of pure and Applied Mathematics*, 1(2) (2012) 186-191.
12. R.Nagarajan and K.Balamurugan, On interval-valued bi-cubic vague subgroups, *Annals of pure and Applied Mathematics,* 6(2) (2014) 125-132.
13. A.L.Narayanan and T.Manikantan, Interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in semi-groups, *J. App. Maths and Computing*, 20 (2006) 455-465.
14. P.Rajarajeswari and P.Dhanalakshmi, Interval-valued fuzzy soft matrix theory, *Annals of pure and Applied Mathematics*, 7(2) (2014) 61-72.

15. A.Rosenfeld, Fuzzy groups, *Journal of Mathematical Analysis and Applications*, 35(1971) 51-517
16. M.Shabir and Israr Ali Khan, Interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in ordered semi-groups, *Mathware and Soft Computing*, 15 (2008) 263-272.
17. N.Thillaigovindan and V.Chinnadurai, Interval-valued fuzzy generalized bi-ideals, Proceedings of the National Conference on Algebra, *Graph theory and their Applications*, Department of Mathematics, Manonmaniam Sundaranar University, Narosa, (2009) 85-98.
18. L.A.Zadeh, Fuzzy Sets, *Inform. and Control*, 8(3)(1965) 338-353.