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$m_j(P_4,G)$ for all Graphs G on 4 Vertices

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Abstract. Let $j \geq 3$. Given that $m_j(H,G)$ denotes the smallest positive integer s such that $K_{j\times s}\to(H,G)$. In this paper, we exhaustively find $m_j(P_4,G)$ for all 11 non-isomorphic graphs G on 4 vertices, out of which 6 graphs G are connected and the others are disconnected.

Keywords: Ramsey theory, Multipartite Ramsey numbers

AMS Mathematics Subject Classification (2010): 05C55, 05D10

1. Introduction

In this paper, we consider simple graphs containing no loops or parallel edges. We define the complete balance multipartite graph $K_{j\times s}$ consisting of j partite sets (where the m^{th} $(1 \leq m \leq j)$ partite set V_m consisting of the vertex set $\{v_{m,i} \mid 1 \leq i \leq s\}$) as a graph, in which, there is an edge between every pair of vertices belonging to different partite sets. That is

$$V(K_{j\times s}) = \bigcup_{1\leq m\leq j} \{v_{m,i} \mid 1 \leq i \leq s\} \text{ and}$$

$$E(K_{j\times s}) = \bigcup \{(v_{m,i}, v_{m',i'}) \mid 1 \leq i,i' \leq s,\ 1 \leq m,m' \leq j \text{ and } m \neq m'\}.$$

Let the graph P_i represent a path on i vertices and G be any graph on 4 vertices.
Given any two coloring (consisting of say red and blue colors) of the edges of a graph $K_{j\times s}$, we say that $K_{j\times s}\to(P_4,G)$, if there exists a red copy of P_4in $K_{j\times s}$ or a blue copy of Gin $K_{j\times s}$. The size Ramsey multipartite number $m_j(P_4,G)$ is defined as the smallest natural number t such that $K_{j\times t}\to(P_4,G)$ (see [1,3,4,5,6,7] for general cases of $m_j(H,G)$). In this paper, we exhaustively find $m_j(P_4,G)$ for all 11 non-isomorphic graphs G on 4 vertices. The summary of our findings is illustrated in Table 1.

The next section deals with finding the entries of Table 1. Clearly the rows corresponding to row 1, row 2, row 4, row 5 and row 7 follows from Syafrizal et al. (see [7]).

$m_j(P_4,G)$	$j=$ / Graph G	3	4	5	6	7	8	9	Greater than or equal to 10
Row 1	$4K_1$	2	1	1	1	1	1	1	1
Row 2	$P_2 U 2K_1$	2	1	1	1	1	1	1	1
Row 3	$2K_2$	2	2	1	1	1	1	1	1
Row 4	$P_3 U K_1$	2	1	1	1	1	1	1	1
Row 5	P_4	2	2	1	1	1	1	1	1
Row 6	$K_{1,3}$	3	2	1	1	1	1	1	1
Row 7	$C_3 U K_1$	3	2	2	1	1	1	1	1
Row 8	C_4	3	2	1	1	1	1	1	1
Row 9	$K_{1,3} + x$	3	2	2	2	1	1	1	1
Row 10	B_2	4	2	2	1	1	1	1	1
Row 11	K_4	∞	4	2	2	2	2	2	1

Table 1: Values of $m_j(P_4,G)$.

2. Size Ramsey numbers $m_j(P_4,G)$ when G is connected graph on 4 vertices

Lemma 2.1. *If $j \geq 3$, then*

$$m_j(P_3,C_4) = \begin{cases} 2 & j=3 \\ 1 & j\geq 4 \end{cases}$$

Proof: Since $r(P_3, C_4) = 4$ (see [2]),we obtain that $m_3(P_3,C_4) \geq 2$. Next consider any red/blue coloring given by $K_{3\times 2} = H_R \oplus H_B$, such that H_R contains no red P_3 and H_B contains no blue C_4. Then since there is no red P_3, we get $\delta(H_B) \geq 3$. But then by the degree condition $\delta(H_B) \geq 3$; $v_{1,1},v_{1,2}$ will have two common neighbors in H_B say x and y. Thus $v_{1,1}$, x, $v_{1,2}$, y, $v_{1,1}$ will be a blue C_4. i.e., $m_3(P_3,C_4) \geq 2$. Therefore, $m_3(P_3,C_4) = 2$.
For $j \geq 4$, since $r(P_3, C_4) = 4$ (see [2]), we get $m_j(P_3, C_4) = 1$.

Theorem 2.1. *If $j \geq 3$, then*

$$m_j(P_4,C_4) = \begin{cases} 3 & j=3 \\ 2 & j=4 \\ 1 & j\geq 5 \end{cases}$$

Proof: Since $r(P_4,C_4) = 5$ (see [2]), we obtain that $m_3(P_4,C_4) \geq 3$.
To show, $m_3(P_4,C_4) \leq 3$, consider any red/blue coloring given by $K_{3\times 3} = H_R \oplus H_B$, such that H_R contains no red P_4 and H_B contains no blue C_4. But as $m_3(P_3,C_3) = 2$ we get that there exists red P_3 with end points x and y. Let z and w be two points not in this red P_3 and not belonging to the partite sets x, y belong to. But then as H_R contains no red P_4, we will obtain that x, z, y, w, x is a blue C_4, a contradiction. Therefore $m_3(P_4,C_4) = 3$, as required.

Next consider the coloring of $K_{4\times1} = H_R \oplus H_B$, generated by $H_R = C_3$. Then, $K_{4\times1}$ has no red P_4 or a blue C_4. Therefore, we obtain that $m_4(P_4,C_4) \geq 2$. To show $m_4(P_4,C_4) \leq 2$, consider any red/blue coloring given by $K_{4\times2} = H_R \oplus H_B$, such that H_R contains no red P_4 and H_B contains no blue C_4. But as $r(P_3,C_4) = 4$ we get that there exists red P_3. Without loss of generality assume that this red path is given by $v_{1,1},v_{2,1},v_{3,1}$. But then as H_R contains no redP_4, we will obtain that $v_{1,1},v_{2,2},v_{3,1},v_{4,2},v_{1,1}$ is a blue C_4, a contradiction. Therefore $m_4(P_4,C_4) = 2$.

If $j \geq 5$, since $r(P_4,C_4) = 5$ (see [2]), we get $m_j(P_4,C_4) = 1$.

Theorem 2.2. *If* $j \geq 3$, *then*

$$m_j(P_4,K_{1,3}) = \begin{cases} 3 & j=3 \\ 2 & j=4 \\ 1 & j\geq 5 \end{cases}$$

Proof: Let $j=3$. Consider the coloring of $K_{3\times2} = H_R \oplus H_B$, generated by $H_R = 2K_3$. Then, $K_{3\times2}$ has no red P_4 or a blue $K_{1,3}$. Therefore, $m_3(P_4,K_{1,3}) \geq 3$.

To show $m_3(P_4,K_{1,3}) \leq 3$, consider any red/blue coloring given by $K_{3\times3} = H_R \oplus H_B$, such that H_R contains no red P_4 and H_B contains no blue $K_{1,3}$. As H_B contains no blue $K_{1,3}$ both $v_{1,1}$, $v_{1,2}$ will satisfy $\deg_R(v_{1,1}) \geq 4$ and $\deg_R(v_{1,2}) \geq 4$. Therefore, this will force $v_{1,1}$ and $v_{1,2}$ to have common red neighbors say x and y. Then we get that $v_{1,1}$, x, $v_{1,2}$, y is a red P_4, a contradiction.

That is, $m_3(P_4,K_{1,3}) \leq 3$. Therefore, $m_3(P_4,K_{1,3}) = 3$.

Since $r(P_4,K_{1,3}) = 5$ (see [2]), we obtain that, $m_4(P_4,K_{1,3}) \geq 2$.

To show $m_4(P_4,K_{1,3}) \leq 2$, consider any red/blue coloring given by $K_{3\times2} = H_R \oplus H_B$, such that H_R contains no red P_4 and H_B contains no blue $K_{1,3}$. By [7], as we get that H_R contains a red P_3. This gives rise to two possibilities, namely $v_{1,1}$ is adjacent to $v_{2,1}$, $v_{2,2}$ in red or $v_{1,1}$is adjacent to $v_{2,1}$, $v_{3,1}$. But then in both cases as $v_{2,1}$ cannot be a root of a blue $K_{1,3}$, we would get a red P_4.

Clearly, $m_j(P_4,K_{1,3}) = 1$ when $j \geq 5$ as $r(P_4,K_{1,3}) = 5$ (see [2]).

Theorem 2.3. *If* $j \geq 3$, *then*

$$m_j(P_4,K_{1,3}+e) = \begin{cases} 3 & \text{if } j=3 \\ 2 & \text{if } j\in\{4,5,6\} \\ 1 & \text{if } j\geq 7 \end{cases}$$

Proof: Consider the coloring of $K_{3\times2} = H_R \oplus H_B$, generated by $H_R = 2K_3$. Then, $K_{3\times2}$ has no red P_4 or a blue $K_{1,3}$+e. Therefore, $m_3(P_4,K_{1,3}+e) \geq 3$.

To show $m_3(P_4,K_{1,3}+e) \leq 3$, consider any red/blue coloring given by $K_{3\times3} = H_R \oplus H_B$, such that H_R contains no red P_4 and H_B contains no blue $K_{1,3}$+e. By [4] as, $m_3(P_4,C_3) = 3$, we get that H_B, contains aC_3 say without loss of generality induced by $v_{1,1}$,$v_{2,1}$,$v_{3,1}$. But then this gives rise to two possible scenarios, namely, one vertex of $\{v_{1,1}, v_{2,1}, v_{3,1}\}$ is adjacent to a vertex of $\{v_{1,2}, v_{1,3}, v_{2,2}, v_{2,3}, v_{3,2}, v_{3,3}\}$ in blue and the other scenario where no vertices of $\{v_{1,1}, v_{2,1}, v_{3,1}\}$ are adjacent to any vertices of $\{v_{1,2}, v_{1,3}, v_{2,2}, v_{2,3}, v_{3,2}, v_{3,3}\}$ in blue. The first scenario clearly gives a blue $K_{1,3}+e$. The second scenario forces a red P_4, consisting of $v_{1,2}, v_{2,1}, v_{1,3}, v_{3,1}$. Hence we get $m_3(P_4, K_{1,3}+e) \le 3$, and thus can conclude that $m_3(P_4, K_{1,3}+e) = 3$.

Since $r(P_4, K_{1,3}+e) = 7$ (see [2]) we obtain that, $m_6(P_4, K_{1,3}+e) \ge 2$.

Next to show $m_4(P_4, K_{1,3}+e) \le 2$, consider any coloring of $K_{4\times 2} = H_R \oplus H_B$, such that H_R contains no red P_4 and H_B contains no blue $K_{1,3}+e$. By [4], as $m_4(P_4, C_3) = 2$, we get that H_B contains a C_3. Without loss of generality assume that this blue C_3 induced by $v_{1,1}$, $v_{2,1}$, $v_{3,1}$. But then as this C_3cannot be extended to a blue $K_{1,3}+e$, all edges given by $(v_{1,1}, v_{2,2})$, $(v_{3,1}, v_{2,2})$ and $(v_{1,1}, v_{3,2})$ will have to be red. This gives $v_{3,2}, v_{1,1}, v_{2,2}, v_{3,1}$ is a red P_4, a contradiction. Hence, $m_4(P_4, K_{1,3}+e) \le 2$. That is, $m_4(P_4, K_{1,3}+e) = 2$.

That is, we get that

$$2 \le m_6(P_4, K_{1,3}+e) \le m_5(P_4, K_{1,3}+e) \le m_4(P_4, K_{1,3}+e) \le 2.$$

Therefore, we can conclude that $m_j(P_4, K_{1,3}+e) = 2$ if $j = \{4, 5, 6\}$.

Clearly, $m_j(P_4, K_{1,3}+e) = 1$ when $j \ge 7$, as $r(P_4, K_{1,3}+e) = 7$ (see [2]).

Since all values of $m_j(P_4, B_2)$ are known (see [3]), we are left with finding $m_j(P_4, K_4)$. This case is considered in the following theorem.

Theorem 2.4. *If $j \ge 3$, then*

$$m_j(P_4, K_4) = \begin{cases} \infty & \text{if } j = 3 \\ 4 & \text{if } j = 4 \\ 2 & \text{if } j \in \{5,6,7,8,9\} \\ 1 & \text{if } j \ge 10 \end{cases}$$

Proof: Let t be an arbitrary integer. Consider the coloring of $K_{3\times t} = H_R \oplus H_B$, generated by $H_B = K_{3\times t}$. Then, $K_{3\times t}$ has no red P_4 or a blue K_4. Hence, $m_3(P_4, K_4) > t$, for any integer t. Therefore, we can conclude that $m_3(P_4, K_4) = \infty$.

For $j = 4$ case, consider the coloring of $K_{4\times 3} = H_R \oplus H_B$, generated by H_R illustrated in the following graph. Then, $K_{4\times 3}$ has no red P_4 or a blue K_4. Therefore, $m_4(P_4, K_4) \ge 4$.

Next, we need to show that $m_4(P_4, K_4) \le 4$. Consider any red/blue coloring given by $K_{4\times 4} = H_R \oplus H_B$, such that H_R contains no red P_4 and H_B contains no blue K_4. By [4] as, $m_3(P_4, C_3) = 3$, we get that H_B, contains aC_3 say without loss of generality induced by $S = \{v_{2,1}, v_{3,1}, v_{4,1}\}$. Next as each of the four vertices $v_{1,1}$, $v_{1,2}, v_{1,3}$, $v_{1,4}$ does not induce a blue K_4 with S, by pigeon hole principle without loss of generality we may assume that $(v_{1,1}, v_{2,1})$ and $(v_{1,2}, v_{2,1})$are both red edges. Next applying $m_3(P_4, C_3) = 3$, (see [4]) to

$\bigcup_{2\leq m\leq 4} \{v_{m,i} \mid 2 \leq i \leq 4\}$, we get that H_B, contains aC_3 say without loss of generality induced by say S'={$v_{2,2}$, $v_{3,2}$,$v_{4,2}$}. But then as S'∪{$v_{1,1}$} doesn't induce a blue K_4 we get that ($v_{1,1}$, x) is a red edge for some x in S'. Thus, $v_{1,2}$, $v_{2,1,}v_{1,1}$, x is a P_4, a contradiction. Thus, $m_4(P_4, K_4) \geq 4$. Therefore, $m_4(P_4, K_4) = 4$.

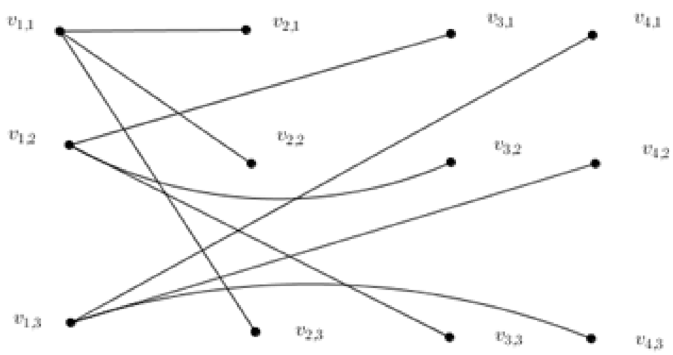

Figure 1: The H_R red colored graph

Since $r(P_4, K_4) = 10$ (see [2]), we see that $m_9(P_4, K_4) \geq 2$.

Next we will show that $m_5(P_4, K_4) \leq 2$. Consider any red/blue coloring given by $K_{5\times 2}$= $H_R \oplus H_B$, such that H_R contains no red P_4 and H_B contains no blue K_4. By [4] as, $m_4(P_4, C_3) = 2$, we get that H_B, contains aC_3 say without loss of generality induced by S={$v_{3,1}$, $v_{4,1}$,$v_{5,1}$}. Next as each of the four vertices of S'={$v_{1,1}$, $v_{1,2}$,$v_{2,3}$,$v_{2,2}$} does not induce a blue K_4 with S, by pigeon hole principle without loss of generality we may assume that one of the following three cases occur.

Case 1: At least three vertices of S' are adjacent in red to $v_{3,1}$.
This case is illustrated in the following diagram.

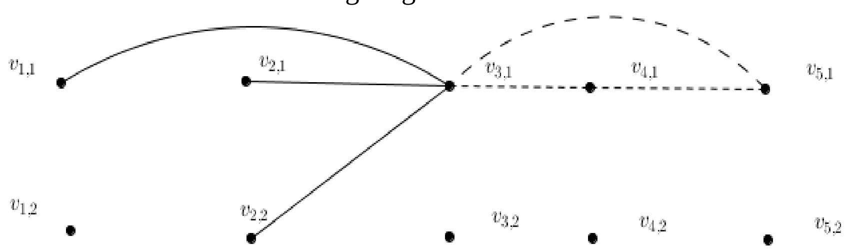

Figure 2: Illustrates Case 1

In this case $v_{1,1}$, $v_{2,1}$, $v_{4,1}$, $v_{5,1}$ must not induce a blue K_4, all possible options will give us a red P_4, a contradiction.

Case 2: Exactly two vertices of $S'$are adjacent in red to $v_{3,1}$.
In this case we get one of the following two scenarios as illustrated in the following diagram.

In the first scenario since $v_{1,1},v_{2,2},v_{3,2},v_{5,1}$ must not induce a blue K_4, the edge $(v_{3,2},v_{5,1})$ will be forced to be a red, as in all other options will give us a red P_4.

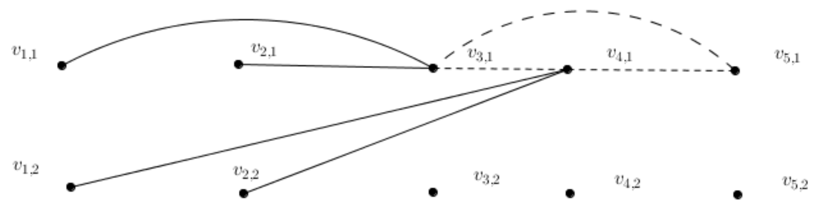

Figure 3: Illustrates Case 2: The first scenario

However, in this scenario when $(v_{3,2},v_{5,1})$ is red as $v_{1,2}$, $v_{2,1},v_{3,2},v_{4,2}$ must not induce a blue K_4, as before the edge $(v_{3,2},v_{4,2})$ will be forced to be a red, as in all other options will give us a red P_4. But now since $v_{1,1},v_{2,2},v_{4,2},v_{5,2}$ must not induce a blue K_4, all possible options will give us a red P_4, a contradiction.

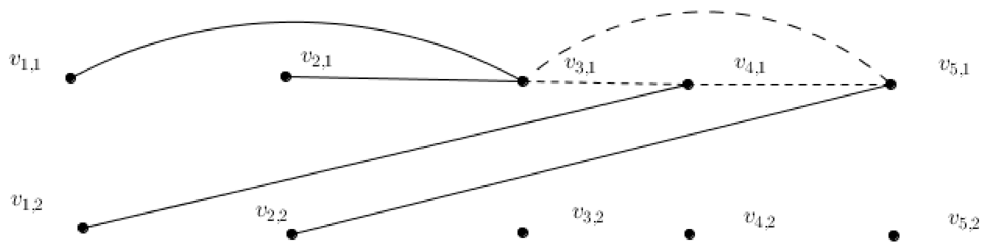

Figure 4: Illustrates Case 2: The second scenario

In the second scenario since $v_{1,1},v_{2,2},v_{4,1},v_{5,2}$ must not induce a blue K_4, the edge $(v_{2,2},v_{5,2})$ or else the edge $(v_{4,1},v_{5,2})$ will be forced to be a red, as in all other options will give us a red P_4.

In the scenario when $(v_{2,2},v_{5,2})$ is red as $v_{1,2}$, $v_{2,1},v_{4,2},v_{5,2}$ must not induce a blue K_4, it directly results that $(v_{1,2},v_{4,2})$ is red as in all other options will give us a red P_4. But now since $v_{2,1},v_{3,2},v_{4,2},v_{5,2}$ must not induce a blue K_4, all possible options will give us a red P_4, a contradiction. Next, in the scenario when $(v_{4,1},v_{5,2})$ is red as $v_{1,1},v_{2,2},v_{4,2},v_{5,2}$ must not induce a blue K_4, it directly results that $(v_{2,2},v_{4,2})$ is red as in all other options will give us a red P_4. But, now since $v_{1,2},v_{2,1},v_{3,2},v_{4,2}$ must not induce a blue K_4, all possible options will give us a red P_4, a contradiction.

Case 3: Exactly two vertices of S' belonging to one partite set are adjacent in red to $v_{3,1}$.
In the first scenario since $v_{1,1},v_{2,1},v_{3,2},v_{5,1}$ must not induce a blue K_4, the edge $(v_{3,2},v_{5,1})$ will be forced to be a red, as in all other options will give us a red P_4.

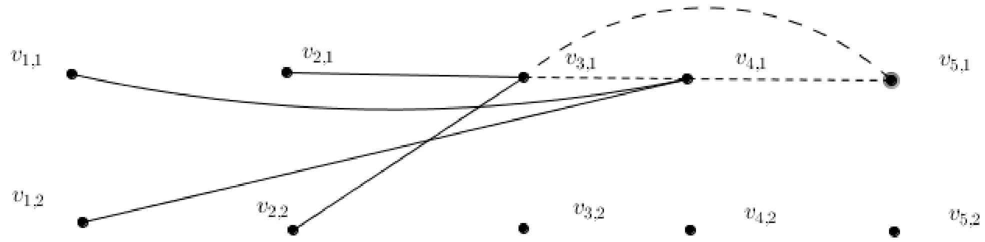

Figure 5: Illustrates Case 3: The first scenario

However, in this scenario when ($v_{3,2}$,$v_{5,1}$) is red, as $v_{1,1}$,$v_{2,1}$,$v_{3,2}$,$v_{4,2}$ must not induce a blue K_4, as before the edge ($v_{3,2}$,$v_{4,2}$) will be forced to be a red, as in all other options will give us a red P_4. But now since $v_{1,1}$, $v_{2,1}$,$v_{4,2}$,$v_{5,2}$ must not induce a blue K_4, all possible options will give us a red P_4, a contradiction.

In the second scenario since $v_{1,1}$,$v_{2,1}$,$v_{3,2}$,$v_{5,1}$ must not induce a blue K_4, the edge ($v_{3,2}$,$v_{5,1}$) or the edge ($v_{1,1}$,$v_{3,2}$) will be forced to be a red, as in all other options will give us a red P_4.

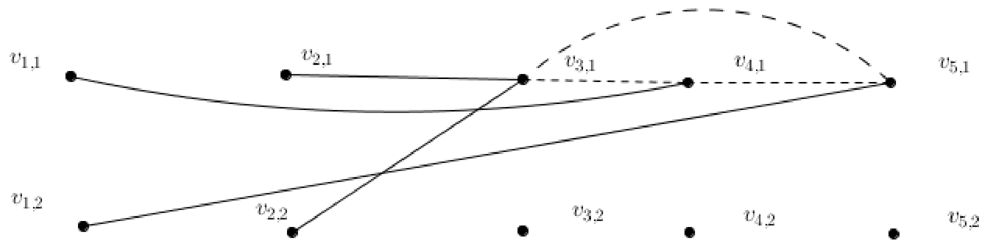

Figure 6: Illustrates Case 3: The second scenario

In this scenario when ($v_{3,2}$,$v_{5,1}$) is red, as $v_{1,1}$,$v_{2,1}$,$v_{3,2}$,$v_{4,2}$ must not induce a blue K_4, as before the edge ($v_{1,1}$,$v_{4,2}$) will be forced to be a red, as in all other options will give us a red P_4. But now since $v_{2,2}$, $v_{3,2}$,$v_{4,2}$,$v_{5,2}$ must not induce a blue K_4, all possible options will give us a red P_4, a contradiction.

Next in this scenario when ($v_{1,1}$,$v_{3,2}$) is red, as $v_{1,2}$,$v_{2,1}$,$v_{3,2}$,$v_{5,2}$ must not induce a blue K_4, as before the edge ($v_{1,2}$, $v_{5,2}$) will be forced to be a red, as in all other options will give us a red P_4. But now since $v_{2,1}$,$v_{3,2}$,$v_{4,2}$,$v_{5,2}$ must not induce a blue K_4, all possible options will give us a red P_4, a contradiction.

That is, we get that

$$2 \geq m_5(P_4,K_4) \geq m_6(P_4,K_4) \geq m_7(P_4,K_4) \geq m_8(P_4,K_4) \geq m_9(P_4,K_4) \geq 2.$$

Therefore, we can conclude that $m_j(P_4,K_4) = 2$ if $j = \{5,...,9\}$.

Finally, $m_j(P_4,K_4) = 1$ when $j \geq 10$, as $r(P_4,K_4) = 10$ (see [2]).

3. Size Ramsey numbers $m_j(P_4,G)$ when G is disconnected graph on 4 vertices

We have already dealt with all cases excluding $G = 2K_2$. We will deal with this in the following theorem.

Theorem 3.1. *If* $j \geq 3$, *then*

$$m_j(P_4, 2K_2) = \begin{cases} 2 & \text{if } j \in \{3,4\} \\ 1 & \text{if } j \geq 5 \end{cases}$$

Proof: Since, $r(P_4, 2K_2) = 5$ (see [2]), we obtain that $m_j(P_4, 2K_2) \geq 2$.

To show $m_3(P_4, 2K_2) \leq 2$, consider any red/blue coloring given by $K_{3\times 2} = H_R \oplus H_B$, such that H_R contains no red P_4 and H_B contains no blue $2K_2$.

Since $m_3(P_4,P_4) = 2$ (see [7]), we get that H_B has a P_4 and thus a $2K_2$. That is, $m_3(P_4, 2K_2) \leq 2$. Therefore, $m_3(P_4, 2K_2) = 2$.

Clearly, $m_j(P_4, 2K_2) = 1$ when $j \geq 5$, as $r(P_4, 2K_2) = 5$ (see [2]).

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