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On Solutions to the Diophantine Equation $p^x + q^y = z^4$

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Dedicated to Ali Burshtein

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Abstract. It is shown for all primes $p \geq 2$ and $y = 1$, that for each and every value $x \geq 1$, the title equation has infinitely many solutions. When x is even, then for $p = 2$, 3, the equation has exactly one solution in which q is prime, and in all other solutions when $p \geq 2$ q is composite. When x is odd, then for $p \geq 2$ the equation has solutions in which q is either prime or composite. Numerical solutions are also exhibited for $p \geq 2$ with odd and even values x.

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1. Introduction

The field of Diophantine equations is very old, very large, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 3, 4, 5, 7, 8, 10, 13]. The title equation stems from $p^x + q^y = z^2$.

In this paper we discuss solutions to the Diophantine equation

$$p^x + q^y = z^4 \tag{1}$$

for all primes $p \geq 2$ when q is odd, prime or composite, and x, y, z are positive integers.

In Section 2 for $p = 2$ with $y = 1$, and in Section 3 for all primes $p \geq 3$ with $y = 1$, the infinitude of solutions for each and every integer $x \geq 1$ is established.

In both Sections 2 and 3, we consider even values x as $x = 2n$, whereas odd values x as $x = 2n + 1$ when $n \geq 1$ is an integer. Although, in some places we could use x even or x odd instead of $x = 2n$ or $x = 2n + 1$, for the sake of uniformity of each theorem the notation is kept throughout.

2. The equation $p^x + q^y = z^4$ when $p = 2$ and $y = 1$

In this case our main interest is to determine the solutions of this equation and in particular when q is prime. Nevertheless, solutions in which q is composite are also established. This is done in the following Theorem 2.1.

Theorem 2.1. Suppose in equation (1) $p = 2$ and $y = 1$. Then the equation

$$2^x + q = z^4 \qquad (2)$$

has:

(a) For each and every even value x exactly one solution in which q is prime, and infinitely many solutions in which q is composite.

(b) For each and every odd value x infinitely many solutions in which q is prime or composite.

Proof: A priori q is odd, therefore $z \geq 3$ is odd.

(a) Suppose that x is even**.** Denote $x = 2n$ where $n \geq 1$ is an integer. From (2) we have

$$2^{2n} + q = z^4 \qquad (3)$$

implying that $q = z^4 - (2^n)^2 = (z^2 - 2^n)(z^2 + 2^n)$. If q is prime, then $z^2 - 2^n = 1$ and $z^2 + 2^n = q$. Since $z^2 - 2^n = 1$ yields $z^2 - 1 = (z - 1)(z + 1) = 2^n$, it follows that the only solution of this equation is $z = 3$ and $n = 3$. Hence, for $x = 6$ and $q = 17$ prime

$$2^6 + 17 = 3^4$$

is the only solution of equation (3) when $x \geq 2$ is even and q is prime.

As a consequence, in every other solution of equation (3), the value q is composite.

We will now show that for each and every value n equation (3) has infinitely many solutions. Let $n \geq 1$ be any fixed value, and hence 2^{2n} is fixed. For each fixed value 2^{2n}, denote by $\bar{z}$ the smallest possible value z, such that $\bar{z}^4$ exceeds 2^{2n} for the first time. Respectively, denote $\bar{q} = \bar{z}^4 - 2^{2n}$. For each value $\bar{z}$, there exist infinitely many consecutive odd values $z > \bar{z}$, and respectively odd values $q > \bar{q}$, such that equation (3) is satisfied. Thus, the fixed value n implies the existence of infinitely many solutions to equation (3). Since we consider each and every value $n \geq 1$, it therefore follows that equation (3) has infinitely many solutions for each and every value $n \geq 1$ in which q is composite as asserted.

It is noted that in the solution $2^6 + 17 = 3^4$, $\bar{q} = 17$ and $\bar{z} = 3$.

The above argument may now be illustrated for example in the cases: $n = 1$ ($x = 2$), $n = 2$ ($x = 4$) and $n = 3$ ($x = 6$).

$n = 1$:	$2^2 + 77 = 3^4$	$\bar{z} = 3$	$\bar{q} = 77$	composite.
	$2^2 + 621 = 5^4$	$z = 5$	$q = 621$	composite.
$n = 2$:	$2^4 + 65 = 3^4$	$\bar{z} = 3$	$\bar{q} = 65$	composite.
	$2^4 + 609 = 5^4$	$z = 5$	$q = 609$	composite.
	$2^4 + 2385 = 7^4$	$z = 7$	$q = 2385$	composite.
$n = 3$:	$2^6 + 561 = 5^4$	$z = 5$	$q = 561$	composite.

Part **(a)** is complete.

(b) Suppose that x is odd. If $x = 1$, we have from equation (2)

$$2 + q = z^4.$$

For each and every odd value $z \geq 3$, certainly there exists an odd value q so that the above equation is satisfied. For the first five odd consecutive values z, we demonstrate the following five solutions, namely:

$$2 + 79 = 3^4, \qquad 2 + 2399 = 7^4, \qquad 2 + 14639 = 11^4,$$

and 79, 2399, 14639 are primes. Whereas

$$2 + 623 = 5^4, \qquad 2 + 6559 = 9^4,$$

and 623, 6559 are composites.

Hence, when $x = 1$, equation (2) has infinitely many solutions in which q is either prime or composite.

If $x > 1$, denote $x = 2n + 1$ where $n \geq 1$ is an integer. From (2) we have

$$2^{2n+1} + q = z^4, \tag{4}$$

where q, z are odd. The proof that equation (4) has infinitely many solutions for each and every value n is the same as the proof of equation (3) when $x = 2n$ ($2n$ is replaced by $2n + 1$), with one distinction, namely: the prime q occurs more than once.

We exhibit this case for the following two values $n = 1$ ($x = 3$) and $n = 2$ ($x = 5$).

$n = 1$:	$2^3 + 73 = 3^4$	$\overline{z} = 3$	$\overline{q} = 73$	prime.
	$2^3 + 617 = 5^4$	$z = 5$	$q = 617$	prime.
	$2^3 + 2393 = 7^4$	$z = 7$	$q = 2393$	prime.
$n = 2$:	$2^5 + 49 = 3^4$	$\overline{z} = 3$	$\overline{q} = 49$	composite.
	$2^5 + 593 = 5^4$	$z = 5$	$q = 593$	prime.
	$2^5 + 2369 = 7^4$	$z = 7$	$q = 2369$	composite.

Evidently, equation (4) has infinitely many solutions for each and every value $n \geq 1$. For any given value $n \geq 1$ in equation (4), the question when is $\overline{q}$ or q equal to a prime is still unsettled.

This concludes part **(b)**, and the proof of Theorem 2.1. □

Remark 2.1. Following part **(b)** of Theorem 2.1., we conjecture that for each odd value $z \geq 3$, there exists at least one odd value x and q prime satisfying $2^x + q = z^4$.

3. The equation $p^x + q^y = z^4$ when $p \geq 3$ is prime and $y = 1$

In the following Theorem 3.1., we consider the equation $p^x + q^y = z^4$ when $p \geq 3$ is prime and $y = 1$. The infinitude of solutions for each and every fixed prime $p \geq 3$ with every value $x \geq 1$ is established.

Theorem 3.1. Let $y = 1$ in equation (1). The equation

$$p^x + q = z^4 \qquad p \geq 3 \quad \text{is prime} \tag{5}$$

for each and every prime $p \geq 3$ has:

(a) For each and every even value x exactly one solution when $p = 3$ and q prime, and infinitely many solutions in which q is composite.

(b) For each and every odd value x infinitely many solutions in which q is prime or composite.

Proof: The value q is odd, therefore $z \geq 2$ is even.

(a) Suppose that x is even**.** Denote $x = 2n$ where $n \geq 1$ is an integer. From (5) we have

$$p^{2n} + q = z^4, \tag{6}$$

and hence $q = z^4 - (p^n)^2 = (z^2 - p^n)(z^2 + p^n)$. If q is prime, it follows that $z^2 - p^n = 1$ and $z^2 + p^n = q$. When $z^2 - p^n = 1$, then $z^2 - 1 = p^n$ or $(z-1)(z+1) = p^n$. Let c be a non-negative integer. Denote $z - 1 = p^c$ and $z + 1 = p^{n-c}$. Then we obtain $p^c \cdot (p^{n-2c} - 1) = 2$ where $n > 2c$. Hence, $p^c = 1$ or $p^c = 2$. If $p^c = 1$, then $c = 0$ and $z = 2$. Thus, $z + 1 = 3 = p^n$ implies that $p = 3$ and $n = 1$. The case $p^c = 2$ is impossible.

For all primes $p \geq 3$, $x = 2n$, and q is prime, equation (6) has the only solution $p = 3$, $n = 1$ $(x = 2)$, $q = 7$ and $z = 2$, namely:

$$3^2 + 7 = 2^4.$$

Except for the above values, for all other values $p \geq 3$ and $n \geq 1$, the value q in equation (6) is composite. The first few such numerical solutions are:

$3^2 + 247 = 4^4$, $3^4 + 175 = 4^4$, $5^2 + 231 = 4^4$, $5^4 + 671 = 6^4$, $7^2 + 207 = 4^4$.

Evidently, equation (6) has infinitely many solutions with q composite as asserted.

This concludes part **(a)**.

(b) Suppose that x is odd**.** If $x = 1$, we have from equation (5)

$$p + q = z^4.$$

Certainly, for each and every fixed prime $p \geq 3$, there exists a value q prime or composite, such that the above equation is satisfied. The solutions for the first three consecutive primes p with primes q and the respective values $z = 2, 4, 6$ are:

$$3 + 13 = 2^4, \qquad 5 + 251 = 4^4, \qquad 7 + 1289 = 6^4.$$

Whereas, for $p = 3, 5, 7$ and q is composite, we have:

$$3 + 253 = 4^4, \qquad 5 + 9995 = 10^4, \qquad 7 + 9 = 2^4.$$

Thus, for each and every prime $p \geq 3$, the above equation has infinitely many solutions in which q is either prime or composite.

If $x > 1$, denote $x = 2n + 1$ where $n \geq 1$ is an integer. From (5) we have

$$p^{2n+1} + q = z^4. \tag{7}$$

We will show that equation (7) has infinitely many solutions for every prime p with each and every value n. Let p and n be any fixed values. Thus p^{2n+1} is fixed. For each fixed value p^{2n+1}, denote by $\bar{z}$ the smallest possible value z, such that $\bar{z}^4$ exceeds p^{2n+1} for the first time. Respectively, denote $\bar{q} = \bar{z}^4 - p^{2n+1}$. For each value $\bar{z}$, there exist infinitely many consecutive even values $z > \bar{z}$, and respectively odd values $q > \bar{q}$, so that equation (7) is satisfied. Hence, for the fixed prime p, the fixed value n implies that there exist infinitely many solutions to equation (7). Since we consider the infinite set of all primes $p \geq 3$, and the infinite set of all values $n \geq 1$, it follows for every prime p with each and every value n, that equation (7) has infinitely many solutions. The value q is either prime or composite.

The above argument may now be illustrated for example in the following three solutions of $p^{2n+1} + q = z^4$ when $p = 5$ is fixed, and $n = 1, 2, 3$ ($x = 3, 5, 7$).

$n = 1$:	$5^3 + 131 = 4^4$	$\bar{z} = 4$,	$\bar{q} = 131$	prime.
$n = 2$:	$5^5 + 971 = 8^4$	$\bar{z} = 8$,	$\bar{q} = 971$	prime.
$n = 3$:	$5^7 + 26851 = 18^4$	$\bar{z} = 18$,	$\bar{q} = 26851$	composite.

Hence, for each and every fixed prime $p \geq 3$ and $n = 1, 2,\ldots,k,\ldots,$ there exist infinitely many respective values $z \geq \bar{z}$ satisfying equation (7) in which the value $q \geq \bar{q}$ is either prime or composite.

For $p \geq 3$ and $x \geq 1$, the question when is q prime or composite is not pursued here since it is beyond the scope of our study.

This completes the proof of part **(b)** and of Theorem 3.1. □

4. Conclusion

In the following six solutions the value q is prime.

$3^1 + 13 = 2^4$, $3^3 + 229 = 4^4$, $3^5 + 13 = 4^4$, $5^1 + 11 = 2^4$, $5^3 + 131 = 4^4$, $5^5 + 971 = 8^4$.

Two questions may now be raised.

Question 1. Does $p^x + q = z^4$ has at least one solution for each and every prime $p \geq 3$, $x \geq 1$ odd and q prime ?

The answer is affirmative for $p = 3, 5, 7$.

Question 2. Does $p^x + q = z^4$ has a solution for any fixed prime $p \geq 3$, with each and every odd $x \geq 1$ and q prime ?

The answer is affirmative for $p = 5$ when $x = 1, 3, 5$.

We presume that other interesting questions concerning equation (1) may be raised.

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