Annals of Pure and Applied Mathematics
Vol. 14, No. 2, 2017, 207-211
ISSN: 2279-087X (P), 2279-0888(online)
Published on 24 August 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v14n2a1

Annals of **Pure and Applied Mathematics**

All the Solutions of the Diophantine Equation $p^3 + q^2 = z^3$

Nechemia Burshtein

117 Arlozorov Street, Tel Aviv 6209814, Israel
Email: anb17@netvision.net.il

Received 9 August 2017; accepted 22 August 2017

Abstract. In this paper, it is established that the title equation has exactly four solutions, all of which are exhibited.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In the huge field of Diophantine equations, no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 3, 4, 5, 6, 9, 11, 13]. The title equation stems from $p^x + q^y = z^2$.

In this paper, the values x, y, are fixed positive integers. Our main objective revolves around the existence and the number of solutions of the equation $p^3 + q^2 = z^3$.

2. The main result

In this section, we determine all the solutions of the equation $p^3 + q^2 = z^3$, where p, q, z, and all other values represent positive integers. This is done in Theorem 2.1.

Theorem 2.1. Suppose that p is prime and $q > 1$. Then the equation

$$p^3 + q^2 = z^3 \tag{1}$$

has exactly four solutions in all of which $p = 7$. In one solution q is prime, and in all other solutions q is composite.

Proof: From (1) we obtain

$$q^2 = z^3 - p^3 = (z - p)(p^2 + pz + z^2). \tag{2}$$

Denote $z - p = T$ where $T \geq 1$. Substituting $z = p + T$ into (2) results in

$$q^2 = T(3p^2 + 3pT + T^2). \tag{3}$$

We distinguish two cases for which equality (3) may be satisfied, namely: (i) When $T > 1$ and $3p^2 + 3pT + T^2$ are squares simultaneously. (ii) When $T \geq 1$ and $3p^2 + 3pT + T^2$ are not necessarily squares simultaneously.

We will now show that case (i) is actually impossible.

(i) Suppose that $T > 1$ and $3p^2 + 3pT + T^2$ are squares simultaneously. Denote $T = U^2$ and $3p^2 + 3pT + T^2 = V^2$. Then

$$3p^2 + 3pT + T^2 = 3p^2 + 3pU^2 + (U^2)^2 = V^2$$

or

$$3p(p + U^2) = V^2 - (U^2)^2 = (V - U^2)(V + U^2). \tag{4}$$

It follows from (4) that p divides at least one of the values $(V - U^2)$, $(V + U^2)$. We will now show that this statement does not hold.

If $p \mid (V - U^2)$, denote $pR = V - U^2$ or $V = pR + U^2$. Then from (4) we have

$$3p(p + U^2) = p^2R^2 + 2pRU^2 + (U^2)^2 - (U^2)^2$$

or

$$p^2(R^2 - 3) + pU^2(2R - 3) = 0$$

which is impossible for all values R. Hence $p \nmid (V - U^2)$.

If $p \mid (V + U^2)$, denote $pS = V + U^2$ or $V = pS - U^2$. From (4) we obtain

$$p^2(S^2 - 3) - pU^2(2S + 3) = 0$$

implying

$$p = U^2 \cdot \frac{2S+3}{S^2-3}. \tag{5}$$

The divisors of p are 1 and p, and since $T > 1$ therefore $T = U^2 > 1$ or $U > 1$. Then from (5) it follows that: either (a) $\frac{U^2}{S^2-3} = 1$ and $2S + 3 = p$, or (b) $U = p$ and $\frac{U(2S+3)}{S^2-3} = \frac{p(2S+3)}{S^2-3} = 1$. If (a), then $\frac{U^2}{S^2-3} = 1$ or $U^2 = S^2 - 3$. But $S^2 - U^2 = 3$ has the only solution $U = 1$ and $S = 2$ which is impossible. If (b), then $\frac{p(2S+3)}{S^2-3} = 1$ or $p = \frac{S^2-3}{2S+3}$ which is impossible since $\frac{S^2-3}{2S+3}$ is never an integer. Thus $p \nmid (V + U^2)$, and case (i) is complete.

(ii) Suppose that $T \geq 1$ and $3p^2 + 3pT + T^2$ are not necessarily squares simultaneously. In equality (3) set $3p^2 + 3pT + T^2$ as

$$3p^2 + 3pT + T^2 = TA^2 \tag{6}$$

for some value A which guarantees that equality (3) is indeed a square $q^2 = (TA)^2$. Then, from (6) it follows that $T \mid 3p^2$. The value T may assume all possible divisors of $3p^2$, namely: $T = 1$, $T = 3$, $T = p$, $T = 3p$, $T = p^2$, $T = 3p^2$. The six cases are considered separately.

The case $\boldsymbol{T = 1}$. Substituting $T = 1$ in (3) yields

$$q^2 = 3p^2 + 3p + 1, \tag{7}$$

from which

$$q^2 - 1 = (q - 1)(q + 1) = 3p(p + 1).$$

Therefore, either $p \mid (q - 1)$ or $p \mid (q + 1)$. Note that $p \neq 2$.

If $p \mid (q - 1)$, denote $Bp = q - 1$ where $B \geq 1$. Substituting $q = Bp + 1$ into (7) results in

$$B^2p^2 + 2Bp + 1 = 3p^2 + 3p + 1,$$

and after simplifications implies that $p = \dfrac{3-2B}{B^2-3}$. The term $\dfrac{3-2B}{B^2-3}$ is negative for all values $B \geq 1$, and therefore is impossible. Thus, $p \nmid (q-1)$ and $p \mid (q+1)$.

If $p \mid (q+1)$, denote $Cp = q + 1$ where $C \geq 1$. Then $q = Cp - 1$, and from (7) it follows that

$$q^2 = (Cp-1)^2 = C^2p^2 - 2Cp + 1 = 3p^2 + 3p + 1. \tag{8}$$

After simplifications of (8), one obtains that

$$p = \frac{2C+3}{C^2-3}.$$

Evidently, the only value that C may assume is $C = 2$. Hence, $C = 2$ yields $\dfrac{2\cdot 2+3}{2^2-3} = 7 = p$. The values $p = 7$, $q = 2p - 1 = 13$ prime, and $z = p + 1 = 8$ form a solution of equation (1).

The case $T = 1$ is complete.

The case $\boldsymbol{T = 3}$. From (3) we obtain $q^2 = 3(3p^2 + 9p + 9)$ or $q^2 = 3^2(p^2 + 3p + 3)$ implying that $p^2 + 3p + 3$ must equal a square, say A^2. If $p^2 + 3p + 3 = A^2$, then

$$A^2 - p^2 = (A-p)(A+p) = 3(p+1). \tag{9}$$

We now show that $3 \nmid (A-p)$ and $3 \nmid (A+p)$ implying that $T \neq 3$. If $3 \mid (A-p)$, denote $3D = A - p$ where $D \geq 1$. Hence, from (9) $3D(A+p) = 3(p+1)$ or $D(A+p) = p + 1$. Since $A > p$, this equality is impossible and $3 \nmid (A-p)$. If $3 \mid (A+p)$ then $3E = A + p$. We have from (9) that $(3E - 2p)3E = 3(p+1)$ or $(3E-2p)E = p + 1$. Thus, $p(2E+1) = 3E^2 - 1$, and $p = \dfrac{3E^2-1}{2E+1}$. But, this fraction never equals an integer, and therefore it follows that $3 \nmid (A+p)$. Hence $T \neq 3$.

As an immediate consequence, it follows that for every prime p, $p^2 + 3p + 3$ is never equal to a square.

The case $\boldsymbol{T = p}$. With $T = p$ in (3), we obtain $q^2 = p(7p^2) = 7p^3$ implying that $p = 7$ and $q^2 = 7^4$. Hence, the values $p = 7$, $q = 7^2$ and $z = 2p = 14$ yield a solution of equation (1).

The case $\boldsymbol{T = 3p}$. When $T = 3p$ in (3), then $q^2 = 3p\cdot 21p^2 = 3^2\cdot 7p^3$. Thus, $p = 7$ and $q^2 = 3^2p^4 = 3^2\cdot 7^4$. The values $p = 7$, $q = 3\cdot 7^2$ and $z = 4p = 28$ form a solution of equation (1).

The case $\boldsymbol{T = p^2}$. From (3) we have

$$q^2 = p^2(3p^2 + 3p^3 + p^4) = p^4(3 + 3p + p^2).$$

It now follows that the value $p^2 + 3p + 3$ must equal a square say M^2, so that $q^2 = (p^2M)^2$. But, $p^2 + 3p + 3 \neq M^2$ as was shown in the case $T = 3$. Thus $T \neq p^2$.

The case $\boldsymbol{T = 3p^2}$. From (3) we obtain

$$q^2 = 3p^2(3p^2 + 9p^3 + 9p^4) = 9p^4(1 + 3p + 3p^2).$$

Therefore, the value $3p^2 + 3p + 1$ must be equal to a square, say N^2, in order that $q^2 = (3p^2N)^2$. The value $3p^2 + 3p + 1$ appears in equality (7) of the case $T = 1$, and is indeed equal to a square only when $p = 7$ for which a solution of equation (1) exists.

Hence, the values $p = 7$, $q = 3p^2(3p^2 + 3p + 1)^{1/2} = 3\cdot 7^2\cdot 13$ and $z = p + 3p^2 = p(3p+1) = 2\cdot 7\cdot 11$ yield a solution to equation (1).

The four solutions of equation (1) have been established and exhibited.
This concludes the proof of Theorem 2.1. □

As a consequence of Theorem 2.1 we have:

Remark 2.1. The unique solution of the square $K^2 = 3p^2 + 3pT + T^2$ consists of the value $T = 1$ and the primes $p = 7$ and $K = 13$.

As a summary, and for the convenience of the readers, we now demonstrate the four solutions in the order of their occurrence.

Solution 1. $7^3 + 13^2 = (2^3)^3$.
Solution 2. $7^3 + (7^2)^2 = (2\cdot 7)^3$.
Solution 3. $7^3 + (3\cdot 7^2)^2 = (2^2\cdot 7)^3$.
Solution 4. $7^3 + (3\cdot 7^2\cdot 13)^2 = (2\cdot 7\cdot 11)^3$.

3. Conclusion

We conclude by giving a glimpse on the equation $p^3 + q^m = z^3$ when $m = 1, 2$ and 3.

It is easily seen that infinitely many solutions exist for the equation $p^3 + q^1 = z^3$ when p is prime and q is prime/composite. Few such examples are:

$2^3 + 19 = 3^3$, $3^3 + 37 = 4^3$, $5^3 + 91 = 6^3$, $7^3 + 386 = 9^3$.

In this paper, the equation $p^3 + q^2 = z^3$ yields quite surprisingly only four solutions in all of which $p = 7$ and in only one of them q is prime.

In 1637, Fermat (1601 – 1665) stated that the Diophantine equation $x^n + y^n = z^n$, with integral $n > 2$, has no solutions in positive integers x, y, z. This is known as Fermat's "Last Theorem". In 1995, 358 years later, the validity of the Theorem was established and published by A. Wiles. Thus, the equation $p^3 + q^3 = z^3$ has no solutions in positive integers p, q, z.

REFERENCES

1. J.B.Bacani and J.F.T.Rabago, The complete set of solutions of the diophantine equation $p^x + q^y = z^2$ for twin Primes p and q, *Int. J. Pure Appl. Math.*, 104 (2015) 517 – 521.
2. N.Burshtein, On the infinitude of solutions to the diophantine equation $p^x + q^y = z^2$ when p=2 and p=3, *Annals of Pure and Applied Mathematics*, 13 (2) (2017) 207 – 210.
3. N.Burshtein, On the diophantine equation $p^x + q^y = z^2$, *Annals of Pure and Applied Mathematics*, 13 (2) (2017) 229 – 233.
4. N.Burshtein, On solutions to the diophantine equation $p^x + q^y = z^4$, *Annals of Pure and Applied Mathematics*, 14 (1) (2017) 63 – 68.
5. N.Burshtein, All the solutions of the diophantine equation $p^3 + q^2 = z^2$, *Annals of Pure and Applied Mathematics*, 14 (1) (2017) 115 – 117.
6. S.Chotchaisthit, On the diophantine equation $4^x + p^y = z^2$, where p is a prime number, *Amer. J. Math. Sci.*, 1 (1) (2012) 191 – 193.

7. S.Chotchaisthit, On the diophantine equation $2^x + 11^y = z^2$, *Maejo Int. J. Sci Technol.*, 7 (2013) 291 – 293.
8. Md. A.- A. Khan, A.Rashid and Md. S.Uddin, Non-negative integer solutions of two diophantine equations $2^x + 9^y = z^2$ and $5^x + 9^y = z^2$, *Journal of Applied Mathematics and Physics*, 4 (2016) 762 – 765.
9. B.Poonen, Some diophantine equations of the form $x^n + y^n = z^m$, *Acta Arith.*, 86 (1998) 193 – 205.
10. J.F.T.Rabago, A note on two diophantine equations $17^x + 19^y = z^2$ and $71^x + 73^y = z^2$, *Math. J. Interdisciplinary Sci.*, 2 (2013) 19 – 24.
11. B.Sroysang, More on the diophantine equation $8^x + 19^y = z^2$, *Int. J. Pure Appl. Math.*, 81(4) (2012) 601 – 604.
12. B.Sroysang, On the diophantine equation $5^x + 7^y = z^2$, *Int. J. Pure Appl. Math.*,89 (2013) 115 – 118.
13. A.Suvarnamani, Solution of the diophantine equation $p^x + q^y = z^2$, *Int. J. Pure Appl. Math.*, 94 (4) (2014) 457 – 460.
14. A.Suvarnamani, On the diophantine equation $p^x + (p+1)^y = z^2$, *Int. J. Pure Appl. Math.*, 94(5) (2014) 689 – 692.