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Types and Properties of Characteristic Classes

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Abstract. This paper aims to study the basic properties, behaviors and types of characteristic classes. The study gives the importance of characteristic classes and how they develop cohomology theory as a classification method in algebraic topology. The paper deals also with the relations between several characteristic classes with their comparison.

***Keywords**:* Fiber bundles, vector bundles, characteristic classes, cohomology theory

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1. Introduction

For studying vector bundles we have a technique of characteristic classes. Any bundle ξ in the cohomology of the base space $B(\xi)$ which is the natural setting for characteristic classes [1], so that characteristic classes behave well with respect to bundle maps.

The theory of characteristic classes is associated to the names of Whitney-Stiefel, Pontryagin and Chern, and was developed further by Weil, Bott, Thom and many others. Whitney and Stiefel introduced characteristic classes in [2,3]. The Whitney product theorem is introduced by Whitney in (1940-1941) and due Wu in (1948) [3], Stiefel studied the homology classes determined by the tangent bundle of a smooth manifold and invented co- homology theory, whereas Whitney discussed the case of sphere bundles, which have the advantage of having compact fibers. Pontryagin constructed the classes which bear his name by studying the homology of so- called Grassmann manifolds. Pontryagin's work goes back to (1942)[2]. In (1946), Chern defined characteristic classes for complex vector bundles, and showed that complex Grassman manifolds are easier to understand than the real ones [2,3]. Hopf had discovered in (1927) that the number of zeroes of a smooth vector field on a compact oriented manifold is equal to its Euler characteristic; Thom and Wu (1986) proved that the integrals of the highest-dimensional Chern class equals the Euler characteristic, and Hirzebruch constructed associated to the tangent bundle of a$4k$-dimensional real manifold (compact and oriented) called the L-genus, and proved that it is equal to another integer, called the signature. In the case of 4-dimensional manifolds it turns out to be equal to one-third of the integral of the first Pontryagin class of the manifold [2]. There have been many generalizations, such as the extension of Hopf's result to sections of complex vector bundles by Bott and Chern and the various "index theorems", the most famous of which is the Atiyah-Singer index theorem, which relates the index manifold to the index of an elliptic differential operator (the Laplacian) on that manifold [4].

Our discussion of characteristic classes is rather heuristic and follows mainly the ideas of Stiefel-Whitney classes and Chern. And only briefly mention how they are related to Pontryagin and Euler classes.

2. Basics of characteristic classes

Characteristic classes are cohomology classes associated to vector bundles. They measure in some way how a vector bundle is twisted, or nontrivial. There are four main types of characteristic classes:

1. Stiefel-Whitney classes $w_i(E) \in H(B; \mathbf{Z}_2)$ for a real typical vector bundle $E \to B$
2. Chern classes $c_i(E) \in H^{2i}(B;\mathbf{Z})$ for a complex typical vector bundle $E \to B$
3. Pontryagin classes $p_i\,(E) \in H^{4i}(B;\mathbf{Z})$ for a real typical vector bundle $E \to B$
4. The Euler class e(E) $\in H^n(B;\ \mathbf{Z})$ for an oriented n -dimensional real typical vector bundle $E \to B$

The Stiefel-Whitney and Chern classes have same behaviors so they are formally quite similar. But when you take $\mathbf{Z}$ rather than $\mathbf{Z}_2$ coefficients, Pontryagin classes can be expressed as a refinement of Stiefel-Whitney classes, and the Euler class is a further refinement in the orientable case.

Definition 1. For a real vector bundle ξ, the *ith* Stiefel-Whitney class of bundle ξ, denoted $w_i(\xi)$, is $x_i(\xi) \in H^i(B\,(\xi), \mathbf{Z}_2)$ [4,5].
For a complex vector bundle ξ, the *ith* Chern class of ξ, denoted $c_i(\xi)$, is
$x_i(\xi) \in H^{2i}(\mathrm{B}\,(\xi), \mathbf{Z})$ [5,6].
In addition, $w(\xi) = 1 + w_1(\xi) + ... + w_n(\xi)$ is called the total Stiefel-Whitney class and $c(\xi) = 1 + c_1(\xi) + ... + c_n(\xi)$ is total Chern class [6].

3. Properties of Stiefel-Whitney classes

For any real vector bundle (over a space B there is a class $w(\xi) \in H^i(B(\xi), \mathbf{Z}_2)$, with the following properties:
(P_0) We have $1 + w_1(\xi) + ... + w_n(\xi)$ where $w_i(\xi) \in H^i(B\,(\xi), \mathbf{Z}_2)$ and $w_i(\xi) = 0$ for
$i >$ dim ξ= n .(P_1) If ξ and η are B-isomorphic, it follows that $w(\xi) = w(\eta)$, and if $f\colon B_1 \to B$ is a map, then we have $f^*(w\,(\xi)) = w(f^*(\xi))$. ($P_2$) (Whitney sum formula) For two vector bundles ξ and η over B, the relation $w(\xi \oplus \eta) = w(\xi)w(\eta)$ (cup multiplication) holds, and so $\overline{w}(\xi)w(\xi \oplus \eta) = w(\eta)$ unique solution!, where $\overline{w}$ the inverse of w.(P_3) For line bundle λ over $S^1 = \mathbf{R}P^1$, the element $w_1(\lambda)$is nonzero in $H^1(S^1, \mathbf{Z}_2) = \mathbf{Z}_2$.
(P'_3) For line bundle γ_1 over $\mathbf{R}P^\infty$, the element $w_1(\gamma_1)$is the result generator of the polynomial ring $H^*(\mathbf{R}P^\infty, \mathbf{Z}_2)$.Using property ($P_1$) and the inclusion $\mathbf{R}P^1 \to \mathbf{R}P^\infty$, we find that (P_3) and (P'_3) are equivalent to each other. We choose a generator of $H^2(S^2, \mathbf{Z})$ which in turn defines a generator of $H^2(CP^n,\mathbf{Z})$ for each n with $1 \le n \le +\infty$. This element z will generate the polynomial ring $H^*(CP^\infty,\mathbf{Z})$ [5,7].

Remark 1. Properties (P_0), (P_1) and (P'_3) hold for Stiefel-Whitney.

4. Properties of Chern classes

For each complex vector bundle ξ over a space B there is a class $c(\xi) \in H^*(B, \mathbf{Z})$ (where $H^*$is universal cohomology ring) with the following properties:

(C_o) We have $c(\xi) = 1 + c_1(\xi) + ... + c_n(\xi)$ where $c_i(\xi) \in H^{2i}(B, \mathbf{Z})$ and $c_i(\xi) = 0$ for $i > \dim \xi$.

(C_1) If ξ and η are B-isomorphic, it follows that $c(\xi) = c(\eta)$, and if $f: B_1 \rightarrow B$ is a map, then we have $f^*(c(\xi)) = c(f^*(\xi))$.($C_2$) For two vector bundles ξ and η over B, the relation $c(\xi \oplus \eta) = c(\xi)c(\eta)$ (cup multiplication) holds.

(C_3) For line bundle λ over $S^2 = \mathbf{C}P^1$, the element $c_1(\lambda)$ isthe given generator of $H^2(S^2, \mathbf{Z})$.

(C'_3) For line bundle γ_1 over $\mathbf{C}P^\infty$, the element $c_1(\gamma_1)$is theresult of the polynomial ring $H^*(\mathbf{C}P^\infty,\mathbf{Z})$. Using property ($C_1$) and the inclusion $\mathbf{C}P^1 \rightarrow \mathbf{C}P^\infty$, we see that (C_3)and (C'_3) are equals. From the parallel character of properties (P_0) to (P_3) and properties (C_o) to (C_3) it is clear that the two sets of characteristic classes have many formal properties in common [3,5].

Remark 2. Properties (C_o), (C_1), and (C'_3) hold for Chern classes.

Theorem 1. The functions w_I: $L_{\mathbf{R}}(B) \rightarrow H^1(B,\mathbf{Z}_2)$ and c_I: $L_{\mathbf{c}}(B) \rightarrow H^2(B,\mathbf{Z})$ *define isomorphisms of co functors* [5,6].

We have $w_1(\xi \otimes \eta) = w_1(\xi)+w_1(\eta)$ and $c_1(\xi \otimes \eta) = c_1(\xi)+c_1(\eta)$, from this theorem. The cohomology ring $H^*(\mathbf{R}P^\infty,\mathbf{Z}_2)$ is generated by $w_1(\gamma_1)$ and $H^*(\mathbf{C}P^\infty,\mathbf{Z})$ by $c_1(\gamma_1)$. Consequently, the characteristic classes of line bundles are uniquely defined by their axiomatic properties [2].

5. Stability properties and examples of characteristic classes

Proposition 1. If ξ is a trivial bundle over B, then $w_i(\xi) = 0$ for $i > 0$ in the real case and $c_i(\xi) = 0$ for $i> 0$ in the complex case [3,5].

Proof: The statement is true for ξover a point because the cohomology in nonzero dimensions is zero, and every trivial bundle is isomorphic to the induced bundle by a map to a point. By properties (P_1,C_1) we have the result. In addition, using properties (P_2,C_2), we have the following result.

Theorem 2. Let ξ and η be two equivalent vector bundles. Then $w(\xi) = w(\eta)$ holds in the real case and $c(\xi) = c(\eta)$ holds in the complex case [5].

Proof: For some n and m, there is an isomorphism between $\xi \oplus \theta^n$ and $\eta \oplus \theta^m$. From this we have the following equalities in the real case:

$w(\xi) = w(\xi)w(\theta^n)=w(\eta \oplus \theta^m)= w(\eta)\, w(\theta^m) = w(\eta)^1= w(\eta)$ or we have $w(\xi)=w(\eta)$.

Similarly, in the complex case we have $c(\xi)= c(\eta)$ (where θ *is trivail bundle)*

Note 1. What happens if $\xi \oplus \eta$ is trivial? Well, by the Whitney product theorem, $w(\xi \oplus \eta)= 0$ for $i > 0$. So,$w_1(\xi) + w_1(\eta) = 0$ (Remember $w_0(\xi) = w_0(\eta) =1$).

$w_2(\xi) + w_1(\xi)\, w_1(\eta) + w_2(\eta) = 0, \Rightarrow w_2(\eta) = (w_1\xi)^2 - w_2(\xi)$[7].

In particular, if $\xi \oplus \eta$ is trivial: $\overline{w}(\xi) = w(\eta)$.

Proposition 2. For the sphere S^n the tangent bundle $\tau(s^n)$, we have $w(\tau(s^n)) = 1$.

Proof: Since $\tau(s^n) \oplus \theta^1$ and θ^{n+1} are isomorphic, the tangent bundle $\tau(s^n)$ is s-trivial. Therefore, proposition1 and theorem2, $w(\tau(s^n)) = 1$ [5,6].

Proposition 3. For the tangent bundle $\tau(\mathbf{R}P^n)$ there are the relations $w(\tau(\mathbf{R}P^n)) = (1 + z)^{n+1}$, where z is the generator of $H^1(\mathbf{R}P^n,\mathbf{Z}_2)$ and $c(\tau(\mathbf{C}P^n)) = (1 + z)^{n+1}$, where z is the generator of $H^2(CP^n,\mathbf{Z})$, where P is projection bundle.
Every vector field on $\mathbf{R}P^n$ defines a vector field on S^n. A nonexistence statement for vector fields on S^n is stronger than a nonexistence statement for vector fields on $\mathbf{R}P^n$, but as an application we include the next proposition which is really an easy consequence [5,6].

Proposition 4. Every tangent vector field on $\mathbf{R}P^{2k}$ has at least one zero [5].
Proof: Observe that $w_{2k}(\tau(\mathbf{R}P^{2k})) = (2k + 1)Z^{2k} = Z^{2k} \neq 0$. If $\tau(\mathbf{R}P^{2k})$ had cross section that was everywhere nonzero, we would have $\tau(RP^{2k}) = \xi \oplus \theta^1$Then $w_{2k}(\tau(RP^{2k})) = w_{2k-1}(\xi)w_1(\theta^1) = 0$, which is a contradiction.

Definition 2. Let ξ be a vector bundle over B. A splitting map of $\xi$is a map $f: B_1 \to B$ such that $f^*(\xi)$ is a sum of line bundles and $f^*: H^*(B, K_c) \to H^*(B_1, K_c)$ is a monomorphism [5].

6. Existing of splitting maps

Proposition 5. If ξ is a vector bundle over B, there exists a splitting map for ξ [5].
Proof: We prove this by induction on the dimension of ξ. For a line bundle, the identity on the base space is a splitting map. In general, let $q: E(P\xi) \to B$ be the associated projective bundle. Then $q^*: H^*(B, K_c) \to H^*(E(P\xi, K_c)$ is a monomorphism, and $q^*(\xi) = \lambda_\xi \oplus \sigma_\xi$. By inductive hypothesis there exists a splitting map $g: B_1 \to E(P(\xi)$for σ_ξ. Then $f = qg$ from B_1 to B is a splitting map for ξ.

Corollary 2. For r vector bundles $\xi_1 \ldots ,\xi_r$ over B which are either all real or all complex. Then there exists a map $f: B_1 \to B$ such that f is a splitting map for each ξ_i where $1 \le i \le r$ [5].

Theorem 3. The properties (P_0) to (P_3) completely determine the Stiefel-Whitney classes, and the properties (C_o) to (C_3) completely determine the Chern classes [5].
Proof: Let w_i and $\overline{W}_i$ be two sets of Stiefel-Whitney classes, and let ξ_n be a vector bundle with splitting map $f: B_1 \to B$. Since $w_1$is uniquely determined for line bundles λ_i, we have $f^*(w(\xi) = w(f^*(\xi) = (1 + w(\lambda_1) \cdots (1 + w_1(\lambda_n)) = (1 + \overline{w}_1(\lambda_1)) \ldots (1 + \overline{w}(\lambda_n)) = \overline{w}(f^*(\xi)) = f^*(\overline{w}(\xi)$, where $f^*(\xi) = \lambda_1 \oplus \cdots \oplus \lambda_n$. Since f^* is a monomorphism, we have $w(\xi) = \overline{w}(\xi)$. The same proof applies to Chern classes.

Theorem 4. For ξ and η, two vector bundles over B, there is the relation $x(\xi \oplus \eta) = x(\xi)x(\eta)$ [5].
(Where x denotes Stiefel-Whitney classes or the Chern classes)

7. Fundamental class of sphere bundles

For a vector bundle ξ with total space $E(\xi)$, let $E_0(\xi)$ denote the open sub-space of nonzero vectors. For $b \in B$, let $j_b: (\mathbf{R}^n, \mathbf{R}^n - \{0\}) \to (E(\xi), E_0(\xi))$ denote the inclusion onto the fibre of ξ over $b \in B$. Each complex n-dimensional vector bundle restricts to a real (2n)-dimensional real vector bundle[5].In previous sections , we developed characteristic classes, using projection bundle $P(\xi)$; now we use $E_0(\xi)$ to define other classes.

Definition 3. A vector bundle is orientable provided its structure group restricts from O(n) to SO(n). An orientation of a vector bundle is a particular restriction of the structure group to $S\,O$(n). An oriented vector bundle is a pair consisting of a vector bundle and an orientation on the bundle. In other words, a vector bundle has an atlas of charts where the linear transformations changing from one chart to another have strictly positive determinants [5].

Example1. Every restriction of a complex vector bundle to a real vector bundle is orientable and has a natural orientation because $U(n) \subset SO(2n) \subset O(2n)$[5]. The next theorem contains the fundamental construction of this section.

Theorem 5. Let ξ be a real vector bundle. The cohomology groups have coefficients in **Z** if the bundle is oriented and in $\mathbf{Z}_2$ in general. Then the following statements hold [5]:
(1) There exists a unique $U_\xi \in H^n(E, E_0)$ such that $j^*_t(U_\xi)$ is a fixed generator of $H^n(\mathbf{R}^n, \mathbf{R}^n - \{0\})$.
(2) For i < n, there is the relation $H^i(E, E_0) = 0$.
(3) The function $a \mapsto p^*(a)\, U_\xi$ (cup product) of $H^i(B) \to H^{i+n}(E, E_0)$ is an isomorphis.

7. The oriented types classes… the Euler class
Definition 4. The Euler class of a real vector bundle cover B, denote by$e(\xi)$,is $p^{*-1}j^*(U_\xi)$, where $p: E \to B$ is the projection of ξ. The term "Euler class" is usually used only in the oriented case and with integral coefficients. Moreover, we have $e(\xi) \in H^n(B)$. The class U_ζ is called the fundamental class [3,4,5].

Definition 5. (The Gysin sequence)
For (n-dimensional real vector bundle) ξ^n there is the following exact sequence of cohomology groups where the coefficients are in $\mathbf{Z}_2$ in general or **Z** for oriented bundlesis called the Gysin sequence of the bundle (E_0, p, B) [8].

$$H^i(B) \xrightarrow{multbye(\xi)} H^{i+n}(B) \xrightarrow{p*} H^{i+n}(E_0) \xrightarrow{\psi} H^{i+n+1}(B)$$

Corollary 3. The Euler class of an odd-dimensional oriented bundle ξ has the property that $2e(\xi) = 0$ [3]. Because of this we assume that the fiber dimension is even when using the Euler classes.

Corollary 4. If the orientation of ξ is reversed, then the Euler class $e(\xi)$ changes sign $e(\xi') =- e(\xi)$ [3].

Proposition 6. Let ξ be a vector bundle over B, and let $f: B' \to B$ be a map. Then $f^*(e(\xi) = e(f^*(\xi)$.
Proof: There is a map $g:(E', E_0) \to (E, E_0)$ inducing f, where E is the total space of ξ and E' is the total space of $f^*(\xi)$. By the uniqueness property of Theorem (10), $f^*(U_\xi)$ is equal to $U_{f^*(\xi)}$, since they are equal on each fibre of $f^*(\xi)$. The proposition follows now from the commutative diagram.

$$\begin{array}{ccccc} H^*(E, E_0) & \to & H^*(E) & \to & H^*(B) \\ \downarrow g^* & & \downarrow g^* & & \downarrow f^* \end{array}$$

$$H^*(E', E'_0) \to H^*(E') \to H^*(B') \quad .$$

Corollary 5. If ξ is a trivial bundle of dimension n ≥ 1, then $e(\xi) = 0$ [3].

Theorem 6. For the Euler class, the relation $e(\xi\oplus\eta) = e(\xi)e(\eta)$ holds [5].
Proof: By Definition (of Euler class), we have $e(\xi\oplus\eta) = p^{*-1}j^*(U)$. Then we calculate $p^{*-1}j^*(U) = p^{*-1}j^*(q_1{}^*(U')q_2{}^*(U'')) = p^{*-1}[p_1{}^*j_1{}^*(U')p_2{}^*j_2{}^*(U'')] = e(\xi)e(\eta)$. This proves the theorem.

Theorem 7. If a vector bundle ξ has an everywhere-nonzero cross section, then $e(\xi)= 0$ [3].
Proof: A vector bundle with an everywhere-nonzero cross section splits off a line bundle, that is, $\xi =\theta^1\oplus\eta$. Then $e(\xi) = e(\theta^1)\oplus e(\eta) = 0e(\eta) = 0$. The Euler class of a trivial bundle is zero.

Proposition 7. For two vector bndle ξ and η the Cartesian product of Euler class is given by $e(\xi\times\eta) = e(\xi)\times w(\eta)$[3].
Proof: see [3].

8. Steenrod operations by Stiefel–Whitney classes

Definition 6. A cohomology operation of degree I with coefficients in group G is a morphism $\theta{:}H^*(,G)\to H^{*+i}(,G)$ of functors [3].

Theorem 8. For cohomology over the field $\mathbf{Z}_2$ of two elements, there is a unique operation $Sq^i{:}H^*(,\mathrm{F}_2)\to H^{*+i}(,\mathbf{Z}_2)$ of degree i such that Sq^i commutes with suspension and $Sq^i(x) =x^2$, the cup square, for $x\in H^i(X,\mathbf{Z}_2)$ [6].

The operation Sq^i is called the Steenrod square.

Theorem 9. The Steenrod squares satisfy the following properties [6].
(1) In degree 0, Sq^0 is the identity, and $Sq^i|H^n(,\mathbf{Z}_2) =0$ for $i>n$.
(2) For $x,y\in H^*(X,\mathbf{Z}_2)$, (Cartan formula) we have

$$Sq^k\ (xy) = \sum_{k=i+j} Sq^i(x)Sq^j(y).$$

Multiproduct version is

$$Sq^q(x_1 \ldots x_r) = \sum_{i(1)+\ldots+i(r)=q} Sq^{i(1)}(x_1)\ldots Sq^{i(r)}(x_r).$$

(3) For $0<a<2b$, (Adem relations) the iterate of squares satisfies

$$Sq^aSq^b = \sum_{j=0}^{[a/2]} \binom{a-1-j}{a-2j} Sq^{a+b-j}Sq^j .$$

On low-dimensional classes for Steenrod operations we have the following theorem.

Theorem 10. We consider dimensions one and two [6].

(1) If $x\in H^1(X,\mathbf{Z}_2)$, then we have $Sq^i(x^m)=\binom{m}{i}x^{m+i}$.

(2) If $y\in H^2(X,\mathbf{Z}_2)$ and if $Sq^1(y)=0$, then we have $Sq^{2i}(y^m)=\binom{m}{1}y^{m+i}$

And $Sq^{2i+1}(y^m)=0$.

Proof: When $m=0$ is clear by using induction on m. Statement (1) then is obtained as follows;

$Sq^i(x^m)=Sq^i(x.x^{m-1})=Sq^0(x).Sq^i(x^{m-1})+Sq^1(x).Sq^{i-1}(x^{m-1})$

$$=\binom{m-1}{i}+\binom{m-1}{i-1}x^{m+i}=\binom{m}{i}x^{m+i}.$$

Theorem11. Using the class $U_\xi\in H^n(D(\xi)\diagup S(\xi))$ and the total Steenrod operation $Sq=\sum_{0\leq i}Sq^i$ we have the following formula for the total Stiefel-Whitney class $Sq(U_\xi)=w(\xi)U_\xi$ *or* $w(\xi)=\varphi^{-1}(Sq(U_\xi))$, where D(ξ), S(ξ) is the associated disk bundle, and the associated sphere bundle respectively.

Proof: *S*ee [6].

Adem relations: (special cases) (1) For $a=1$, we have $1\leq b$, in this case the corresponding sum consists only the term for $j=0$, so that, we have

$$Sq^2Sq^b=\binom{b-1}{1}Sq^{b+1}=\begin{cases}Sq^{b+1} & \text{if } b \text{ is even}\\ 0 & \text{if } b \text{ is odd}\end{cases}$$

With simple cases $Sq^1Sq^1=0$, $Sq^1Sq^2=Sq^3$, $Sq^1Sq^3=0$, and $Sq^1Sq^4=Sq^5$ [6].

(2) For $a=2$, we have $2\leq b$, in this case the sum consists of only the two terms corresponding to $j=0$ and $j=1$. So that, we have

$$Sq^2Sq^b=\binom{b-1}{2}Sq^{b+2}+\binom{b-2}{0}Sq^{b+1}Sq^1.$$

It divides into two cases depending on b mod 4.

$$Sq^2Sq^b=Sq^{b+1}Sq^1+\begin{cases}Sq^{b+2} & \text{for} b\equiv 0,3(\text{mod}4)\\ 0 & \text{for} b\equiv 1,2(\text{mod}4)\end{cases}$$

With some simple cases $Sq^2Sq^2=Sq^3Sq^1$, $Sq^2Sq^3=Sq^4Sq^1+Sq^5$, $Sq^2Sq^4=Sq^5Sq^1+Sq^6$, $Sq^2Sq^5=Sq^6Sq^1$, and $Sq^2Sq^7=Sq^8Sq^1+Sq^9$.

To integers $\mathbf{Z}$, the binomial coefficient $\binom{n}{i}$ is the coefficient of x^i in the polynomial $(1+x)^n\in\mathbf{Z}[x]$. Here, they are understood as numbers modulo 2.

Example 2. (Two mod 2 congruences)

We have the following in the field $\mathbf{Z}_2=\{0,1\}$ of two elements for $c\in\mathbf{Z}$ [6]

$$\binom{c}{1} = \begin{cases} 0 & if \quad c \quad is\ even \\ 1 & if \quad c \quad is\ odd \end{cases}$$

and

$$\binom{c}{2} = \begin{cases} 0 & if \quad c = \quad 0,1 \pmod 4 \\ 1 & if \quad c = \quad 2,3 \pmod 4 \end{cases}$$

Definition 7. The Thom space of a vector bundle ξ, denoted ξ_B, is the quotient D(ξ)/S(ξ). The projection σ:$H^{i+n}(D(\xi), S(\xi) \to H^{i+n}$ (ξ_B) is an isomorphism, and we define the Thorn morphism ψ: H^i(B) → $\bar{H}^{i+n}$ (ξ_B) to be $\psi = \sigma\phi'$ [5].

Theorem 12. (Thom) The Stiefel- Whitney class $w_i(\xi)$ is given by$\phi^{-1}(Sq^iU_\xi)$.
$w_i(\xi) = \phi^{-1}(Sq^iU_\xi)$ [1].
By other mean the total Stiefel-Whitney class $w(\xi) = w_0(\xi) + w_1(\xi) + ...$ is given by $\phi^{-1}(Sq^i)$ $\phi(1)$, In terms of the total square Sq, and $w_i(\xi)$ is the unique class such that $\phi(w_i(\xi))=Sq^i\,\phi(1)$.

Theorem 13. The natural isomorphism $H^n(B; \mathbf{Z}) \to H^n(B; \mathbf{Z}_2)$ carries the Euler class $e(\xi)$ to the top Stiefel-Whitney class $w_n(\xi)$ [3].
Proof: If we apply this homomorphism (induced by the coefficient surjection $\mathbf{Z} \to \mathbf{Z}_2$) to both sides of $e(\xi) = \varphi^{-1}(u \sqcup u)$ then evidently the integer cohomology class u maps to the mod 2 cohomology class u defined before and $u \sqcup u$ maps to $Sq^n(u)$. Hence φ^{-1} $(u \sqcup u)$ maps to $\varphi^{-1}Sq^n(u) = w_n(u)$.

Definition 10. (Poincaré duality)
Let M be a compact m-dimensional manifold and let $\omega \in H^r(M)$ and $\mu \in H^{m-r}(M)$. Noting that $\omega\wedge\mu$ is a volume element, we define an inner product

$$\langle , \rangle : H^r(M) \times H^{m-r}(M) \to \mathbf{R} \text{ by } \langle \omega, \mu \rangle = \int_M \omega\wedge\mu \qquad (1)$$

The inner product is bilinear. Moreover, it is non-singular, that is, if $\omega = 0$ or $\mu \neq 0$, $\langle \omega, \mu \rangle$ cannot vanish identically. Thus, equation (1) defines the duality of $H^r(M)$ and $H^{m-r}(M)$,$H^r(M) \cong H^{m-r}(M)$ called the Poincaré duality [9].

9. Wu’s formula for the Stiefel–Whitney classes of a manifold

The Steenrod squares $Sq = \sum_i Sq^i$ and the Stiefel–Whitney class $w(\xi)$ of a bundle are related by the form $w(\xi) = \varphi^{-1}(Sq(U_\xi))$. Using Poincaré duality and its relation to U_M, we have the Wu class and its relation to the Stiefel–Whitney classes of the tangent bundle [3,6].

Corollary 6. Let Sq^{tr}: $H(X) \to H(X)$ denote the transpose of the (total) Steenrod square Sq:$H_*(X) \to H_*(X)$. In particular, we have $Sq(a),b = a\ ,Sq^{tr}(b)$ for $a \in H^*(X)\ ,b \in H_*(X)$ [6].

Definition 8. Let M be closed manifold with Poincaré duality is omorphism D: $H^i(M) \to H_{n-i}(M)$ and fundamental class $[M]$. The Wu class of M is $v = D^{-1}(Sq^{tr}([M]))$ [5,6].

The Wu class has the property that $\langle a, D(v)\rangle = \langle a, Sq^{tr}([M])\rangle = \langle Sq(a), [M]\rangle = \langle a, v \frown [M]\rangle = \langle av, [M]\rangle$.($\frown$) means cap product.

Theorem 14. Let M be a closed smooth manifold. Then, the Stiefel-Whitney class $w(M)=w(T(M))$ of the tangent bundle is given as the Steenrod square of the Wu class $w(M) = Sq(v)$ [6].
Proof: See [5]**.**

Corollary 7. The Stiefel–Whitney classes of closed manifolds are homotopy in- variants of the manifold [6].

Corollary 8. If $v=\sum_i v_i$,where $v_i \in H^i(M)$, then we have $v_i = 0$ *for* $2i > \dim(M)$ [5,6].
Proof: *See* [5].

9. The Atiyah class

We discuss the notion of holomorphic connection, which should not be confused with the notion compatible with the holomorphic structure. This notion is much more restrictive, but it generalize to pure algebraic setting [10].

Definition 9. And let $\{U_i\}$ be an open covering of X such that there are holomorphic trivializations $^{(12)}\psi_i:\xi_{U_i} \to U \times \mathbf{C}^r$, and transition function $\psi_{i,j} = \psi_i \psi_j^{-1}$: $\mathbf{C}^r \to \mathbf{C}^r$. Consider the differentials $d\psi_{i,j}:\mathbf{C}^r \to \mathbf{C}^r$, and the compositions $\psi_j^{-1}(\psi_j^{-1} d\psi_{i,j})\psi_j$.
Since $\{\psi_{i,j}\}$ is a cocycle also $\{\psi_j^{-1}(\psi_j^{-1} d\psi_{i,j})\psi_j\}$ is a cocycle. The class given by the *Čeach* cocycle $\{\psi_j^{-1}(\psi_j^{-1} d\psi_{i,j})\psi_j\}$ is denoted by $A(\xi) = \{U_{i,j}, \psi_j^{-1}(\psi_j^{-1} d\psi_{i,j})\psi_j\} \in H^1(X, \Omega_X \otimes End(\xi))$, and is called the Atiyah class of the holomorphic vector bundle ξ [10].

Proposition 8. A holomorphic vector bundle ξ admits a holomorphic connection if and only if its Atiyah class $A(\xi) \in H^1(X, \Omega_X \otimes End(\xi))$ is trivial [10].
Proof: *See* [10].

9.1. The Atiyah-Singer index theorem

Let D: $\Gamma(\xi) \to \Gamma(\eta)$ be an elliptic differential operator between vector bundles ξ and η on a compact oriented differentiable manifold X of dimension n, then [10];

(1) The topological index $\gamma(D)$ of the operator D is $ch(D)Td(X)$ $[X]$.
(2) The elliptic differential operator D has a pseudo inverse, it is a Fredholm operator. It's analytic index is defined as the difference between the finite dimension of $Ker(D)$ and the finite dimension Coker(D) i.e. $Index(D) = dimKer(D)\ dimCoKer(D) = dim\ Ker(D) dim\ Ker(D^*)$[4] *(where* D^* the adjoin of D)

Remark 3.

(i) $Td(X)$ is the Todd class of X,

(ii) $ch(D)$ is equal to $\phi^{-1}(ch(d(p^*\xi, p^*\eta, \sigma(D)),$

(iii) ϕ is the Thom isomorphism from $H^k(X,Q)$ to $H^{n+k}(B(X)/S(X),Q),$

(iv) $B(X)$ is the unit ball bundle of the cotangent bundle of X, and $S(X)$ is its boundary, and p is the projection to X.

(v) ch is the Chern character from K-theory K(X) to the rational cohomology ring $H(X,Q)$.

(vi) $d(p^*\xi, p^*\eta, \sigma(D))$ is the" difference element" of $K(B(X)/S(X))$ associated to two vector bundles $p^*\xi$ and $p^*\eta$ on $B(X)$ and an isomorphism $\sigma(D)$ between them on the subspace $S(X)$.

(vii) $\sigma(D)$ is the symbol of D.

Theorem15. (***Atiyah-Singer)***

Let D :$\Gamma(\xi) \to \Gamma(\eta)$ be an elliptic differential operator between vector bundles ξ and η on a compact oriented differentiable manifold of dimension n. Then the analytic index and the topological index of D are equal [4,10],

$$Index(D) = \gamma(D).$$

10. Relations between real and complex vector bundles

We have considered the operation of conjugation ξ^* of a complex vector bundle ξ, we can also restrict the scalars of ξ to **R**.

This yields a group homomorphism

ε_0: $K_c(X) \to K_R(X)$.

The process of tensoring a real vector bundle η with **C** yields a complex vector bundle $\eta \oplus$**C**, called the complexification of η. This yields a ringmorphism

ε_U: $K_R(X) \to K_c(X)$.

Clearly, there are the relations, $\varepsilon_0(\varepsilon_U(\eta))=2\eta$ and $\varepsilon_u(\varepsilon_0(\xi))= \xi + \xi^*$.

Observe that for a real vector bundle η the complex vector bundles $\eta\oplus$**C** and $(\eta\oplus$**C**$)^*$ are isomorphic [2].

Proposition 9. For a complex vector bundle ξ, the relation $c_i(\xi^*) = (-l)^i c_i(\xi)$ holds [5,11].

Proof: The proposition is true for line bundles. Let f: $B_1 \to B$ be a splitting map, where $f^*(\xi) = \lambda_1\oplus\cdots\oplus\lambda_n$. Then $c(f^*(\xi^*))= c(\lambda^*_1)\cdots c(\lambda^*_n) = (1-c_1(\lambda_1))\cdots (1-c_1(\lambda_n)) =\sum_{0\le i}(-1)^i c^i(\xi^*)$.

This proves the result.

Corollary 9. If a complex vector bundle ξ is isomorphic to ξ^*, then $2c_{2i+1}(\xi)=0$ for $0\le$ i(5,11).

Proof: We have $c_{2i+1}(\xi) = - c_{2i+1}(\xi) = - c_{2i+1}(\bar{\xi})$ or $2c_{2i+1}(\xi) = 0$.

The above corollary applies to the complexification $(\eta\oplus$**C**$)^*$ of a real vector bundle.

Theorem 16. Let ξ be the canonical (real) line bundle on $\mathbf{R}P^{2n-1}$ and η the canonical (complex) line bundle on $\mathbf{C}P^{n-1}$. Then the following statements apply [5].

(1) The bundle $q^*(\eta))$ is isomorphic to $\varepsilon_U(\eta)$.
(2) The class $w_2(\varepsilon_0(\eta))$ is the mod 2 reduction of $c_1(\eta)$.
(3) $c_l(\eta)) = e(\varepsilon_0(\eta))$.

Corollary10. Over a para compact space, $w_2(\varepsilon_0(\eta))$ is equal to the mod 2 restriction of $c_1(\eta)$ for any complex line bundle η [5].
Proof: The corollary is true for the universal bundle by condition (2) in theorem (16) and therefore for all complex line bundles.

Corollary11. Over a paracompact space, $e(\varepsilon_0(\eta))$ is equal to $c_1(\eta)$for any complex line bundle η [5].
Both corollaries are true for the universal bundles by theorem (5), and therefore they are verified for all line bundles, using the classifying maps.

Corollary 12. If a complex vector bundle ξ of rank n over B is given a canonical orientation then $e(\xi) = c_n(\xi) \in H^{2i}(B, \mathbf{Z})$ [3].

Theorem 17. Let ξ be a real vector bundle over a space B. Then ξ is orientable if and only if $w_I(\xi) =0$ [5].
Proof: First, we consider the case where $\xi = \lambda_1 \oplus \cdots \oplus \lambda_n$ is a Whitney sum of line bundles. Then the line bundle $\Lambda^n\xi$ has as coordinate transformations the determinant of the coordinate transformations of ξ. Therefore, ξ is orientable if and only if $\Lambda^n\xi$ is trivial, that is, $w_I(\Lambda^n\xi) = 0$. But $w_I(\Lambda^n\xi) = w_I(\lambda_1 \oplus \cdots \oplus \lambda_n) = w_I(\lambda_I) + ... + w_I(\lambda_n) = w_I(\xi)$. Therefore, ξ is orientable if and only if $w_I(\xi) = 0$.
For the general case, let $j: B_l \rightarrow B$ be a splitting map for ξ. Then ξ is orientable if and only if $f^*(\xi)$ is orientable, because $f^*(w_I(\Lambda^n\xi)) = w_I(\Lambda^n f^*(\xi)$ holds and f^* is a monomorphism. Finally, we have $w_I(\xi) = 0$ if and only if $w_I(f^*(\xi)) = 0$**.** This proves the theorem [5].

11. Pontrjagin class

Definition10. The ith Pontrjagin class of a real vector bundle ξ, denoted $P_i(\xi)$, is $(-1)^i c_{2i}(\xi \oplus \mathbf{C})$ which is a member of $H^{4i}(B, \mathbf{Z})$ [3,5].
We define $P(\xi) = 1 + P_1(\xi) + ... \in H^*(B,\mathbf{Z})$ to be the total Pontrjagin class of the vector bundle ξ. The Whitney sum theorem holds only in the following modified form:

$$2(p(\xi\oplus\eta) - p(\xi)p(\eta)) = 0$$

Let $q: \mathbf{R}P^{2n-1} \rightarrow \mathbf{C}P^{n-1}$be the map that assigns to each real line determined by {x, -x} for z $\in S^{2n-1}$ the complex line determined by z.
By definition we get $P_i(\xi_{\mathbf{R}}) = (-1)^i c_{2i}(\xi \oplus \mathbf{C})$ where$(\xi \oplus \mathbf{C})$ is complex get real [6].
In particular $P_1(\xi) = c_1(\xi)^2 - 2c_2(\xi)$ [6,12].

Theorem 18. For any orientable 2n-vector bundle ξ, $p_n(\xi) = e(\xi)^2$ [12].
Proof: $P_n(\xi) = (-1)^n c_{2n}(\xi_{\mathbf{C}}) == (-1)^n e(\xi_{CR}) = \pm e(\xi^n \oplus \xi^n) = == (-1)^n e(\xi)^2$**.**
For the tangent bundle, it is commonto write *p(M)* instead of curvature two form, we have Each Pontrjagin class is given by $p_0(M) = 1$.
Although $p_2(M)$ vanishes as a differential form [9,13,14].

Lemma 13. For any complex vector bundle ξ with fiber dimension n, the Chern classes determine the Pontrjagin classes by the formula [8,15].
$1-P_1(\xi) + P_2(\xi)-... =(1-c_1(\xi) +c_2(\xi) -\ldots)(1+c_1(\xi) +c_2(\xi) +\ldots)$.
Proof: $c(\xi\oplus_{\mathbf{R}}\mathbf{C}) =c(\xi)c(\xi^*) =\sum_{i=0}^{\infty} c_i(\xi)\sum_{i=0}^{\infty}(-1)^i c_i(\xi)$. Moreover, if k=1 (mod 2) then $c_k(\xi\oplus\mathbf{C})= \sum_{0\le i\le k}(-1)^i c_i(\xi)\, c_{k-1}(\xi) =0$. So the total Chern class is just the sum of all even Chern class [15].

Now the others useful characteristic classes associated with real or complex vector bundles are Atyiah class, Wu class and Atyiah-Singer class as we saw in the study, also how they interacted and developed the main characteristic classes in different terms.

Note: that the following two tables (1,2) give a good comparison between all characteristic classes which we have discussed.

Table 1: This table shows a comparison the four main characteristic classes which we had discussed.

Characteristic Class	Type	Method	Chern	Stiefel-Whitney	Euler
Chern	Complex Coefficient:**Z**Group =U(K)	$c_0(\xi)=1$ $c_1(\xi)=0$ $c_i(\xi)=0$ i>dimξ		All properties are similar	$c_1(L)=e(L)_R$
Stiefel-Whitney	Real Coefficient:$\mathbf{Z}_2$ Group=O(K)	$w_0(\xi)=1$ $w_1(\xi)=0$ $w_i(\xi)=0$ i>dimξ	How to compute the total-are similar		$w_n(\xi)=e(\xi)$
Euler	Orientable Real Coefficient: **Z** Group=O(K)	$2e(\xi)=0$ if dimξ is odd $e(\xi)=0$ if ξ nonzero cross sectionor trivial	$e(\xi)_R=c_n(\xi)$	$e(\xi)_R=w_1(\xi)$.	
Ponterjagin	Orientable Real into complex Coefficient **Z** Group=SO(K),O(K).	$P_0(M)=1$,where M tangent bundle $P_i(\xi)=0$ i>n/2	$P_n(\xi)=(1)^n c_{2n}(\xi_{\mathbf{C}})$ $P(\xi_R)=c(\xi)c(\xi^*)$ $P_1(\xi)=c_1(\xi)^2=-2c_2(\xi)$		$P_n(\xi)=e(\xi)^2$.

Table 2: This table shows the relations between Wu class, Atiyah class and Atiyah-Singer class with Stiefel-Whitney classes, concepts and theorems which we had discussed

Characteristic class	Type	Method	Index theorem	Holomorphic	Steenrod operation

Types and Properties of Characteristic Classes

Stiefel-Whitney	Real Coefficient: Z_2 Group=O(K)	$w_0(\xi)=1$ $w_1(\xi)=0$ $w_i(\xi)=0$ $i>\dim\xi$			$w(\xi)=\varphi^{-1}(SqU_\xi)$
Atiyah	complex			$A(\xi)\in H^1(X, \Omega_X\otimes End(\xi))$	
Wu	real	$v_i=0$ $for 2i>dim(M)$			
Atiyah - Singer	Oriented real		$Index(D)=\gamma(D)$		$w(M)=Sq(v)$

12. Conclusions

Our study shows a deep and strong relation between several types of characteristic classes, and how each of them develop anther by their relations, namely Ponterjagin and Euler classes to chern class, Steenrot square to Stiefel-Whitney, Steenrot with Wu class, atiyah-singer class with index theorem and atiyah class admits holomorhpic. So we have the real type of characteristic class in term of complex type.

The study also appears that complex characteristic are more easy and more useful to get new relationships.

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