

## On the Infinitude of Covering Systems with Least Modulus Equal to 2

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*Received 19 September 2017; accepted 22 September 2017*

*Dedicated to the memory of the late Prof. Paul Erdős friend and mentor*

**Abstract.** A finite set of residue classes  $a_i \pmod{n_i}$  with  $1 < n_1 < n_2 < \dots < n_s$  is called a covering system of congruences if every integer satisfies at least one of the congruences  $x \equiv a_i \pmod{n_i}$ . An example is the set  $\{0 \pmod{2}, 1 \pmod{3}, 3 \pmod{4}, 5 \pmod{6}, 9 \pmod{12}\}$ . A covering system all of whose moduli are odd called an odd covering system is a famous unsolved conjecture of Erdős and Selfridge. In this paper, we establish that there exist infinitely many even covering systems in which the least modulus is 2 and all other moduli are even. In each such even covering system, the number of the moduli and their prime factors are determined. Moreover, we construct a covering system with nine moduli, the smallest modulus is 2, and the lcm of the moduli is divisible by only the primes 2 and 5. With the smallest modulus 2, this is an attempt in the direction of constructing covering systems none of whose moduli is a product of the prime 3.

**Keywords:** Congruences, covering systems, exactly covering systems

**AMS Mathematics Subject Classification (2010):** 11B25, 11A07

### 1. Introduction

The concept of Covering systems was introduced and developed by Erdős. Many authors and numerous articles have been written on this interesting and non-trivial topic, and quite a wide literature exists on the subject.

We begin with some definitions and notations.

**Definition 1.1.** The system of congruences

$$x \equiv a_i \pmod{n_i}, \quad 1 < n_1 < n_2 < \dots < n_t, \quad 0 \leq a_i < n_i \quad (1)$$

is called a Covering System (abbreviated CS), if every integer satisfies at least one of the congruences (1).

**Definition 1.2.** The system of congruences

$$x \equiv a_i \pmod{n_i}, \quad 1 < n_1 \leq n_2 \leq \dots \leq n_k, \quad 0 \leq a_i < n_i \quad (2)$$

is called an Exactly Covering System (abbreviated ECS), if every integer satisfies exactly one of the congruences (2).

A system of congruences corresponding to any of the above two definitions, will be denoted as a set of ordered pairs of integers of the form  $(a, n)$ , where  $a$  is the residue class and  $n$  is the modulus.

The least common multiple of the moduli in a CS/ECS will be denoted by LCM.

The smallest and simplest example of a CS is the following set of ordered pairs

$$\{(0, 2), (0, 3), (1, 4), (5, 6), (7, 12)\}. \quad (3)$$

Hereafter, we shall refer to the set (3) as the basic CS.

A necessary condition [7] on the moduli  $n_1, n_2, \dots, n_t$  of a CS is

$$\sum_{i=1}^t \frac{1}{n_i} > 1.$$

**Definition 1.3.** An ECS, in which for every value  $m$  there are at most  $M$  moduli which are equal to  $m$ , will be called an  $\text{ECS}(M)$ .

It is well known [8, 7], that the moduli  $n_1 \leq n_2 \leq \dots \leq n_k$  of any ECS satisfy:

$$\sum_{i=1}^k \frac{1}{n_i} = 1 \quad (n_i, n_j) > 1 \quad (4)$$

and

$$n_{k-1} = n_k. \quad (5)$$

Condition (5) implies moreover that  $M = 1$  is impossible.

The simplest examples of ECS's are the following sets

$$\{(0, 2), (1, 2)\}, \quad \{(0, 2), (1, 4), (3, 4)\}, \quad \{(0, 2), (1, 4), (3, 8), (7, 8)\}. \quad (6)$$

Hereafter, we shall refer to sets described in (6) as basic ECS's. The moduli of each set in (6) satisfy the conditions in (4) and (5).

The basic ECS's in (6) are also known in the literature as Natural ECS's (abbreviated NECS's). As for  $\text{ECS's}(M)$  we also have accordingly  $\text{NECS's}(M)$ . Many results on ECS's and NECS's may also be found in [1, 2, 3, 4, 10, 16, 17, 18, 20, 21].

We shall now cite some known results, in particular those concerned with the least modulus  $n_1$  of a CS. Erdős asked whether there are CS's in which  $n_1 \geq R$  for arbitrary  $R$ . Over the last 60 years the value  $R$  increased. In 1968, Churchhouse [6] using a computer found CS's with  $n_1 = 2, 3, \dots, 9$ . In 1971, Krukenberg [13] gave examples of CS's for all values  $n_1$  up to 18 inclusive.

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Choi [5] found a CS with  $n_1 = 20$ . The current record belongs to Owens [15] with  $n_1 = 42$ . Hough showed the amazing result [12] that  $n_1$  is at most  $10^{16}$ . For this achievement he won the 2017 David P. Robbins prize. Other results may also be found for instance in [9, 13, 16, 19, 21].

## 2. On the infinitude of even CS's when the least modulus is 2

In this section, we exhibit a connection between an ECS and a CS in each of which the smallest modulus is 2. First, this is illustrated in two self-contained examples, namely Example 2.1 and Example 2.2 using two of the basic ECS's in (6) and the basic CS in (3). Secondly, the pattern described in these examples enables us to establish the general case, i.e., the existence of infinitely many even CS's with smallest modulus 2. This is done in Theorem 2.1.

**Example 2.1.** The basic ECS in (6) whose moduli are  $\{2, 4, 4\}$  combined with the basic CS in (3) whose moduli are  $\{2, 3, 4, 6, 12\}$  yield the set of seven moduli

$$\{2, 4, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 4 \cdot 6, 4 \cdot 12\},$$

which are even, distinct and  $\sum_{i=1}^7 \frac{1}{n_i} > 1$ . The respective CS is

$$\{(0, 2), (3, 4), (5, 8), (5, 12), (9, 16), (9, 24), (1, 48)\}.$$

**Example 2.2.** The basic ECS in (6) whose moduli are  $\{2, 4, 8, 8\}$  combined with the basic CS in (3) whose moduli are  $\{2, 3, 4, 6, 12\}$  yield the set of eight moduli

$$\{2, 4, 8, 8 \cdot 2, 8 \cdot 3, 8 \cdot 4, 8 \cdot 6, 8 \cdot 12\},$$

which are even, distinct and  $\sum_{i=1}^8 \frac{1}{n_i} > 1$ . The respective CS is

$$\{(0, 2), (3, 4), (1, 8), (5, 16), (13, 24), (29, 32), (45, 48), (77, 96)\}.$$

Example 2.1 and Example 2.2, clearly show the pattern which connects a basic ECS with the basic CS. The general case will now be established for all basic ECS's of the form described in (6) combined with the basic CS in (3).

**Theorem 2.1.** For each and every value  $t \geq 1$ , the set of ordered pairs

$$\{(0, 2), (1, 4), (3, 8), (7, 16), \dots, (2^{t-1} - 1, 2^t), (2^t - 1, 2^t)\} \quad (7)$$

represents a basic ECS. Let  $t \geq 1$  be any fixed value. Then:

- (i) The set
$$\{2, 4, 8, 16, \dots, 2^t, 2^t \cdot 2, 2^t \cdot 3, 2^t \cdot 4, 2^t \cdot 6, 2^t \cdot 12\} \quad (8)$$
is a set of moduli of a CS.
- (ii) There are infinitely many CS's whose moduli satisfy (8).
- (iii) All the moduli in (8) are even and distinct.
- (iv) In (8), the LCM is  $2^t \cdot 12$  whose prime divisors are 2 and 3.

(v) The number  $k$  of the moduli in (8) is equal to  $k = t + 5$ .

**Proof:** For (i), recall from (3) the moduli  $\{2, 3, 4, 6, 12\}$  of the basic CS whose  $\text{LCM} = 12$ . It is noted that if the integers  $1 - 12$  are covered, then all the integers are covered, and the set of moduli is a CS. As for the moduli  $2, 4$  in (3), the moduli  $2 \cdot 2^t, 4 \cdot 2^t$  in (8) and their values  $a_i$  yield up to the new  $\text{LCM} = 12 \cdot 2^t$  respectively six and three integers all of which are distinct with no overlaps. The other three moduli in (8), namely  $3 \cdot 2^t, 6 \cdot 2^t$  and  $12 \cdot 2^t$  and their appropriate values  $a_i$ , each yields exactly one integer up to  $12 \cdot 2^t$ , whereas all other obtained integers are overlaps. Thus, all twelve integers are covered up to  $12 \cdot 2^t$  as required, and (8) is a set of moduli of a CS as asserted.

As for (ii), each value  $t \geq 1$  in (7) yields a basic ECS. Hence, there are infinitely many basic ECS's. Then, for each and every value  $t \geq 1$  there exists a CS. Thus, there exist infinitely many CS's whose moduli satisfy (8).

The statements in (iii), (iv) and (v) follow directly from (8).

This completes the proof of Theorem 2.1.  $\square$

The following Corollary 2.1 follows from Theorem 2.1.

**Corollary 2.1.** The sum of the reciprocals of the five moduli in (3) is

$\sum_{i=1}^5 \frac{1}{n_i} = \frac{4}{3}$ . The sum of the reciprocals of the  $(t + 5)$  moduli in (8) is equal to

$$\sum_{i=1}^{t+5} \frac{1}{n_i} = \frac{2^t - 1}{2^t} + \frac{1}{2^t} \cdot \frac{4}{3} = \frac{3 \cdot 2^t + 1}{3 \cdot 2^t} = 1 + \frac{1}{3 \cdot 2^t}.$$

For arbitrarily large  $t$ , it follows that the reciprocals of the moduli of the CS in (8) have a sum which is as close to 1 as we wish, but is never equal to 1.

**Remark 2.1.** In Theorem 2.1, we have used the basic CS having the five moduli  $2, 3, 4, 6, 12$ . As the least modulus increases, the larger are the number of the moduli and so are their prime factors. Evidently, for the purposes of Theorem 2.1, any known CS will suffice instead of the basic CS. However, constructing a CS union of a basic ECS and the basic CS, has its advantages primarily in (i) the smallest number of obtained moduli, and (ii) their largest prime divisor which is  $p = 3$ .

### 3. On a CS whose least modulus is 2 and the LCM is $2^4 \cdot 5$

In the literature, all CS's with least modulus 2, have LCM's which are divisible by  $p = 3$ . Hough and Nielsen have shown that every CS has a modulus divisible by either 2 or 3.

In this section, we exhibit Example 3.1 i.e., a CS of nine moduli containing the least modulus 2, the modulus 5, and the  $\text{LCM } 2^4 \cdot 5 = 80$ . This is the first CS none of whose moduli is divisible by  $p = 3$ .

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**Example 3.1.** The following set of nine ordered pairs is a CS in which except for 1, the nine moduli are all the divisors of 80.

$\{(1, 2), (0, 4), (0, 5), (2, 8), (6, 10), (6, 16), (14, 20), (22, 40), (78, 80)\}$ .

A CS with least modulus 2 and the only odd modulus 5 does not exist when  $k < 9$ . Hence, Example 3.1 is a CS whose number of moduli  $k = 9$  is minimal. For any other CS of the same nature, it follows that  $k > 9$ . Moreover, as mentioned earlier, Example 3.1 now implies that there exists at least one CS which does not contain a modulus divisible by 3, but rather, moduli divisible by 2.

**Final Remark.** In view of Example 3.1, we may now presume that for primes  $p \geq 7$ , there exists a CS containing the least modulus 2 and the  $\text{LCM} = 2^r \cdot p$ .

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