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On the Wiener Index of Some Total Graphs

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Abstract. The total graph $T(G)$ of a graph $G=(V,E)$ is that graph whose vertex set is $V \cup E$, and two vertices are adjacent if and only if they are adjacent or incident in G. For a fan graph $F_{1,n}$, the graph $F_{1,n}.F_{1,m}$ is obtained by identifying a vertex corresponding to $\overline{K}_1$ of $F_{1,n}$ to one vertex corresponding to $\overline{K}_1$ of $F_{1,m}$. In this note, we investigate the Wiener index of the total graphs of $F_{1,n}.F_{1,m}$, $K_{m,n}$, $F_{1,n}$, $F_{2,n}$ and $K_2 \square C_n$.

Keywords: Wiener index; total graph; fan graph

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1. Introduction

A fan graph $F_{m,n}$ is defined as the graph join $\overline{K}_m + P_n$, where $\overline{K}_m$ is the empty graph on m nodes and P_n is the path graph on n vertices. The case $m=1$ corresponds to the usual fan graphs, while $m=2$ corresponds to the double fan graph. A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V, i.e., U and V are each independent sets. A *complete bipartite graph* $K_{m,n}$ is a special kind of bipartite graph where every vertex of the first vertex set is adjacent to every vertex of the second vertex set, where m and n are number of vertices in the first and second vertex set respectively. Let G and H be two graphs. The *cartesian product* or *product* $G \square H$ of G and H is a graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and any two vertices (u,u') and (v,v') are adjacent in $G \square H$ if and only if either

- $u=v$ and u' is adjacent with v' in H, or
- $u'=v'$ and u is adjacent with v in G.

For a graph $G=(V,E)$ if u, $v \in V(G)$, then the *distance* $d(u,v)$ between u and v is defined as the length of a shortest u-v path in G. The *Wiener index* of G

is defined as

$$W(G) = \sum_{\{u,v\}\subset V} d(u,v).$$

It is the oldest topological index and its mathematical properties and chemical applications have been extensively studied. The Wiener index was introduced by the chemist Harold Wiener in 1947 for explaining the correlation between the boiling points of paraffins and the structure of their molecules. It is a graph invariant much studied in both mathematical and chemical literature (*see* [12-14,17,21]). The concept of graph operator has found various applications in chemical research (*see* [8-11,16,18,19,22]).

The *total graph* $T(G)$ of a graph $G = (V, E)$ is that graph whose vertex set is $V \cup E$, and two vertices are adjacent if and only if they are adjacent or incident in G. It is introduced by Behzad & Chartrand [5]. Several properties of total graphs are investigated in the literature (*see* [1-4,6,7,15,20]). The total graph $H = T(G)$ of G is shown in Figure 1.

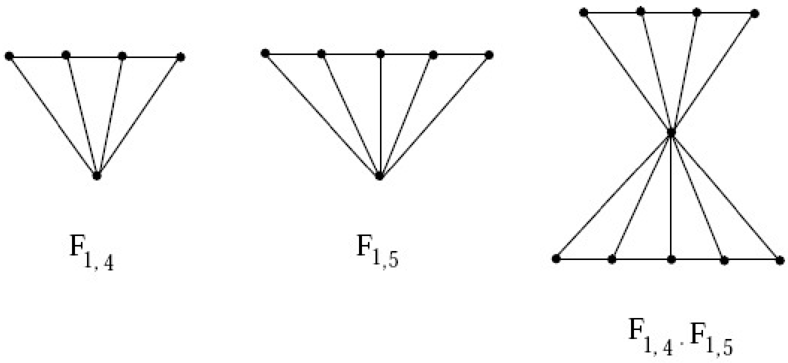

Figure 1: A graph G and its total graph $H = T(G)$

We define a new graph operator $F_{1,n}.F_{1,m}$ defined as follows: For a fan graph $F_{1,n}$, the graph $F_{1,n}.F_{1,m}$ is obtained by identifying a vertex corresponding to $\overline{K}_1$ of $F_{1,n}$ to one vertex corresponding to $\overline{K}_1$ of $F_{1,m}$. The graphs $F_{1,4}$, $F_{1,5}$ and $F_{1,4}.F_{1,5}$ are shown in Figure 2.

2. **Wiener index of total graph of fan graphs**

Theorem 2.1. *The Wiener index of the graph* $G = F_{1,n}$,

$$W(G) = n^2 - n + 1.$$

Proof: We know that

$$W(F_{1,n}) = \sum_{u,v\in V(F_{1,n})} d(u,v).$$

Therefore,

$$= n.1 + (n-1).1 + [(n-2) + (n-3) + ... + 1].2$$
$$= n^2 - n + 1.$$

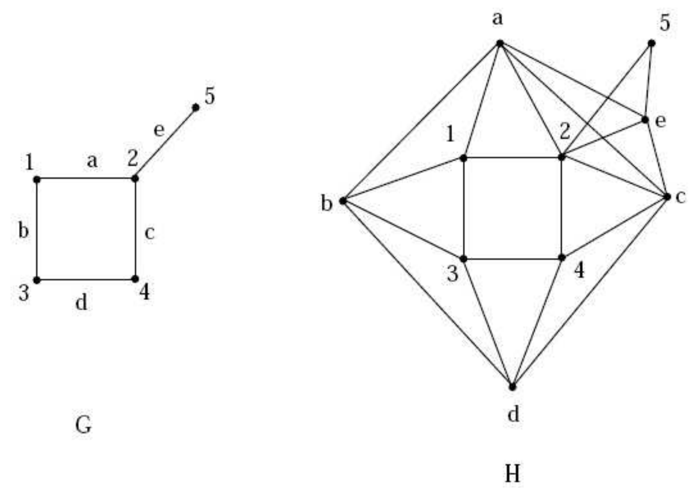

Figure 2: The graphs $F_{1,4}$, $F_{1,5}$, $F_{1,4}.F_{1,5}$

Theorem 2.2. *The Wiener index of graph* $G = T(F_{1,n})$,

$$W(G) = 10n^2 - 20n + 19.$$

Proof: We know that

$$W(T(F_{1,n})) = \sum_{u,v \in V(T(F_{1,n}))} d(u,v)$$

$$= \sum_{u,v \in V(F_{1,n})} d(u,v) + \sum_{\substack{u \in V(F_{1,n}) \\ e \in E(F_{1,n})}} d(u,e) + \sum_{e,f \in E(F_{1,n})} d(e,f)$$

$$= W(F_{1,n}) + \sum_{\substack{u \in V(F_{1,n}) \\ e \in E(F_{1,n})}} d(u,e) + \sum_{e,f \in E(F_{1,n})} d(e,f).$$

Here,

$$W(F_{1,n}) = n^2 - n + 1 \quad \text{(usingTheo rem2.1)}$$

$$\sum_{\substack{u \in V(F_{1,n}) \\ e \in E(F_{1,n})}} d(u,e) = 5n^2 - 7n + 6$$

and

$$\sum_{e,f \in E(F_{1,n})} d(e,f) = 4n^2 - 12n + 12.$$

Thus,

$$W(T(F_{1,n})) = (n^2 - n + 1) + (5n^2 - 7n + 6) + (4n^2 - 12n + 12)$$
$$= 10n^2 - 20n + 19.$$

Hence,

$$W(T(F_{1,n})) = 10n^2 - 20n + 19.$$

Theorem 2.3. *The Wiener index of graph* $G = F_{2,n}$,

$$W(G) = n^2 + 3.$$

Proof: We know that

$$W(F_{2,n}) = \sum_{u,v \in V(F_{2,n})} d(u,v)$$
$$= n.1 + n.1 + (n-1).1 + 2 + [(n-2) + (n-3) + ... + 2 + 1].2$$
$$= n^2 + 3.$$

Hence,

$$W(F_{2,n}) = n^2 + 3.$$

Theorem 2.4. *The Wiener index of graph* $G = T(F_{2,n})$,

$$W(G) = \frac{33n^2 - 37n + 42}{2}.$$

Proof: We know that

$$W(T(F_{2,n})) = \sum_{u,v \in V(T(F_{2,n}))} d(u,v)$$
$$= \sum_{u,v \in V(F_{2,n})} d(u,v) + \sum_{\substack{u \in V(F_{2,n}) \\ e \in E(F_{2,n})}} d(u,e) + \sum_{e,f \in E(F_{2,n})} d(e,f)$$
$$= W(F_{2,n}) + \sum_{\substack{u \in V(F_{2,n}) \\ e \in E(F_{2,n})}} d(u,e) + \sum_{e,f \in E(F_{2,n})} d(e,f).$$

Here,

$$W(F_{2,n}) = n^2 + 3 \quad \text{(usingTheorem2.3)}$$

$$\sum_{\substack{u \in V(F_{2,n}) \\ e \in E(F_{2,n})}} d(u,e) = 7n^2 - n + 4$$

and

$$\sum_{e,f \in E(F_{2,n})} d(e,f) = \frac{17n^2 - 35n + 28}{2}.$$

Thus,

$$W(T(F_{2,n})) = (n^2 + 3) + (7n^2 - n + 4) + \frac{17n^2 - 35n + 28}{2}$$

$$= \frac{33n^2 - 37n + 42}{2}.$$

Hence,

$$W(T(F_{2,n})) = \frac{33n^2 - 137n + 42}{2}.$$

Theorem 2.5. *The Wiener index of graph* $G = F_{1,m}.F_{1,n}$,

$$W(G) = (m+n)^2 - (m+n) + 2.$$

Proof: We know that

$$W(F_{1,m}.F_{1,n}) = \sum_{u,v \in V(F_{1,m}.F_{1,n})} d(u,v)$$

$$= m.1 + n.1 + (m-1).1 + [(m-2) + (m-3) + ... + 1].2 + mn.2 + (n-1).1 + [(n-2) + (n-3) + ... + 1].2$$

$$= (m+n)^2 - (m+n) + 2.$$

Hence,

$$W(F_{1,m}.F_{1,n}) = (m+n)^2 - (m+n) + 2.$$

Theorem 2.6. *The Wiener index of graph* $G = T(F_{1,m}.F_{1,n})$,

$$W(G) = 10(m+n)^2 - 28(m+n) + 41.$$

Proof: We know that

$$W(T(F_{1,m}.F_{1,n})) = \sum_{u,v \in V(T(F_{1,m}.F_{1,n}))} d(u,v)$$

$$= \sum_{u,v \in V(F_{1,m}.F_{1,n})} d(u,v) + \sum_{\substack{u \in V(F_{1,m}.F_{1,n}) \\ e \in E(F_{1,m}.F_{1,n})}} d(u,e) + \sum_{e,f \in E(F_{1,m}.F_{1,n})} d(e,f)$$

$$= W(F_{1,m}.F_{1,n}) + \sum_{\substack{u \in V(F_{1,m}.F_{1,n}) \\ e \in E(F_{1,m}.F_{1,n})}} d(u,e) + \sum_{e,f \in E(F_{1,m}.F_{1,n})} d(e,f).$$

Here,

$$W(F_{1,m}.F_{1,n}) = (m+n)^2 - (m+n) + 2 \quad \text{(usingTheorem2.5)}$$

$$\sum_{\substack{u \in V(F_{1,m}.F_{1,n}) \\ e \in E(F_{1,m}.F_{1,n})}} d(u,e) = 5(m+n)^2 - 10(m+n) + 12$$

and

$$\sum_{e,f \in E(F_{1,m}.F_{1,n})} d(e,f) = 4(m+n)^2 - 17(m+n) + 27.$$

Thus,

$$W(T(F_{1,m}.F_{1,n})) = ((m+n)^2-(m+n)+2)+(5(m+n)^2-10(m+n)+12)$$
$$+4(m+n)^2-17(m+n)+27)$$
$$=10(m+n)^2-28(m+n)+41.$$

Hence,

$$W(T(F_{1,m}.F_{1,n}) = 10(m+n)^2-28(m+n)+41.$$

3. Wiener index of total graph of $K_2 \mathrm{W} C_n$

Theorem 3.1. *The Wiener index of the graph* $G = K_2 \mathrm{W} C_n$,

$$W(G) = \begin{cases} \dfrac{n(n^2+2n-1)}{2}, nisodd \\ \dfrac{n^2(n+2)}{2}, niseven. \end{cases}$$

Proof: Since,

$$W(K_2 \mathrm{W} C_n) = \sum_{u,v\in V(2C_n)} d(u,v)$$

To calculate $W(K_2 \mathrm{W} C_n)$, we consider two cases:

Case 1: Suppose n is odd. Then,

$$W(K_2 \mathrm{W} C_n) = 3n.1+4n\left[2+3+.....+\frac{n-1}{2}\right]+2n.\left(\frac{n-1}{2}+1\right).$$

Thus,

$$W(K_2 \mathrm{W} C_n) = \frac{n(n^2+2n-1)}{2}.$$

Case 2: Suppose n is even. Then,

$$W(K_2 \mathrm{W} C_n) = 3n.1+4n\left[2+3+......+\left(\frac{n}{2}-1\right)\right]+3n.\frac{n}{2}+n.\left(\frac{n}{2}+1\right)$$

Thus,

$$W(K_2 \mathrm{W} C_n) = \frac{n^2(n+2)}{2}.$$

Theorem 3.2. *The Wiener index of graph* $G = T(K_2 \mathrm{W} C_n)$,

$$W(T(K_2 \mathrm{W} C_n)) = 3n^3+9n^2+2.$$

Proof: We know that

$$W(T(K_2 \mathrm{W} C_n)) = \sum_{u,v\in V(T(K_2 \mathrm{W} C_n))} d(u,v)$$
$$= \sum_{u,v\in V(K_2 \mathrm{W} C_n)} d(u,v) + \sum_{\substack{u\in V(K_2 \mathrm{W} C_n) \\ e\in E(K_2 \mathrm{W} C_n)}} d(u,e) + \sum_{e,f\in E(K_2 \mathrm{W} C_n)} d(e,f)$$

$$= W(K_2 \mathrm{W} C_n) + \sum_{\substack{u \in V(K_2 \mathrm{W} C_n) \\ e \in E(K_2 \mathrm{W} C_n)}} d(u,e) + \sum_{e,f \in E(K_2 \mathrm{W} C_n)} d(e,f)$$

To calculate $W(T(K_2 \mathrm{W} C_n))$, we consider two cases:

Case 1: Suppose n is odd. Then,

$$W(K_2 \mathrm{W} C_n) = \frac{n(n^2 + 2n - 1)}{2} \quad \text{(usingTheorem3.1)}$$

$$\sum_{\substack{u \in V(K_2 \mathrm{W} C_n) \\ e \in E(K_2 \mathrm{W} C_n)}} d(u,e) = \frac{n(3n^2 + 12n + 1)}{2}$$

$$\sum_{e,f \in E(K_2 \mathrm{W} C_n)} d(e,f) = n^3 + 2n^2 + n.$$

Therefore,

$$W(T(K_2 \mathrm{W} C_n)) = \frac{n(n^2 + 2n - 1)}{2} + \frac{n(3n^2 + 12n + 1)}{2} + (n^3 + 2n^2 + n)$$

$$= 3n^3 + 9n^2 + n.$$

Case 2: Suppose n is even. Then,

$$W(K_2 \mathrm{W} C_n) = \frac{n^2(n+2)}{2} \quad \text{(usingTheorem3.1)}$$

$$\sum_{\substack{u \in V(K_2 \mathrm{W} C_n) \\ e \in E(K_2 \mathrm{W} C_n)}} d(u,e) = \frac{3n^2(n+4)}{2}$$

and

$$\sum_{e,f \in E(K_2 \mathrm{W} C_n)} d(e,f) = n^3 + 2n^2 + n.$$

Therefore,

$$W(T(K_2 \mathrm{W} C_n)) = \frac{n^2(n+2)}{2} + \frac{3n^2(n+4)}{2} + (n^3 + 2n^2 + n)$$

$$= 3n^3 + 9n^2 + n.$$

Hence,

$$W(T(K_2 \mathrm{W} C_n)) = 3n^3 + 9n^2 + n.$$

4. Wiener index of total graph of $K_{m,n}$

Theorem 4.1. *The Wiener index of the graph* $G = T(K_{m,n})$,

$$W(G) = \frac{1}{2}\left[(2m^2 + 2)(n^2 + 1) + (3mn - 2)(m + n) - 2(mn + 1)\right]$$

Proof: Since,

$$W(T(K_{m,n})) = \sum_{u,v \in V(T(K_{m,n}))} d(u,v)$$

$$= \sum_{u,v \in V(K_{m,n})} d(u,v) + \sum_{e,f \in E(K_{m,n})} d(e,f) + \sum_{\substack{u \in V(K_{m,n}), \\ e \in E(K_{m,n})}} d(u,e).$$

Now,

$$\sum_{u,v \in V(K_{m,n})} d(u,v) = mn(1) + \left({}^{m+n}C_2 - mn\right)(2)$$

$$\sum_{\substack{u \in V(K_{m,n}), \\ e \in E(K_{m,n})}} d(u,e) = 2mn(1) + \left(m(mn-n) + n(mn-m)\right)(2)$$

$$\sum_{e,f \in E(K_{m,n})} d(e,f) = \frac{m^2n + n^2m - 2mn}{2}(1) + \frac{m^2n^2 - m^2n - n^2m + mn}{2}(2).$$

Thus,

$$W(T(K_{m,n})) = \left(mn(1) + \left({}^{m+n}C_2 - mn\right)(2)\right) + \left(2mn(1) + \left(m(mn-n) + n(mn-m)\right)(2)\right)$$

$$+ \left(\frac{m^2n + n^2m - 2mn}{2}(1) + \frac{m^2n^2 - m^2n - n^2m + mn}{2}(2)\right)$$

$$= \frac{1}{2}\left[(2m^2 + 2)(n^2 + 1) + (3mn - 2)(m+n) - 2(mn+1)\right].$$

5. Conclusion

In this article, we have investigated the results related to the Wiener index of the total graph of five particular graphs, namely, $F_{1,n}$, $F_{2,n}$, $F_{1,n}.F_{1,m}$, $K_2 W C_n$ and $K_{m,n}$.

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