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Separation Axioms and Graphs of Functions in Nano Topological Spaces via Nano β-open Sets

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Abstract. In this paper, we have introduced some new separation axioms in nano topological spaces in terms of nano β-open sets along with their basic properties. Two new types of graphs, viz. nano β-closed graphs and strongly nano β-closed graphs of functions between two nano topological spaces are initiated in terms of nano β-open sets. We have established some characterizations of functions having these type of graphs. Moreover, some applications of these graphs on the separation axioms defined here are also achieved.

Keywords: nano β-closure, $n\beta$ - $T_i (i = 1,2)$ spaces; $n\beta$-Urysohn, nano β-closed graph, strongly nano β-closed graph

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1. Introduction

The concept of nano topology and nano open sets were introduced by Thivagar [4] in terms of approximations and boundary region of a universal set using an equivalence relation on it. Some recent works on nano topological spaces can be found in [5, 7, 10, 11]. Beside these, Nasef et al. [8] have investigated some of the properties of nano near open sets and nano continuity and have shown some application examples in nano topology in real life situation. Recently, Azzam [3] have introduced the concept of grill in nano topological spaces and discussed about some usefulness of nano topology. On the other hand, Monsef et al. [1] introduced the notion of β-open sets (=semi-preopen sets [2]) and since its introduction such sets along with some of their relevant concepts have been investigated by many researcher.

In the present paper, we have introduced nano β-closures and which have been used in investigating certain concepts developed in the subsequent sections. In section 4, some new separation axioms have been introduced in a nano topological space using nano β-open sets along with various characterizations and properties. Furthermore, in the last two sections, two new types of functions, namely nano β-closed graph and strongly nano β-closed graph have been introduced between two nano topological spaces. Some characterizations and basic properties along with possible applications of such functions are also investigated.

2. Preliminaries

Let Ω be a nonempty finite set called the universe and R be an equivalence relation on Ω. Then the pair (Ω, R) is called an approximation space. The equivalence of $x \in \Omega$ is denoted by $R(x)$. Let $X \in P(\Omega)$. Then we define the sets

$$L_R(X) = \bigcup_{x\in\Omega}\{R(x): R(x) \subset X\}, H_R(X) = \bigcup_{x\in\Omega}\{R(x): R(x) \cap X \neq \varnothing\}$$

and $B_R(X) = L_R(X) - H_R(X)$. Here the sets $L_R(X)$, $H_R(X)$ and $B_R(X)$ are called lower approximation of $(\Omega, \tau_R(X))$, upper approximation of $(\Omega, \tau_R(X))$ and boundary region of X with respect to R respectively. Then $\tau_R = \{\Omega, \varnothing, H_R(X), B_R(X), L_R(X)\}$ is a topology on Ω with base $\tau_R(X) = \{\Omega, L_R(X), B_R(X)\}$ [4]. This topology is called a nano topology with respect to the subset $(\Omega, \tau_R(X))$ of the universe Ω and the pair $(\Omega, \tau_R(X))$ is called a nano topological space with respect to the subset X of the universe Ω. The members of $\tau_R(X)$ are called nano open sets [4] and their complements are called nano closed sets [4]. Let A be a subset of a nano topological space $(\Omega, \tau_R(X))$. Then the largest nano open set contained in A is called the nano interior of A [4] and is denoted by $nint(A)$ and the smallest nano closed set containing A is called the nano closure of A [4] and is denoted by $ncl(A)$.

A subset A of a nano topological space $(\Omega, \tau_R(X))$ is called nano β-open [9] if $A \subset ncl(nint(ncl(A)))$. The family of all nano β-open subsets of a nano topological space $(\Omega, \tau_R(X))$ is denoted by $N\beta O(\Omega, R, X) = N\beta O(\Omega, X)$. The family of all nano β-open subsets of a nano topological space $(\Omega, \tau_R(X))$ containing $x \in \Omega$ is denoted by $N\beta O(\Omega, R, X; x) = N\beta O(\Omega, X; x)$. The complement of a nano β-open set is called a nano β-closed set. The family of all nano β-closed subsets of a nano topological space $(\Omega, \tau_R(X))$ is denoted by $N\beta C(\Omega, R, X) = N\beta C(\Omega, X)$.

3. Nano β-closure operators

Some of the concepts and results developed here will be used in the subsequent sections.

Definition 3.1. *A nano topological space $(\Omega, \tau_R(X))$ is said to satisfy a property nP if $ncl(A \cap B) = ncl(A) \cap ncl(B)$ for every pair of subsets A and B of a nano topological space $(\Omega, \tau_R(X))$.*

Theorem 3.2. *(a) Arbitrary union of nano β-open sets of a nano topological space $(\Omega, \tau_R(X))$ is a nano β-open set.*

(b) If a nano topological space $(\Omega, \tau_R(X))$ satisfies the property nP, then the intersection of any two nano β-open sets is nano β-open and so $n\beta O(\Omega, X)$ is a topology on Ω finer than nano topology $\tau_R(X)$.

Proof: (a): Obvious.

(b) Let A and B are any two nano β-open sets. Then $A \subset ncl(nint(ncl(A)))$ and $B \subset ncl(nint(ncl(B)))$.

Now $A \cap B \subset ncl(nint(ncl(B))) = ncl(nint(ncl(A)) \cap nint(ncl(B)))$
$= ncl(nint(ncl(A) \cap ncl(B))) = ncl(nint(ncl(A \cap B)))$ and so $A \cap B$ is nano β-open.

Definition 3.3. *Let A be a subset of a nano topological space $(\Omega, \tau_R(X))$. Then $n\beta$-interior (resp. $n\beta$-closure) of A is denoted by $n\beta int(A)$ (respectively $n\beta cl(A)$) and is defined as the set $n\beta int(A) = \cup\{G \subset A : G \in n\beta O(\Omega, X)\}$ (respetively $n\beta cl(A) = \cap\{F \supset A : \Omega - F \in n\beta O(\Omega, X)\}$).*

Theorem 3.4. *For any subsets A and B of a nano topological space $(\Omega, \tau_R(X))$, the following statements hold:*

(i) $x \in n\beta cl(A)$ if and only if $A \cap U \neq \varnothing$ for each $U \in N\beta O(\Omega, X; x)$;
(ii) A is nano β-open if and only if $A = n\beta int(A)$;
(iii) A is nano β-closed if and only if $A = n\beta cl(A)$;
(iv) If $A \subset B$ then $n\beta int(A) \subset n\beta int(B)$ and $n\beta cl(A) \subset n\beta cl(B)$;
(v) $n\beta cl(\Omega - A) = \Omega - n\beta int(A)$;
(vi) $nint(A) \subset n\beta int(A)$;
(vii) $ncl(A) \supset n\beta cl(A)$.

Proof: Straightforward.

4. Separation axioms in terms of nano β-open sets

Definition 4.1. *A space $(\Omega, \tau_R(X))$ is called*

(i) $n\beta$-T_1 if for each pair of distinct points $x, y \in \Omega$, there exist an $U_x \in N\beta O(\Omega, X; x)$ and an $U_y \in N\beta O(\Omega, X; y)$ such that $x \notin U_y$ and $y \notin U_x$.

(ii) $n\beta$ - T_2 if for each pair of distinct points $x, y \in \Omega$, there exist an $U_x \in N\beta O(\Omega, X; x)$ and an $U_y \in N\beta O(\Omega, X; y)$ such that $U_x \cap U_y = \varnothing$.

(iii) $n\beta$-Urysohn if for each pair of distinct points $x, y \in \Omega$, there exist an $U_x \in N\beta O(\Omega, X; x)$ and an $U_y \in N\beta O(\Omega, X; y)$ such that $n\beta cl(U_x) \cap n\beta cl(U_y) = \varnothing$.

(iv) $n\beta$-regular if for each point $x \in \Omega$ and each nano β-closed set F such that $x \notin F$, there exist a $V \in N\beta O(\Omega, X; x)$ and a $W \in N\beta O(\Omega, X)$ such that $F \subset W$ and $V \cap W = \varnothing$.

Remark 4.2. *A* $n\beta$ *-Urysohn nano topological space is a* $n\beta$ *-* T_2 *nano topological space and a* $n\beta$ *-* T_2 *nano topological space is a* $n\beta$ *-* T_1 *nano topological space.*

The following characterizations of $n\beta$ - T_1, $n\beta$ - T_2 and $n\beta$ -regular spaces are straightforward.

Theorem 4.3. *A nano topological space* $(\Omega, \tau_R(X))$ *is* $n\beta$ *-* T_1 *if and only if singletons of* $(\Omega, \tau_R(X))$ *are nano* β *-closed.*

Theorem 4.4. *For a nano topological space* $(\Omega, \tau_R(X))$*, the following statements are equivalent:*
(a) $(\Omega, \tau_R(X))$ is $n\beta$ - T_2;
(b) For each $x \in \Omega$ and $y(\neq x) \in \Omega$, there exist an $U_x \in N\beta O(\Omega, X; x)$ such that $y \notin n\beta cl(U_x)$.
(c) For each $x \in \Omega$, $\cap\{n\beta cl(U) : U \in n\beta O(\Omega, X; x)\} = \{x\}$.

Theorem 4.5. *A nano topological space* $(\Omega, \tau_R(X))$ *is* $n\beta$ *-regular if and only if for each* $x \in \Omega$ *and each* $U \in N\beta O(\Omega, X; x)$*, there exists a* $V \in N\beta O(\Omega, X; x)$ *such that* $n\beta\text{-}cl(V) \subset U$.

Theorem 4.6. *Every* $n\beta$ *-regular* $n\beta$ *-* T_2 *nano topological space is* $n\beta$ *-Urysohn.*
Proof: Let the nano topological space $(\Omega, \tau_R(X))$ is $n\beta$ -regular and $n\beta$ - T_2. Consider any two distinct points $x, y \in \Omega$. Since $(\Omega, \tau_R(X))$ is $n\beta$ - T_2, there exist an $U_x \in n\beta O(\Omega, X; x)$ and an $U_y \in n\beta O(\Omega, X; y)$ such that $U_x \cap U_y = \varnothing$ and so $n\beta cl(U_x) \cap U_y = \varnothing$. Then $U = \Omega - n\beta cl(U_x) \in n\beta O(\Omega, X; y)$. Since $(\Omega, \tau_R(X))$ is $n\beta$ -regular, by Theorem 4.5, we can find a $V_y \in n\beta O(\Omega, X; y)$ such that $n\beta cl(V_y) \subset U$. Thus $n\beta cl(V_y) \cap n\beta cl(U_x) = \varnothing$. So $(\Omega, \tau_R(X))$ is $n\beta$ -Urysohn.

Definition 4.7. *Let* $(\Omega, \tau_R(X))$ *and* $(\Lambda, \tau_{R^\star}(Y))$ *be two nano topological spaces. Then a function* $\psi : \Omega \to \Lambda$ *is called nano* β *-open if* $\psi(U) \in N\beta O(\Lambda, R^\star, Y)$ *for every* $U \in N\beta O(\Omega, R, X)$.

Remark 4.8. *Let* $(\Omega, \tau_R(X))$ *and* $(\Lambda, \tau_{R^\star}(Y))$ *be two nano topological spaces. Then a surjection* $\psi : \Omega \to \Lambda$ *is nano* β *-open if and only if* $\psi(U) \in N\beta C(\Lambda, R^\star, Y)$ *for every* $U \in N\beta C(\Omega, R, X)$.

Definition 4.9. *Let* $(\Omega, \tau_R(X))$ *and* $(\Lambda, \tau_{R^\star}(Y))$ *be two nano topological spaces. Then a function* $\psi : \Omega \to \Lambda$ *is called quasi nano* β*-irresolute if for each* $x \in \Omega$ *and for each* $V \in N\beta O(\Lambda, R^\star, Y; \psi(x))$, *there exists an* $U \in N\beta O(\Omega, R, X; x)$ *such that* $\psi(U) \subset n\beta cl(V)$.

Theorem 4.10. *Let* $(\Omega, \tau_R(X))$ *and* $(\Lambda, \tau_{R^\star}(Y))$ *be two nano topological spaces. Let* $\psi : \Omega \to \Lambda$ *be a quasi nano* β*-irresolute and injective mapping, where* Λ *be* $n\beta$*-Urysohn. Then* $(\Omega, \tau_R(X))$ *is* $n\beta$*-*T_2.

Proof: Let p_1 and p_2 be any two distinct points of $(\Omega, \tau_R(X))$. Since ψ is injective, $\psi(p_1) \neq \psi(p_2)$. Again since Λ is $n\beta$-Urysohn, there exist a $V_{p_1} \in N\beta O(\Lambda, R^\star, Y; \psi(p_1))$ and a $V_{p_2} \in N\beta O(\Lambda, R^\star, Y; \psi(p_2))$ such that $n\beta cl(V_{p_1}) \cap n\beta cl(V_{p_2}) = \varnothing$. Also since ψ is quasi nano β-irresolute, there exist a $S_{p_1} \in N\beta O(\Omega, R, X; p_1)$ and a $S_{p_2} \in N\beta O(\Omega, R, X; p_2)$ such that $\psi(S_{p_1}) \subset n\beta cl(V_{p_1})$ and $\psi(S_{p_2}) \subset n\beta cl(V_{p_2})$ and hence $\psi(S_{p_1}) \cap \psi(S_{p_2}) \subset n\beta cl(V_{p_1}) \cap n\beta cl(V_{p_2}) = \varnothing$. Thus $\psi(S_{p_1}) \cap \psi(S_{p_2}) = \varnothing$ and so $S_{p_1} \cap S_{p_2} = \varnothing$. Therefore $(\Lambda, \tau_{R^\star}(Y))$ is $n\beta$-T_2.

Theorem 4.11. *Let* $(\Omega, \tau_R(X))$ *and* $(\Lambda, \tau_{R^\star}(Y))$ *be two nano topological spaces. Let the function* $\psi : \Omega \to \Lambda$ *be nano* β*-open and bijective, where* $(\Omega, \tau_R(X))$ *be* $n\beta$*-Urysohn. Then* Λ *is* $n\beta$*-Urysohn.*

Proof: Let q_1 and q_2 be any two distinct points of Λ. Since ψ is bijective, there exist p_1, $p_2 \in \Omega$ with $p_1 \neq p_2$, $\psi(p_1) = q_1$ and $\psi(p_2) = q_2$. Since $(\Omega, \tau_R(X))$ is $n\beta$-Urysohn, there exist a $S_{p_1} \in N\beta O(\Omega, R, X; p_1)$ and a $S_{p_2} \in N\beta O(\Omega, R, X; p_2)$ such that $n\beta cl(S_{p_1}) \cap n\beta cl(S_{p_2}) = \varnothing$. But the nano β-open-ness of ψ ensures that $\psi(\Omega - n\beta cl(S_{p_1})) = \Lambda - \psi(n\beta cl(S_{p_1}))$ and $\psi(\Omega - n\beta cl(S_{p_2})) = \Lambda - \psi(n\beta cl(S_{p_2}))$ are nano β-open sets and so $\psi(n\beta cl(S_{p_1}))$ and $\psi(n\beta cl(S_{p_2}))$ are $n\beta$-closed in $(\Lambda, \tau_{R^\star}(Y))$.

Now $n\beta cl(\psi(S_{p_1})) \cap n\beta cl(\psi(S_{p_2})) \subset n\beta cl(\psi(n\beta cl(S_{p_1}))) \cap n\beta cl(\psi(n\beta cl(S_{p_2})))$ $= \psi(n\beta cl(S_{p_1})) \cap \psi(n\beta cl(S_{p_2})) = \psi(n\beta cl(S_{p_1}) \cap n\beta cl(S_{p_2})) = \varnothing$. Also, since ψ is nano β-open, $\psi(S_{p_1}) \in n\beta O(\Lambda, R^\star, Y; q_1)$ and $\psi(S_{p_2}) \in N\beta O(\Lambda, R^\star, Y; q_2)$. Thus

$(\Lambda, \tau_{R^\star}(Y))$ is $N\beta$-Urysohn.

5. Nano β-closed and strongly nano β-closed graphs

Definition 5.1. *Let $(\Omega, \tau_R(X))$ and $(\Lambda, \tau_{R^\star}(Y))$ be two nano topological spaces and $\psi : \Omega \to \Lambda$ be a function. Then its graph $G(\psi)$ is called nano β-closed if for each $(x, y) \in \Omega \times \Lambda - G(\psi)$, there exist an $U \in N\beta O(\Omega, X; x)$ and a $V \in N\beta O(\Lambda, R^\star, Y; y)$ such that $(U \times V) \cap G(\psi) = \varnothing$.*

Lemma 5.2. *Let $(\Omega, \tau_R(X))$ and $(\Lambda, \tau_{R^\star}(Y))$ be two nano topological spaces and $\psi : \Omega \to \Lambda$ be a function. Then following conditions are equivalent.*

(i) ψ has nano β-closed graph,

(ii) for each $(x, y) \in \Omega \times \Lambda - G(\psi)$, there exist an $U \in N\beta O(\Omega, R, X; x)$ and a $V \in N\beta O(\Lambda, R^\star, Y; y)$ such that $\psi(U) \cap V = \varnothing$.

Proof: The proof is straightforward and thus omitted.

Definition 5.3. *Let $(\Omega, \tau_R(X))$ and $(\Lambda, \tau_{R^\star}(Y))$ be two nano topological spaces. A function $\psi : \Omega \to \Lambda$ is called nano β-irresolute if for each $x \in \Omega$ and for each $V \in N\beta O(\Lambda, R^\star, Y; \psi(x))$, there exists an $U \in N\beta O(\Omega, R, X; x)$ such that $\psi(U) \subset V$.*

Theorem 5.4. *Let $(\Omega, \tau_R(X))$, $(\Lambda, \tau_{R^\star}(Y))$ be two nano topological spaces and $\psi : \Omega \to \Lambda$ be nano β-irresolute, where Λ is $n\beta$-T_2. Then the graph $G(\psi)$ is nano β-closed.*

Proof: Let $(x, y) \in \Omega \times \Lambda - G(\psi)$. Then $y \neq \psi(x)$. Since Λ is $n\beta$-T_2, there exist an $U_1 \in N\beta O(\Lambda, R^\star, Y; \psi(x))$ and a $V \in N\beta O(\Lambda, R^\star, Y; y)$ such that $U_1 \cap V = \varnothing$. Also, since ψ is nano β-irresolute, $U = \psi^{-1}(U_1) \in N\beta O(\Omega, R, X; x)$ and so $\psi(U) \cap V = \varnothing$. Therefore the graph $G(\psi)$ is nano β-closed.

Theorem 5.5. *Let $(\Omega, \tau_R(X))$, $(\Lambda, \tau_{R^\star}(Y))$ be two nano topological spaces and $\psi : \Omega \to \Lambda$ be a surjective mapping having nano β-closed graph. Then $(\Lambda, \tau_{R^\star}(Y))$ is $n\beta$-T_1.*

Proof: Let $q_1, q_2 \in \Lambda$ with $q_1 \neq q_2$. Since ψ is surjective, there exists $p_1 \in \Omega$ such that $\psi(p_1) = q_1$ and $\psi(p_1) \neq q_2$. Then $(p_1, q_2) \in \Omega \times \Lambda - G(\psi)$ and so by the Lemma 5.2, we can find $U_{p_1} \in N\beta O(\Omega, R, X; p_1)$ and a $V_{q_2} \in N\beta O(\Lambda, R^\star, Y; q_2)$ such that $\psi(U_{p_1}) \cap V_{q_2} = \varnothing$. Thus $q_1 \in \psi(U_{p_1})$ and so $q_1 \notin V_{q_2}$. Similarly, we can ensure the existence of an $p_2 \in \Omega$ such that $\psi(p_2) = q_2$ and $\psi(p_2) \neq q_1$ and an $U_{p_2} \in N\beta O(\Omega, R, X; p_2)$ and a $V_{q_1} \in N\beta O(\Lambda, R^\star, Y; q_1)$ such that $\psi(U_{p_2}) \cap V_{q_1} = \varnothing$. Then $q_2 \in \psi(U_{p_2})$ and $q_2 \notin V_{q_1}$. So $(\Lambda, \tau_{R^\star}(Y))$ is $n\beta$-T_1.

Theorem 5.6. *Let* $(\Omega, \tau_R(X))$, $(\Lambda, \tau_{R^\star}(Y))$ *be two nano topological spaces and the function* $\psi : \Omega \to \Lambda$ *be nano* β*-open and surjective. If the graph* $G(\psi)$ *is* β*-closed, then* $(\Lambda, \tau_{R^\star}(Y))$ *is* $n\beta$-T_2.

Proof: Let $(\Omega, \tau_R(X))$ and $(\Lambda, \tau_{R^\star}(Y))$ be two nano topological spaces. Let $q_1, q_2 \in \Lambda$ with $q_1 \neq q_2$. Since ψ is surjective, there exist $p_1 \in \Omega$ such that $\psi(p_1) = q_1$ but $\psi(p_1) \neq q_2$. Thus $(p_1, q_2) \in \Omega \times \Lambda - G(\psi)$ and so by Lemma 5.2, we can find an $U_{p_1} \in N\beta O(\Omega, R, X; p_1)$ and a $V_{q_2} \in N\beta O(\Lambda, R^\star, Y; q_2)$ such that $\psi(U_{p_1}) \cap V_{q_2} = \varnothing$. Again since ψ is nano β-open, $\psi(U_{p_1}) \in N\beta O(\Lambda, R^\star, Y; q_1)$. So $(\Lambda, \tau_{R^\star}(Y))$ is $n\beta$-T_2.

Theorem 5.7. *Let* $(\Omega, \tau_R(X))$, $(\Lambda, \tau_{R^\star}(Y))$ *be two nano topological spaces and* $\psi : \Omega \to \Lambda$ *be injective. If the graph* $G(\psi)$ β*-closed,* $(\Omega, \tau_R(X))$ *is* $n\beta$-T_1.

Proof: Let $p_1, p_2 \in \Omega$ and $p_1 \neq p_2$. Since ψ is injective, $\psi(p_1) \neq \psi(p_2)$.
So $(p_1, \psi(p_2)) \in \Omega \times \Lambda - G(\psi)$. Now, by Lemma 5.2, there exist an $U_{p_1} \in N\beta O(\Omega, R, X; p_1)$ and a $V_{p_2} \in N\beta O(\Lambda, R^\star, Y; \psi(p_2))$ such that $\psi(U_{p_1}) \cap V_{p_2} = \varnothing$. Therefore $\psi(p_2) \notin (U_{p_1})$ and so $p_2 \notin U_{p_1}$. Hence $p_1 \in U_{p_1}$ but $p_2 \notin U_{p_1}$. Again since $(p_2, \psi(p_1)) \in \Omega \times \Lambda - G(\psi)$, we can find $U_{p_2} \in N\beta O(\Omega, R, X; p_2)$ and $p_1 \notin U_{p_2}$. Hence Ω is $n\beta$-T_1.

Theorem 5.8. *Let* $(\Omega, \tau_R(X))$, $(\Lambda, \tau_{R^\star}(Y))$ *be two nano topological spaces and* $\psi : \Omega \to \Lambda$ *be a nano* β*-irresolute injection. If the graph* $G(\psi)$ *is* β*-closed,* $(\Omega, \tau_R(X))$ *is* $n\beta$-T_2.

Proof: Let $p_1, p_2 \in \Omega$ and $p_1 \neq p_2$. Since ψ is injective, $\psi(p_1) \neq \psi(p_2)$. So $(p_1, \psi(p_2)) \in \Omega \times \Lambda - G(\psi)$. Then Lemma 5.2 ensures the existence of an $U_{p_1} \in N\beta O(\Omega, R, X; p_1)$ and a $V_{p_2} \in N\beta O(\Lambda, R^\star, Y; \psi(p_2))$ such that $\psi(U_{p_1}) \cap V_{p_2} = \varnothing$. Since ψ is a nano β-irresolute, $\psi^{-1}(V_{q_2}) \in N\beta O(\Omega, R, X; p_2)$. So $(\Omega, \tau_R(X))$ is $n\beta$-T_2.

Definition 5.9. *[8] Let* $(\Omega,\tau_R(X))$ *and* $(\Lambda,\tau_{R^\star}(Y))$ *be two nano topological spaces. A function* $\psi:\Omega\to\Lambda$ *is called nano* β*-continuous if* $\psi^{-1}(V)\in N\beta O(\Omega,R,X)$ *for every* $V\in NO(\Lambda,R^\star,Y)$.

Theorem 5.10. *Let a nano topological space* $(\Omega,\tau_R(X))$ *satisfy the property* nP *and* $(\Lambda,\tau_{R^\star}(Y))$ *be an arbitrary nano topological spaces. And* $\psi:\Omega\to\Lambda$ *has nano* β*-closed graph. Then* ψ *is nano* β*-continuous.*

Proof: Let $V\in NO(\Lambda,R^\star,Y)$ and any $x\in\psi^{-1}(V)$. Then for each $y\in\Lambda-V$, $(x,y)\in\Omega\times\Lambda-G(\psi)$. Since the graph of ψ is nano β-closed, there exists an $U_y\in N\beta O(\Omega,R,X;x)$ and a $V_y\in N\beta O(\Lambda,R^\star,Y;y)$ such that $\psi(U_y)\cap V_y=\varnothing$. Since $(\Omega,\tau_R(X))$ is finite, we can find $q_1,q_2,...,y_k\in\Lambda-V$ such that $Y=(\cup_{i=1}^k V_{y_i})\cup V$ and so $\Lambda-V\subset\cup_{i=1}^k V_{y_i}$. Since $(\Omega,\tau_R(X))$ satisfy the property nP, $S_x=\cap_{i=1}^k U_{y_i}\in N\beta O(\Omega,R,X;x)$ and $\psi(S)\cap(\Lambda-V)=\varnothing$. So $\psi^{-1}(V)=\cup\{S_x:x\in\psi^{-1}(V)\}\in N\beta O(\Omega,R,X)$. Therefore ψ is nano β-continuous.

Definition 5.11. *Let* $(\Omega,\tau_R(X))$, $(\Lambda,\tau_{R^\star}(Y))$ *be two nano topological spaces and* $\psi:\Omega\to\Lambda$ *be a function. Then its graph* $G(\psi)$ *is called strongly nano* β*-closed if for each* $(x,y)\in\Omega\times\Lambda-G(\psi)$*, there exist an* $U\in N\beta O(\Omega,R,X;x)$ *and a* $V\in N\beta O(\Lambda,R^\star,Y;y)$ *such that* $(U\times n\beta cl(V))\cap G(\psi)=\varnothing$.

Clearly, every function possessing strongly nano β-closed graph has nano β-closed graph.

Lemma 5.12. *Let* $(\Omega,\tau_R(X))$ *and* $(\Lambda,\tau_{R^\star}(Y))$ *be two nano topological spaces. Then for a function* $\psi:\Omega\to\Lambda$*, following conditions are equivalent:*

(i) the graph $G(\psi)$ is nano β-closed;

(ii) for each $(x,y)\in\Omega\times\Lambda-G(\psi)$, there exist an $U\in N\beta O(\Omega,R,X;x)$ and a $V\in N\beta O(\Lambda,R^\star,Y;y)$ such that $\psi(U)\cap n\beta cl(V)=\varnothing$.

Proof: The proof is straightforward and so omitted.

Definition 5.13. *A filter base* $\mathcal{F}$ *on a nano topological space* $(\Omega,\tau_R(X))$ *is said to nano* β*-*θ*-converge (respectively nano* β*-converge) to a point* $x\in\Omega$ *if for each* $V\in N\beta O(\Omega,R,X;x)$*, there exists an* $F\in\mathcal{F}$ *such that* $F\subset n\beta cl(V)$ *(respectively* $F\subset V$ *).*

Theorem 5.14. *Let* $(\Omega,\tau_R(X))$ *and* $(\Lambda,\tau_{R^\star}(Y))$ *be two nano topological spaces. Let* $(\Lambda,\tau_{R^\star}(Y))$ *be* $n\beta$*-regular and* $\psi:\Omega\to\Lambda$ *be any function. Then the following* *statements are equivalent:*

(i) $G(\psi)$ is strongly nano β-closed;

(ii) If a filter base $\mathcal{F}$ on $(\Omega,\tau_R(X))$, nano β-converges to x and $\psi(\mathcal{F})$ nano β-θ-converges to y in $(\Lambda,\tau_{R^\star}(Y))$, then $y=\psi(x)$.

Proof: (i) $\Rightarrow$ (ii): Let $\mathcal{F}$ be a filter base on $(\Omega,\tau_R(X))$ that nano β-converges to x and $\psi(\mathcal{F})$ nano β-θ-converges to y. If possible, let $y\neq\psi(x)$.
Then $(x,y)\in\Omega\times\Lambda-G(\psi)$. Clearly, $N\beta O(\Omega,R,X;x)\subset\mathcal{F}$ and $\{n\beta cl(V):V\in N\beta O(\Lambda,R^\star,Y;y)\}\subset\psi(\mathcal{F})$. So, for each $U\in N\beta O(\Omega,R,X;x)$ and each $V\in N\beta O(\Lambda,R^\star,Y;y)$, there exist $P_1\in\mathcal{F}$ and $P_2\in\mathcal{F}$ such that $P_1\subset U$ and $\psi(P_2)\in n\beta cl(V)$. Hence there exists an $P_0\in\mathcal{F}$ such that $P_0\subset P_1\cap P_2$ and satisfies $P_0\subset U$ as well as $\psi(P_0)\subset n\beta cl(V)$. Hence $\varnothing\neq\psi(P_0)\subset\psi(U)\cap n\beta cl(V)$. So by Lemma 5.12, $G(\psi)$ is not strongly nano β-closed.

(ii) $\Rightarrow$ (i): Let $(\Lambda,\tau_{R^\star}(Y))$ is $n\beta$-regular and the given condition (ii) holds for ψ. If possible, let $G(\psi)$ is not strongly nano β-closed. Then there exists $(x,y)\in\Omega\times\Lambda-G(\psi)$ such that $(U\times n\beta cl(V))\cap G(\psi)\neq\varnothing$ for each $U\in N\beta O(\Omega,R,X;x)$ and each $V\in N\beta O(\Lambda,R^\star,Y;y)$. Since Λ is $n\beta$-regular, the family
$\mathcal{F}=\{F_{UV}=\{p\in U:(p,\psi(p))\in(U\times n\beta cl(V))\cap G(\psi)\}:U\in N\beta O(\Omega,R,X;x)$ and $V\in N\beta O(\Lambda,R^\star,Y;y)\}$ is a filter base on $(\Omega,\tau_R(X))$. But $\mathcal{F}$ nano β-converges to x in $(\Omega,\tau_R(X))$ and $\psi(\mathcal{F})$ nano β-θ-converges to y and $y=\psi(x)$ — a contradiction.

Theorem 5.15. *Let $(\Omega,\tau_R(X))$, $(\Lambda,\tau_{R^\star}(Y))$ be two nano topological spaces and $\psi:\Omega\to\Lambda$ be a function. If the graph $G(\psi)$ is strongly nano β-closed, then $\psi(x)=\cap\{n\beta cl(\psi(U)):U\in N\beta O(\Omega,R,X;x)\}$ for each $x\in\Omega$.*

Proof: If possible, let there exist an $x\in\Omega$ and an $y(\neq\psi(x))\in\Lambda$ such that $y\in n\beta cl(\psi(U))$ for each $U\in N\beta O(\Omega,R,X;x)$. Since $(x,y)\in\Omega\times\Lambda-G(\psi)$, by Lemma 5.12, we can find a $U_x\in N\beta O(\Omega,R,X;x)$ and a $V_y\in N\beta O(\Lambda,R^\star,Y;y)$ such that $\psi(U_x)\cap n\beta cl(V_y)=\varnothing$ and so $\psi(U_x)\cap V_y\subset\psi(U_x)\cap n\beta cl(V_y)=\varnothing$. Thus $\psi(U_x)\cap V_y=\varnothing$. Then $y\notin n\beta cl(\psi(U_x))$, a contradiction.

Theorem 5.16. *Let $(\Omega,\tau_R(X))$ is an arbitrary nano topological space and $(\Lambda,\tau_{R^\star}(Y))$ is $n\beta$-T_2. Let $\psi:\Omega\to\Lambda$ be nano β-irresolute. Then the graph $G(\psi)$ is strongly nano β-closed.*

Proof: Let $(x,y)\in\Omega\times\Lambda-G(\psi)$. Then $y\neq\psi(x)$. Since $(\Lambda,\tau_{R^\star}(Y))$ is $n\beta$-T_2, there exist a $U\in N\beta O(\Lambda,R^\star,Y;\psi(x))$ and a $V\in N\beta O(\Lambda,R^\star,Y;y)$ such that $U\cap V=\varnothing$. Again since ψ is nano β-irresolute, $K=\psi^{-1}(U)\in N\beta O(\Omega,R,X;x)$ and so $\psi(K)\cap V=\varnothing$, i.e. $V\subset\Lambda-\psi(K)$, i.e. $n\beta cl(V)\subset\Lambda-\psi(K)$, i.e. $\psi(K)\cap n\beta cl(V)=\varnothing$. Hence $G(\psi)$ is strongly nano β-closed.

Theorem 5.17. *Let $(\Omega,\tau_R(X))$, $(\Lambda,\tau_{R^\star}(Y))$ be two nano topological spaces and $\psi:\Omega\to\Lambda$ be a quasi nano β-irresolute injection. If the graph $G(\psi)$ is strongly nano β-closed, then $(\Omega,\tau_R(X))$ is $n\beta$-T_2.*

Proof: Let any two distinct points $p_1,p_2\in\Omega$. Since ψ is injective, $\psi(p_1)\neq\psi(p_2)$. Thus $(p_1,\psi(p_2))\in\Omega\times\Lambda-G(\psi)$ and so by lemma 5.12, there exist a $U_{p_1}\in N\beta O(\Omega,R,X;x)$ and a $V_{p_2}\in N\beta O(\Lambda,R^\star,Y;y)$ such that $\psi(U_{p_1})\cap n\beta cl(V_{p_2})=\varnothing$ and so $\psi^{-1}(n\beta cl(V_{p_2}))\subset\Omega-U_{p_1}$. Since ψ is a quasi nano β-irresolute, there exists $S_{p_2}\in(n)\beta\gamma(\Omega,R,X;p_2)$ such that $\psi(S_{p_2})\in n\beta cl(V_{p_2})$. Then $S_{p_2}\subset\psi^{-1}(n\beta cl(V_{p_2}))\subset\Omega-U_{p_1}$ and hence $S_{p_2}\cap U_{p_1}=\varnothing$. So $(\Omega,\tau_R(X))$ is $n\beta$-T_2.

Theorem 5.18. *Let $(\Omega,\tau_R(X))$ be any nano topological space, $(\Lambda,\tau_{R^\star}(Y))$ be $n\beta$-Urysohn nano topological space and $\psi:\Omega\to\Lambda$ be a quasi nano β-irresolute. Then its graph $G(\psi)$ is strongly nano β-closed.*

Proof: Let $(x,y)\in\Omega\times\Lambda-G(\psi)$. Then $y\neq\psi(x)$. Since Λ is $n\beta$-Urysohn, there exist a $V_1\in N\beta O(\Lambda,R^\star,Y;\psi(x))$ and a $V_2\in N\beta O(\Lambda,R^\star,Y;y)$ such that $n\beta$-$cl(V_1)\cap n\beta cl(V_2)=\varnothing$. Again since ψ is quasi nano β-irresolute, there exists an $U\in N\beta O(\Omega,R,X;x)$ such that $\psi(U)\subset n\beta cl(V_1)\subset\Omega-n\beta cl(V_2)$. Therefore $\psi(U)\cap n\beta cl(V_2)=\varnothing$. So $G(\psi)$ is strongly nano β-closed.

Theorem 5.19. *Let $(\Omega,\tau_R(X))$ be $n\beta$-Urysohn possessing the property nP and $(\Lambda,\tau_{R^\star}(Y))$ be $n\beta$-regular. Let $\psi:\Omega\to\Lambda$ be nano β-open bijection. Then $G(\psi)$ is strongly nano β-closed.*

Proof: Let $(x,y)\in\Omega\times\Lambda-G(\psi)$. Then $y\neq\psi(x)$ and so $x\neq\psi^{-1}(y)$. Since $(\Omega,\tau_R(X))$ is $n\beta$-Urysohn, for each $p\in\psi^{-1}(y)$, there exist a nano β-open set V_x and a nano β-open set V_p containing x and p respectively such that $n\beta$-$cl(V_x)\cap n\beta cl(V_p)=\varnothing$. Then $\{V_p:\psi(p)=y\}$ is a cover of $\psi^{-1}(y)$ by nano β-open sets of $(\Omega,\tau_R(X))$. Since $\psi^{-1}(y)$ is finite, there exist finite number of points

$p_1, p_2, ..., p_k \in \Omega$ with $\psi(p_i) = y$ for each $i \in \{1,2,...,k\}$ such that $\psi^{-1}(y) \subset \cup_{i=1}^{k} V_{p_i}$. Let $G = \cap_{i=1}^{k} V_{x_i}$ and $H = \cup_{i=1}^{k} V_{p_i}$. Since $(\Omega, \tau_R(X))$ satisfies the property nP, $n\beta cl(H) = n\beta cl(\cup_{i=1}^{k} V_{p_i}) = \cup_{i=1}^{k} n\beta cl(V_{p_i})$.

So $n\beta cl(G) \cap n\beta cl(H) = \varnothing$. Again since ψ is nano β-open and bijective, $\psi(H) \in N\beta O(\Lambda, R^{\star}, Y; y)$ and so Theorem 4.5 ensures the existence of an $L \in N\beta O(\Lambda, R^{\star}, Y; y)$ such that $n\beta cl(L) \subset \psi(H)$, i.e. $\psi^{-1}(n\beta cl(L)) \subset H$. Therefore $\psi^{-1}(n\beta cl(L)) \cap G = \varnothing$ and thus $\psi(G) \cap n\beta cl(L) = \varnothing$. Thus by Lemma 5.12, $G(\psi)$ is strongly nano β-closed.

6. Conclusion

Some researchers [3, 6, 8] recently have shown that the concept of nano topology can be used as a tool to study some real life problems. Keeping these in mind, we have extended some separation axioms and graphs of functions via nano β-open sets in nano topology, which may have possible applications in real life situations.

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