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An Improved Solution of the Diophantine Equation $\sum_{i=1}^{k}\frac{1}{Xi}=1$ in Distinct Integers of the Form $X_i \in P^{\alpha}q^{\beta}$

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Abstract. Solutions of the title equation are discussed, and in particular when all x_i are even, $p < q$ are primes, and α,β are positive integers. The author [7] has shown that when all x_i are distinct, even, and of the form $x_i \in p^{\alpha}q^{\beta}$, then the title equation has a solution with $k = 85$. Our main result consists of an improved solution to the title equation when x_i are even, $x_i \in p^{\alpha}q^{\beta}$ and $k = 67$. Moreover, the solution in [7] contains seven distinct odd primes, whereas in the current result this number is reduced to the first six odd primes. Some modifications of the equation and questions are also presented.

Keywords: Diophantine equations, Egyptian fractions

AMS Mathematics Subject Classification (2010): 11D68

1. Introduction

We consider the Diophantine equation in integers

$$\sum_{i=1}^{k}\frac{1}{Xi}=1, \qquad x_1 < x_2 < \cdots < x_k \tag{1}$$

when $x_i = p^{\alpha}q^{\beta}$, $p < q$ are primes and α,β are positive integers.

If all values x_i in equation (1) are of the form $x_i = pq$ $(\alpha = \beta = 1)$, then the above restriction imposed on the values x_i immediately implies the fact that $x_i \nmid x_j$ for $i \neq j$. Moreover, the converse of this statement is false. For if $x_i \nmid x_j$ for $i \neq j$ is true for all values x_i in equation (1), then all values x_i need not be of the form $x_i = pq$. This is shown in [2, Example 2 with $k = 79$] for which the author received a reward of \$10 from Erdös. Barbeau [1] gave an example of equation (1) in which all values x_i are of the form $x_i = pq$ and $k = 101$. In [3, Example 1], the author provided another such solution, but with $k = 63$. Finally, the author [6] constructed a solution of equation (1) in which all values x_i are of the form $x_i = pq$ and $k = 52$. The results of the author in [2] and those in [2, 3, 6] are respectively cited in [11] and [12].

We note that in all the above solutions, the values x_i are composed of odd and even numbers. This leads us to investigate solutions of equation (1) when all x_i are odd, and when all x_i are even numbers.

Among many problems concerned with equation (1) are in particular those in which x_i are odd numbers. It is mentioned [5], that the case of all x_i are odd cannot be applied to equation (1). However, when removing the restriction that $x_i = p^{\alpha}q^{\beta}$, and applying some modifications to equation (1) such as: fixed values k, fixed primes, more than two primes occur and allowing certain exponents to be zero, it is established:

(i) For odd values x_i, it is shown [4] that the smallest possible value k is $k = 9$, and for that value k equation (1) has exactly five solutions with $x_i \in 3^{\alpha}\cdot 5\cdot 7\cdot 11$ where $\alpha \le 3$.

(ii) It is easily verified that equation (1) has no solutions when $x_i \in 3^{\alpha}5^{\beta}$. Therefore, when $k = 11$ and with the three smallest primes, namely $x_i \in 3^{\alpha}5^{\beta}7^{\gamma}$, it is shown in [5] that equation (1) has exactly seventeen solutions.

2. The improved solution

In [7, Example 3], the author demonstrated a solution of equation (1) in which $k = 85$, x_i are even, $x_i \in p^{\alpha}q^{\beta}$ where $p < q$ are primes and α,β are positive integers.

In this section, an improved solution of equation (1) is presented when x_i are even, having the same characteristics as above and $k = 67$. Furthermore, the solution in [7] contains seven distinct odd primes, whereas the improved solution contains the first six odd primes.

The following Table 1 consists of sixty-seven numbers presented in seven rows. The total of the numbers in each row is indicated at the end of the row.

Table 1.

1	6	12	24	48	96	192	384	768	1536	3072	6144	12288	**12**
2	10	20	40	80									**4**
3	14	28	56	112	224	448	896	1792	3584	7168	14336	28672	**12**
4	18	36	72	144	288	576	1152	2304	4608	9216	18432	36864	**12**
5	22	44	88	176	352	704		2816					**7**
6	26	52	104	208	416	832	1664	3328	6656	13312	26624	53248	**12**
7	34	68	136	272	544	1088	2176	4352					**8**

The sixty-seven different numbers contained in rows 1 – 7 of Table 1 have the following two properties:

(i) each number is even and of the form $p^{\alpha}q^{\beta}$,

(ii) the sum of their reciprocals is equal to 1.

This can clearly be verified by means of a computer (Least Common Multiple is 3136573440), but it can also be done directly by observing the following. Consider the two sets of numbers:

$$S = \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096\},$$
$$T = \{3, 5, 7, 9, 11, 13, 17\}.$$

The numbers in S are the first twelve powers of 2, and the set T contains besides the number 9, the first six odd primes. The sets S, T are disjoint sets. Multiplying the twelve numbers in S by each of 3, 7, 9 and 13 in T, respectively results in the numbers in rows 1, 3, 4 and 6. The numbers in row 2 are products of the first four numbers in S by 5 in T. Multiplying the first six numbers and also the eight number in S by 11 in T yield the numbers in row 5. Finally, the numbers in row 7 are the products of the first eight numbers in S by 17 in T.

Due to the property of the sets S and T and to the structure just described of the numbers in rows 1 – 7, it is evident that property (i) follows.

For (ii) observe that: all the multiples appearing in Table 1 of a certain number M where $M \in T$, occur in one and only one row. Computing the sum of the reciprocals in each row, we obtain a fraction, the numerator of which is also a multiple of M. After simplification, the new fraction will have a denominator which is a divisor of 4096. This enables us to carry out the summation without a computer.

The seven partial sums add up to 1.

3. Conclusion

The improved solution folds in itself at least two questions.

Question 1. Let α, β be positive integers, $p < q$ are primes, and $x_i = p^{\alpha} q^{\beta}$ are distinct even integers. If k reciprocals $1 / x_i$ yield a sum equal to 1, what is the smallest possible value k?

We presume that $k > 60$.

Question 2. Let α, β be positive integers, $p < q$ are primes, and $x_i = p^{\alpha} q^{\beta}$ are distinct even integers. If k reciprocals $1 / x_i$ yield a sum equal to 1, what is the minimal number of odd primes required for such a solution ?

We conjecture that this number is six.

REFERENCES

1. E.J.Barbeau, Expressing one as a sum of distinct reciprocals: comments and bibliography, *Eureka* (Ottawa), 3(1977) 178-181.
2. N.Burshtein, On distinct unit fractions whose sum equals 1, *Discrete Mathematics*, 5 (1973) 201-206.
3. N.Burshtein, Improving solutions of $\sum_{i=1}^{k} 1/Xi = 1$ with restrictions as required by Barbeau respectively by Johnson, *Discrete Mathematics*, 306 (2006) 1438-1439.

4. N.Burshtein, The equation $\sum_{i=1}^{9} \frac{1}{Xi} = 1$ in distinct odd integers has only the five known solutions, *Journal of Number Theory*, 127 (2007) 136-144.
5. N.Burshtein, All the solutions of the equation $\sum_{i=1}^{11} \frac{1}{Xi} = 1$ in distinct integers of the form $x_i \in 3^{\alpha}5^{\beta}7^{\gamma}$, *Discrete Mathematics,* 308 (2008) 4286-4292.
6. N.Burshtein, An improved solution of $\sum_{i=1}^{k} \frac{1}{Xi} = 1$ in distinct integers when $x_i \nmid x_j$ for $i \neq j$, *Notes on Number Theory and Discrete Mathematics,* 16(2) (2010) 1-4.
7. N.Burshtein, On the diophantine equation $\sum_{i=1}^{k} \frac{1}{Xi} = 1$ in distinct integers of the form $x_i \in p^{\alpha}q^{\beta}$, *Notes on Number Theory and Discrete Mathematics,* 16(4) (2010) 1-5.
8. N.Burshtein, On solutions of the diophantine equation $\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \cdots + \frac{k}{x_k} = 1$ when $2 \leq x_1 < x_2 < x_3 < \cdots < x_k$ are integers and $k = x_1$, *Notes on Number Theory and Discrete Mathematics,* 23(2) (2017) 30-35.
9. L.E.Dickson, *History of the theory of numbers*, vol. 2 Chelsea, New York, 1966.
10. P.Erdös, On the solutions in integers of $a/b = x_1^{-1} + \cdots + x_n^{-1}$, *Mat. Lapok*, 1 (1950) 192-210.
11. P. Erdös and R.L.Graham, Old and new problems and results in Combinatorial Number Theory, Monographie n° 28 de *L'Enseignement Mathématique* Université de Genève, Imprimerie Kundig Genève, 1980.
12. R.L.Graham, Paul Erdös and Egyptian Fractions, in: L.Lovász, L. Ruzsa, V.Sós (Eds.), *Erdös Centennial*, János Bolyai Mathematical Society, 25(2013) Springer, Chapter 10, pp. 289-309.
13. R.K.Guy, *Unsolved Problems in Number Theory*, Third edition, Springer, New York, 2004.
14. W.Sierpiński, Sur les décompositions de nombres rationnels en fractions primaires, *Mathesis,* 65 (1956) 16-32.