Annals of Pure and Applied Mathematics
Vol. 14, No. 3, 2017, 437-448
ISSN: 2279-087X (P), 2279-0888(online)
Published on 25 October 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v14n3a10

Annals of
Pure and Applied Mathematics

Forbidden Subgraph Characterizations of Extensions of Gallai Graph Operator to Signed Graph

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Received 11 September 2017; accepted 3 October 2017

Abstract. We introduce three types of extension of Gallai graph operator in to signed graphs- Gallai signed graph, product-Gallai signed graph and dot-Gallai signed graph. We find the forbidden subgraph characterizations of Gallai signed graph, product-Gallai signed graph and dot-Gallai signed graph.

Keywords: Gallai graph, signed graph

AMS Mathematics Subject Classification (2010): 05C22

1. Introduction

A signed graph is obtained from a graph when one regards some of the lines as positive and the remaining lines as negative [7]. Precisely, a signed graph is a pair (G,σ) where G is called the underlying graph and $\sigma : E(G) \rightarrow \{+,-\}$ is called the signature function or sign to the edges. The collection of all positive edges and the collection of all negative edges are denoted by $E^+(S)$ and $E^-(S)$, respectively. In social psychology, signed graphs have been used to model social situations (examples in [13], [14] and [9]). A signed graph in which all the edges are positive (negative) is called all-positive (all-negative) signed graph. A signed graph is said to be homogeneous if it is either all-positive or all-negative and heterogeneous, otherwise. A cycle in a signed graph S is said to be positive if the product of the signs of its edges is positive. Otherwise, it is called negative [6]. Similarly, a path in a signed graph is said to be positive, if the product of the signs of the edges is positive and is negative, otherwise. A vertex in a signed graph is considered as a homogeneous vertex if the entire edges incident to it has the same sign. Otherwise, it is a heterogeneous vertex. Further, in [4], every signed graph $S = (G,\sigma)$ can be associated with a signing of its vertices by the function, called the canonical marking of S, defined by the rule,

$$\mu_\sigma(x) = \prod_{E_j \in E_x} \sigma(E_j)$$

where E_x denote the set of all edges of S, which are incident on the vertex *'x'*. In literature signed graph in short called as 'sigraph' [10]. In this paper, canonical marking is used to sign the vertices.

The line graph [8] $L(G)$, of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $L(G)$ if they are incident in G. The Gallai graph $\Gamma(G)$, of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $\Gamma(G)$ if they are incident in G, but do not span a triangle in G [16]. Though, $\Gamma(G)$ is a spanning subgraph of $L(G)$, their behaviors are different. For example, $L(G)$ has a forbidden subgraph characterization, whereas Gallai graphs do not have the vertex hereditary property and hence cannot be characterized using forbidden subgraphs [16]. But, in [3] it has been proved that there exists a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be H-free for any finite graph H. Also it has been proved in [1] that the recognition of anti-Gallai graphs is NP-complete. In [2], the forbidden subgraph characterizations of G for which $\Gamma(G)$ and $\Delta(G)$ is a split graph and is a threshold graph are given.

Signed line graph $L(S)$ of a given signed graph $S = (G,\sigma)$ as defined by Behzad and Chartrand [12] is the signed graph with standard line graph $L(G)$ of G as its underlying graph and whose edges are assigned the signs according to the rule: for any $e_ie_j \in E(L(S))$, $e_ie_j \in E^-(L(S))$ if and only if the edges e_i and e_j of S are both negative in S. For a signed graph S the set of all signed graphs S' with $L(S') \doteqdot S$ is called ***signed line roots*** of S [11].

There are two other notions of a signed line graph of a given signed graph $S = (G,\sigma)$ in [5] - product-line sigraph $L(S)$ and dot-line sigraph $L_{\bullet}(S)$. Both of them have $L(G)$ as its underlying graph and only the rule to assign signs to the edges of $L(G)$ are different. In $L_{\times}$, an edge ee' has sign $\sigma(e)\sigma(e')$ and in $L_{\bullet}(S)$, any edge ee' has the sign of the vertex common to e and e' [6]. For a signed graph S the set of all signed graphs S' with $L(S') \doteqdot S$ is called $\boldsymbol{L_{\bullet}}$***-roots*** of S [5].

1.1. New terminology and definitions

Motivated from the above concepts, we define Gallai signed graph $\Gamma(S)$ of a given signed graph $S = (G,\sigma)$, as the signed graph with the Gallai graph $\Gamma(G)$ as its underlying graph and whose edges are assigned the signs according to the rule: for any e_ie_j in $E(\Gamma(S))$, e_ie_j is negative if and only if the edges e_i and e_j are both negative in S and positive otherwise. Like the concept of signed line roots we introduce the concept of ***Gallai Signed roots*** as the set of all signed graphs S' with $\Gamma(S') \doteqdot S$ is called ***Gallai Signed roots*** of S. In this paper there is no ambiguity to call ***Gallai Signed roots*** as ***roots***.

Similarly, given signed graph $S = (G,\sigma)$, the product-Gallai signed graph $\Gamma_{\times}(S)$ and the dot-Gallai signed graph $\Gamma_{\bullet}(S)$, have $\Gamma(G)$ as their underlying graph and the rule to assign signs to the edges are as follows. In an edge ee' has sign $\sigma(e)\sigma(e')$ and in $\Gamma_{\bullet}(S)$ any edge ee' has the sign of the vertex common to e and e'. Like the concept of $\boldsymbol{L_{\bullet}}$***-roots*** we introduce the concept of $\boldsymbol{\Gamma_{\bullet}}$***-roots***. For a signed graph S the set of all signed graphs S' with $\Gamma_{\bullet}(S') \doteqdot S$ is called $\boldsymbol{\Gamma_{\bullet}}$***-roots*** of S and the set of signed graphs S' with $\Gamma_{\bullet}(S')$ contains S as an induced subgraph is called $\boldsymbol{\Gamma_{\bullet}}$***-semi roots*** of S. Note that, $\boldsymbol{\Gamma_{\bullet}}$***-semi roots*** consist of signed graphs which contains a $\boldsymbol{\Gamma_{\bullet}}$***-root*** as an induced subgraph.

If a graph G has a property P implies that G cannot have an induced sub-graph isomorphic to H, and then H is called a forbidden subgraph for the property P [15].

Though Gallai graphs do not admits forbidden subgraph characterization [16], $\Gamma(S)$, $\Gamma_{\times}(S)$ and $\Gamma_{\bullet}(S)$ admit forbidden subgraphs. In this paper, we obtain forbidden subgraph characterization of Gallai signed graphs, product-Gallai signed graph and dot-Gallai signed graphs.

Given a graph G, G^c denotes the complement of G. The join of two graphs G and H is denoted by $G \vee H$. All graph theoretic notations and terminology not mentioned here are from [15].

2. Forbidden subgraph characterizations

2.1. Gallai signed graph

In this section, we give the forbidden subgraph characterization of Gallai signed graph.

Theorem 2.1.1. The following graphs are the only vertex minimal forbidden subgraphs of the Gallai signed graphs.

(1) A triangle $v_1v_2v_3$ with only one positive edge.

(2) A path $P = v_1v_2v_3v_4$ with signs $\sigma(v_1v_2) = \sigma(v_3v_4) = -$ and $\sigma(v_2v_3) = +$.

(3) A path P as on (2) above together with some or all of the edges v_1v_3, v_2v_4 and v_1v_4; each of them assigned a + sign.

Proof: Let $v_1v_2v_3$ be a triangle in the Gallai signed graph with only one positive edge. Let e_1, e_2, e_3 be the edges corresponding to the vertices v_1, v_2 and v_3. Since v_1v_2 and v_2v_3 are negative e_1, e_2 and e_3 must be negative. But, then in Gallai signed graph, v_1v_3 must also be negative, which is a contradiction.

Let $v_1v_2v_3v_4$ be a path in Gallai signed graph with *signs* $\sigma(v_1v_2) = \sigma(v_3v_4) = -$ and $\sigma(v_2v_3) = +$. Let e_1, e_2, e_3 and e_4 be the edges in the root corresponding to the vertices v_1, v_2, v_3 and v_4. The edges v_1v_2 and v_3v_4 are negative if and only if the corresponding edges e_1, e_2, e_3 and e_4 are all negative. But, then v_2v_3 must also be negative, which is a contradiction.

As evident from the proof, any super graph of the above path is also forbidden and it is vertex minimal forbidden if it does not contain a triangle with only one positive edge. Therefore, this path with some or all of the edges v_1v_3, v_2v_4 and v_1v_4; each of them assigned a + sign are also minimal forbidden.

Now, let $S = (G,\sigma)$ be a signed graph which do not contain the signed graphs mentioned in the theorem as subgraphs. Let $v_1, v_2, \ldots, v_n$ be the vertices of the underlying graph G. Note that, if v_iv_j is a positive edge in G, then either v_i or v_j do not have a negative edge incident on it. We can construct a graph $S' = (H,\mu)$ such that $\Gamma(S')$ contains S, as follows.

Let $H = G^c \vee \{v\}$. The sign function $\mu: E(H) \to \{+,-\}$ is defined as,

$$\mu(e) = \begin{cases} -, & if\ e = vv_i\ and\ v_i\ has\ a\ negative\ edge\ incident\ on\ it\ in\ G \\ +, & otherwise. \end{cases}$$

Clearly the vertices corresponding to $\{vv_1, vv_2, \ldots, vv_n\}$ in S' induce S in $\Gamma(S')$.

2.2. Product-Gallai signed graph

In this section, we prove that the only forbidden subgraphs of product-Gallai sigraph are the cycles.

Lemma 2.2.1. If S contains a negative cycle, then S contains an induced negative cycle.
Proof: Consider a negative cycle C with vertices $v_1, v_2, \ldots, v_n$ in which m edges are negative (note that, m is odd). If C has a chord v_iv_j $(i < j)$ then we get two cycles C_1 with vertex set $v_1, v_2, \ldots, v_i, v_j, v_{j+1}, \ldots, v_n$ and C_2 with vertex set $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$. Clearly, at least one among these cycles is negative, say C_1. If C_1 is an induced cycle then we are done. Otherwise, repeat the procedure with C_1 until we get an induced negative cycle.

Theorem 2.2.2. The only forbidden subgraphs for the product-Gallai sigraph are the negative cycles.
Proof: Let $S = (G,\sigma)$ be a graph without negative cycles. We construct $S' = (H,\mu)$ as follows. Without loss of generality we assume that G is a connected graph. If G is disconnected, repeat the same procedure for each connected component of G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $H = G^c \vee \{v\}$ and $\sigma(vv_1) = +$. Note that, for any two vertices v_i and v_j in G, every v_iv_j path (if more than one exist) is either positive or negative, since otherwise we get a negative cycle in G which is a contradiction. Assign $\sigma(vv_i) = +$ or - according as a path joining v_1 to v_i in S is positive or negative. The remaining edges in H could be assigned any sign. In G, if v_i is adjacent to v_j, then in H, vv_i and vv_j are incident and do not span a triangle. Therefore in $\Gamma_\times(S')$, the vertices corresponding to the edges vv_i and vv_j are adjacent. If v_iv_j is +, then both v_1v_i path and v_1v_j path are either positive or negative, so that vv_i and vv_j are both either + or - in S'. In either cases, in $\Gamma_\times(S')$ the edge joining the vertices corresponding to vv_i and vv_j is +. Similarly, if v_iv_j is -, then every v_1v_i path and v_1v_j path are of opposite signs so that vv_i and vv_j are of opposite sign. Therefore, in $\Gamma_\times(S')$ the, subgraph induced by the vertices corresponding to the edges $vv_1, vv_2, \ldots, vv_n$ is isomorphic to S.

To prove the converse part, let S be a signed graph such that $\Gamma_\times(S)$ contains a cycle $v_1v_2\ldots v_n$. Let $e_1, e_2, \ldots, e_n$ be the edges corresponding to the vertices $v_1, v_2, \ldots, v_n$. Note that v_iv_{i+1} is negative if and only if e_i and e_{i+1} are of opposite signs. Since the number of change of signs for $e_1, e_2, \ldots, e_n, e_1$ is always even, the number of negative edges in the cycles $v_1v_2\ldots v_n$ is also even. That is S cannot have negative cycles. Hence, the theorem is proved.

2.3. Dot-Gallai sigraph

In this section, we characterize dot-Gallai sigraph with respect to the forbidden subgraphs. For convenience we introduce the following terminology. A vertex is of ***type A*** if it belongs to a cycle in which the edges incident to it have the same sign and there is at least one edge of opposite sign incident to it outside the cycle. Otherwise, the vertex is of ***type B.*** We have the following observations regarding, *type* ***A*** and ***type B*** vertices.

(1) If a cycle contained in a signed graph is not homogeneous then there exist at least two ***type B*** vertices.

(2) The edge corresponding to a ***type A*** vertex in a ***$\Gamma_\bullet$–semi root*** has one end vertex positive and the other negative.

(3) If 'n', ***type A*** vertices are adjacent in S then there is a homogeneous path of length $n + 2$ in S. The path connecting any two ***type B*** vertices is homogeneous. Also the edges corresponding to the vertices in this path including the ***type B*** vertices have a common vertex in every ***$\Gamma_{\bullet}$–semi root*** of S.

Lemma 2.3.1. A complete graph which is not homogeneous is a forbidden subgraph for dot-Gallai sigraph.

Proof: Let $S = (G, \sigma)$ be a signed graph which contains a subgraph (K_n, σ). If there exists a signed graph S' such that $\Gamma_{\bullet}(S')$ contains S, then the underlying graph of S' contains $K_{1,n}$. The n^{th}-degree vertex of $K_{1,n}$ is either positive or negative. Therefore the corresponding K_n in the dot-Gallai sigraph is homogeneous.

Lemma 2.3.2. The following signed graphs are not forbidden for dot-Gallai sigraph.

(1) $S_1 = (C_n,\sigma)$.
(2) $S_2 = (K_{1,n},\sigma)$.
(3) Signed trees.
(4) Cycles C_n with at least 5 ***type B*** vertices.
(5) Homogeneous cycles with any number of ***type A*** vertices.

Proof: (1) Let $S_1 = (C_n,\sigma)$, where C_n is a cycle of length n with vertices $\{v_1, v_2, \ldots, v_n\}$ and 'm' negative edges. To construct a ***$\Gamma_{\bullet}$–semi root***, $S_1' = (H,\mu)$, let $e_1, e_2, \ldots, e_n$ be the edges corresponding to $v_1, v_2, \ldots, v_n$ and let these edges induce a cycle in S_1'. Let $\mu(e_i) = +$ for every i and if $\sigma(v_iv_{i+1}) = -$, add a negative pendant edge incident on the common vertex of e_i and e_{i+1}. In this case, $\Gamma_{\bullet}(S_1')$ contains S as an induced subgraph.

(2) Let $S_2 = (K_{1,n},\sigma)$, where $K_{1,n}$ is induced by the vertices $v, v_1, v_2, \ldots, v_n$ where v is the central vertex. We can construct a ***$\Gamma_{\bullet}$–semi root*** $S_2' = (H,\mu)$ as follows. Let $e = uu'$ be the edge corresponding to the central vertex v and let e_i be the edge corresponding to v_i for $i = 1, 2, \ldots, n$. Partition $\{e_1, e_2, \ldots, e_n\}$ to E_1 and E_2 such that $e_i \in E_1$ if $\sigma(vv_i) = +$ in S and $e_i \in E_2$ if $\sigma(vv_i) = -$ in S. Make all edges of E_1 incident to u and that of E_2 to u'. Also, make the remaining end vertices of E_1 (and E_2) induce a complete graph. Let $\mu(e) = +$ and $\mu(e_i) = +, \forall i$. If $E_2 \neq \Phi$, then attach one negative edge to u', where the edges of E_2 are attached. Now the induced sign of u is positive and u' is negative. Clearly, the vertices corresponding to the edges $e, e_1, e_2, \ldots, e_n$ induce $K_{1,n}$ in $\Gamma_{\bullet}(S_2')$.

(3) Let $S_3 = (T,\sigma)$ be a signed graph, where T is a tree. Let v be any vertex of T. The vertex v together with its neighbors induce a signed $K_{1,m}$ in S. Obtain a ***$\Gamma_{\bullet}$–semi root*** for $K_{1,m}$ as described in (2) above. Let v_1 be a neighbor of v in T and let e_1 be the edge corresponding to v_1 in H.

If vv_1 is positive in S, then the common vertex of e and e_1 is positive. Attach positive edges, corresponding to the vertices which have positive adjacency with v_1 to this vertex in ***$\Gamma_{\bullet}$–semi root***. Since there is no adjacency between neighbors of v_1 and neighbors of v, triangulate the edges correspond to the neighbors of v_1 with the edges corresponding to the neighbors of v, if they are incident in the ***$\Gamma_{\bullet}$–semi root***. Similarly attach the edges corresponding to the vertices which have negative adjacency with v_1 to the other end vertex of e_1 and assign positive sign. Also attach one additional negative edge to that vertex. Then the sign of that vertex is negative in ***Γ•–semi root***.

If vv_1 is negative in S then the common vertex of e and e_1 is negative. Attach the edges corresponding to the vertices which have negative adjacency with v_1 to the common vertex of e and e_1 and triangulate the edges correspond to the neighbors of v_1, with the edges correspond to the neighbors of v. Similarly attach the edges corresponding to the positive adjacent vertices to the other end vertex of e_1. Then the sign of that vertex is positive. Repeat the same procedure to obtain a ***Γ•–semi root*** for S.

(4) Let C_n contains exactly 5 ***type B*** vertices. To find a ***Γ•–semi root*** $S' = (H,\mu)$, according to the order in the cycle let the ***type B*** vertices are $v_1, v_2, \ldots, v_5$. By observation 3, since the path between any two successive vertices in the above list is homogeneous the edges corresponding to v_1, v_2 and the ***type A*** vertices between them should have one common end vertex. A similar argument holds for paths between the pairs (v_2,v_3), (v_3,v_4), (v_4,v_5) and (v_5,v_1). Let $e_1, e_2, \ldots, e_5$ be the positive edges corresponding to the vertices $v_1, v_2, \ldots, v_5$. Draw e_1 and e_2 with a common end vertex. If there exists, ***type A*** vertices between v_1 and v_2, corresponding to that vertices attach positive edges to the common vertex of e_1 and e_2. Triangulate all the edges if the corresponding vertices are non adjacent in S. Since e_2, e_3 have a common end vertex and v_2 is not a ***type A*** vertex we can attach e_3 and positive edges corresponding to the ***type A*** vertices between v_2 and v_3 to the other end vertex of e_2. Then triangulate all the edges, if the corresponding vertices in S are non adjacent. If any of the homogeneous path connecting the pairs (v_1,v_2) and (v_2,v_3) is all-negative then add an additional negative edge to the common vertex of the edges corresponding to that pairs. Whatever be the sign of the homogeneous path between v_3 and v_4 attach e_4 and the edges corresponding to the ***type A*** vertices in the path connecting v_3 and v_4 to the other end vertex of e_3. Triangulate all the edges if the corresponding vertices are not adjacent in S. If the homogeneous path is all-negative add one negative edge to the common vertex of e_3 and e_4. Attach positive edges, corresponding to the ***type A*** vertices in the path v_4, v_5 to the other end vertex of e_4. Draw e_5 starting from this vertex meeting at the end vertex of e_1 which is not an incident vertex of e_1 and e_2. Triangulate all the edges if the vertices corresponding to those edges are nonadjacent in S but incident in S'. If the homogeneous path is all-negative attach an additional negative edge to the common vertex of e_4 and e_5 in S'. Then $\Gamma_{\bullet}(S')$ contains S.

If C_n contains more than 5 ***type B*** vertices $v_1,\ldots v_5,\ldots,v_n$. Let $e_1,\ldots e_5,\ldots,e_n$ be the positive edges corresponding to these ***type B*** vertices. For constructing S' draw $e_1, e_2, \ldots, e_n$ in such a way that they form a C_n in S'. Attach the edges corresponding to the ***type A*** vertices between the vertices v_i and v_{i+1} in the common vertex of e_i and e_{i+1}. If any of the homogeneous path between v_i and v_{i+1} is all-negative add an additional negative edge to the common vertex of e_i and e_{i+1}. Triangulate all the edges if the edges are incident in S' but the corresponding vertices are not adjacent in S. Clearly $\Gamma_{\bullet}(S')$ contains S.

(5) Let $v_1, v_2, \ldots, v_n$ denote the vertices of the cycle. To find a ***Γ•–semi root*** consider $H = C_n^{\,c} \vee \{v\}$. Let $e_1, e_2, \ldots, e_n$ denote the edges corresponding to $v_1, v_2, \ldots, v_n$. In H take $e_i = vv_i$ and assign positive sign to $e_1, e_2, \ldots, e_n$. If the cycle is all-negative attach one edge with negative sign to the vertex v. Clearly this graph is a ***Γ•–semi root*** for the cycle. Consider an arbitrary vertex v_i in the cycle if it is a ***type A*** vertex there exists vertices

outside the cycle adjacent to v_i. By the definition of ***type A*** vertex at least one outside edge is of opposite sign. In this case attach positive edge corresponding to the end vertex of this edge to the vertex v_i in H. If the edge is of negative sign attach one additional negative edge to the vertex v_i in H. So the vertices v and v_i receive opposite sign. So depending on the sign of edges incident to v_i attach edges with positive sign to either v or v_i. Repeat the same for other vertices.

Theorem 2.3.3. In a Signed graph S if a heterogeneous cycle C_n contains exactly four ***type B*** vertices, the graph is forbidden for dot-Gallai sigraph if and only if,
(1) all the ***type B*** vertices induce a P_4.
(2) any three ***type B*** vertices induce a P_3 and the fourth ***type B*** vertex is independent to it.
(3) at most one pair of ***type B*** vertices are adjacent (forms an edge) and both the neighboring edges of this edge in the cycle has sign opposite to that of this edge.
Proof: Let the ***type B*** vertices be u, v, w and z. By observation 3, the path connecting the pairs (u,v) is a homogeneous path. Similarly for the other continuous pairs (v,w), (w,z) and (z,u) the paths are homogeneous. Since the cycle is heterogeneous, any one of the path is of opposite sign. Let p, q, r, s denote the edges corresponding to u, v, w, z respectively. Let $S' = (H,\mu)$ denote a ***Γ.–semi root*** of S.

(1) Let u, v, w, z induce a P_4 in S. Since the path connecting z and u is homogeneous and the vertices between them is of ***type A***, by observation 3 the edges corresponding to these vertices and the edges p and s should have a common vertex in S'. Since z and u are non adjacent the edges p and s belong to a K_3. In S if the edge uv is of opposite sign, in S' we can attach the edge q only to the other end vertex of p. But in this case we cannot draw the edge r such that w is adjacent to both v and x in S. So the only possibility is that the edge uv is of the same sign of the edges between the vertices u and z and we can attach the edge q to the common vertex of p and s. Since the vertex w is adjacent to both v and z here the only possibility is join edge r to the common vertex of p, q and s. Then all the edges of the cycle in S receive the same sign of the common vertex of p, q, r and s. Then the cycle becomes a homogeneous cycle. That is a contradiction.

(2) Let the vertices that induce p_3 be u, v, w and z be not adjacent to all of them. As explained above the edges p and s should have a common vertex and belong to a K_3. If the edge uv has opposite sign as that of, the sign of the edges in the path connecting u and z, in S' we can join the edge q only to the other end vertex of p. Since w is adjacent to v in S and by observation 3 the edge r should have common vertex with the edges q and s in S'. But any way we draw r it affects the adjacency between the vertices in S. So, that is not possible. So the only possibility is the edge uv is of the same sign as that of the sign of, the edges in the path connecting u and z. Then we can join the edge q to the common vertex of p and s. Since w is adjacent to v and by observation 3, the edge r should have common vertex with the edges q and s. The only possibility is join r to the common vertex of p, q and s, contradicting the cycle is heterogeneous.

(3) Let the adjacent pair is u and v. In this case uv is adjacent in S implies in S', the edges p and q have a common vertex. Since the path connecting u and z is homogeneous

(observation 3) and is of opposite sign we can join the edge *s* only to the other end vertex of *p*. Since *u* and *z* are nonadjacent in *S* make the edges *p* and *s* belong to a K_3. Since the edges in the homogeneous path connecting *v* and *w* also have the opposite sign as that of the edge *uv* we can join *r* only to the other end vertex of *q* which is not incident with *p*. Also by observation 3, *r* should have common vertex with *s* and both the edges belong to a K_3 since *w* and *z* are nonadjacent in *S*. Anyway *p* and *q* form a K_3 which contradicts the adjacency of *u* and *v* in *S*.

To prove the converse, assume that the 4 ***type B*** vertices do not induce any one of the graphs given in the theorem. Then the following cases arise,

(4) at most one pair of ***type B*** vertices adjacent and at least one of the neighboring edge in the cycle has same sign to that of this edge.
(4) two disjoint pairs of ***type B*** vertices are adjacent.
(5) all the four ***type B*** vertices are nonadjacent.
(6) ***type B*** vertices form a cycle.

In these cases edges means edges with positive sign.

(4) As in the above case we assume the adjacent pair is *u* and *v* and the homogeneous path which has same sign as that of edge *uv* is the path connecting *v* and *w*. *uv* is an edge in *S* implies, in S' the edges *p* and *q* have a common vertex. Also the path connecting *u* and *z* is homogeneous (observation 3) in S'. Whatever be the sign of the edges in the path we can join the edge *s* together with the edges corresponding to the ***type A*** vertices in the *(u,z)* path to other end vertex of the edge *p*. Triangulate the edges if the corresponding vertices are not adjacent in *S*. So the edges *p* and *s* belong to a K_3. Since the edges in the homogeneous path connecting *v* and *w* have the same sign as that of the edge *uv* take *r* as the edge connecting the end vertices of *p* and *s* so that the edges *p, r, s* form a K_3. Join the edges corresponding to the ***type A*** vertices in the homogeneous path connecting *v* and *w* to the common vertex of *p,q* and *r* in S'. Triangulate all the edges if the corresponding vertices are non adjacent in *S*. Also join the edges corresponding to the ***type A*** vertices in the homogeneous path connecting *w* and *z* to the common vertex of *r* and *s* in S'. Triangulate all the edges if the corresponding vertices are non adjacent in *S*. By observation 3 if any path consists of negative edges then add an additional negative edge to the common vertex of the edges corresponding to the ***type A*** vertices in the path. Clearly $\Gamma_{\bullet}(S')$contains *S* as an induced subgraph.

(5) Without loss of generality let *(u,v)* and *(w,z)* be the adjacent pairs. In S'draw edges p and *q* with a common vertex. With respect to the ***type A*** vertices in the homogeneous path connecting *u* and *z* draw edges including *s* in the pendant vertex of *p*. Triangulate the edges if the corresponding vertices are non adjacent in *S*. Then *p* and *s* belongs to a K_3. Corresponding to the ***type A*** vertices in the homogeneous path connecting *v* and *w*, add edges at the pendant vertex of *q*, where *r* is drawn in such a way that *p, q, r* and *s* form a C_4. Triangulate the edges if the corresponding vertices are non adjacent in *S*. If any one of the homogeneous path joining the ***type B*** vertices are all-negative in *S* then in S', add an additional negative edge to the common vertex of the edges corresponding to the vertices in the homogeneous path. Clearly $\Gamma_{\bullet}(S')$contains *S* as an induced subgraph.

(6) The vertices *u, v, w* and *z* are non-adjacent. In S' draw edges *p, q, r* and *s* in such a way that they form a C_4. Since the path connecting every consecutive pairs of vertices are homogeneous, corresponding to the ***type A*** vertices inside each homogeneous path draw edges at the common vertex of, the edges corresponding to the end vertices of the homogeneous path in S'. Triangulate all the edges if the corresponding vertices are non adjacent in *S* but are incident in S'. Then *u, v, w* and *z* form the outer cycle of a K_4. Clearly $\Gamma_{\bullet}(S')$ contains *S* as an induced subgraph.

(7) As discussed in lemma 2.3.2(1) find a ***Γ•–semi root*** for the cycle. Consider an arbitrary vertex in the cycle. For convenience we consider the vertex *u*. If the edges in the cycle incident to *u* are of opposite sign, in ***Γ•–semi root*** one end vertex of *p* is positive and the other is negative. If there exist edges not in the cycle incident to *u*, if an edge is positive (or negative) attach one edge to the positive (or negative) end vertex of *p*. Since the vertex *u* is a ***type B*** vertex, if the edges incident to *u* in the cycle are of the same sign, then the edges incident to *u* outside the cycle are also of the same sign. So, in ***Γ•–semi root*** the end vertices of *p* receive the same sign. In this case we can attach the edges corresponding to the other adjacent vertices of *u* to any end vertex of *p*. Triangulate all the edges which are incident in S' but the corresponding vertices are not adjacent in *S*. Do the same procedure for the remaining vertices. Thus we have a ***Γ•–semi root*** for *S*.

Corollary 2.3.4. A heterogeneous C_5 with one ***type A*** vertex is forbidden.

Theorem 2.3.5. In a Signed graph if a heterogeneous cycle contains three ***type B*** vertices, the graph is forbidden for dot-Gallai sigraph if and only if at least two ***type B*** vertices are adjacent.

Proof: Let the ***type B*** vertices be *u, v* and *w*. By observation 3, the paths connecting the pairs *(u,v)* is a homogeneous path. Similarly for the other continuous pairs *(v,w)* and *(w,u)*. Since the cycle is heterogeneous any one of the path is of opposite sign. For convenience consider the path *(w,u)* is of opposite sign. Let *p, q, r* denote the edges corresponding to *u, v, w* respectively. Let $S' = (H,\mu)$ denote a ***Γ•–semi root*** of *S*. By above discussion, the edges *p* and *q* should have a common vertex. Since the edge *r* has common vertex with both *q* and *p* and the path *(w, u)* is of opposite sign implies that the only way we can draw *r* is, the edges *p, q* and *r* form a K_3. Then in $\Gamma_{\bullet}(S')$ the vertices *u, v* and *w* become non adjacent. So we can find a ***Γ•–semi root*** of the cycle given in the theorem only when all the ***type B*** vertices are not adjacent to each other. That is the given cycle is forbidden if and only if at least two ***type B*** vertices are adjacent.

Corollary 2.3.6. A heterogeneous C_4 with one ***type A*** vertex is forbidden.

Theorem 2.3.7. In a Signed graph if a heterogeneous cycle contains only two ***type B*** vertices then the graph is forbidden for dot-Gallai sigraph.

Proof: Let the ***type B*** vertices be *u* and *v*. By observation 3, the paths connecting the pairs *(u,v)* are homogeneous paths. Since, the cycle is heterogeneous out of two paths connecting u and v, one path consists of negative edges and other with positive edges. Let *p, q* denote the edges corresponding to *u, v* respectively. Let $S' = (H,\mu)$ denote a ***Γ•–semi root*** of *S*. By observation 3, the edges *p* and *q* should have a common positive vertex. Also since the other path is negative the edges *p* and *q* should have a common negative vertex in S', which is not possible. Hence the theorem is proved.

Note: For convenience we call the cycles which are not forbidden viz. lemma 2.3.2, theorems 2.3.3 and 2.3.5 as permissible cycles. Also in the constructions of ***Γ•–semi root*** for the permissible cycles discussed in the above theorems if a vertex is of ***type B*** having opposite adjacency in the cycle then the end vertices of the corresponding edge in the ***Γ•–semi root*** receive opposite signs. And if a vertex is of ***type B*** having same adjacency in the cycle then the end vertices of the corresponding edge in the ***Γ•–semi root*** receive same signs in the constructions discussed in the above theorems except in case 4 of theorem 2.3.3. In this case also except the edge q all the other edges satisfy the conditions noted above. For edge q, one end vertex of q is a pendant vertex with positive sign. If the other end vertex of q is negative, there is no ambiguity to change the sign of pendant vertex as negative by adding an additional negative edge to the pendant vertex. Thus the edge q also satisfies the above condition.

Theorem 2.3.8. If S consist of two permissible cycles whose intersection is a path then $S = (G,\sigma)$ is not forbidden for dot-Gallai sigraph.
Proof: Let S consist of two permissible cycles whose intersection is a path P_n with vertices $v_1, v_2, \ldots, v_n$

Case 1: All the vertices in the cycles are of ***type B***.
To find a ***Γ•–semi root*** for S, corresponding to the common vertices $v_1, v_2, \ldots, v_n$ of the two cycles draw positive edges $e_1, e_2, \ldots, e_n$. In S if the edge v_iv_{i+1} is a negative edge add an additional negative edge to the common vertex of e_i and e_{i+1}. Now by the method discussed in lemma 2.3.2(1) find a ***Γ•–semi root*** of the first cycle. Starting from the edge e_1 again, extend this ***Γ•–semi root*** by adding edges as on lemma 2.3.2(1) to get a ***Γ•–semi root*** of S.

Case 2: There exists ***type A*** vertices.
Consider the first cycle identify the number of ***type B*** vertices. Depending upon the number of ***type B*** vertices since the cycle is a permissible cycle by using any one of the above methods viz. lemma 2.3.2, theorems 2.3.3 and 2.3.5 find a ***Γ•–semi root*** of the first cycle. Consider the edges $e_1, e_2, \ldots, e_n$ corresponding to the vertices $v_1, v_2, \ldots, v_n$ in ***Γ•–semi root*** of the first cycle. The vertices $v_2, v_3, \ldots, v_{n-1}$ have the same behavior in both cycles. That is same behavior means when we consider the cycles independently if v_i, $1 < i < n$ is of ***type A*** (or ***type B***) with respect to the first cycle then with respect to the second cycle also it is of ***type A*** (or ***type B***). So the problem may arise only in the end vertices v_1 and v_n of the path .The behavior of v_1 and v_n may be opposite. That is opposite behavior means when we consider the cycles independently if v_1 (or v_n) is of ***type A*** (or ***type B***) with respect to the first cycle then with respect to the second cycle it is of ***type B*** (or ***type A***).

Consider the case v_1 and v_n have same behavior in both the cycles. If v_1 (or v_n) is of ***type A*** then in the ***Γ•–semi root*** the end vertices of e_1 (or e_n) already receive opposite sign. Then by considering the total number of ***type B*** vertices of the second cycle and by using any one of the above said methods we can extend the above ***Γ•–semi root*** to find S' such that $\Gamma_\bullet(S')$ contains S as an induced subgraph.

If v_1 (or v_n) is of ***type B*** in both cycles then either the edges incident to it should have same sign in both cycles or if the edges of the first cycle incident to it is of opposite sign then the edges of the second cycle incident to it also have opposite sign otherwise it will be of ***type A***. By the above note, in the first case the end vertices of e_1 (or e_n) in $\Gamma_{\bullet}(S')$ have same sign and in the second case, the end vertices of e_1 (or e_n) in $\Gamma_{\bullet}(S')$ have opposite sign. Then by considering the total number of ***type B*** vertices of the second cycle and by using any one of the above said methods we can extend the above ***$\Gamma_{\bullet}$–semi root*** to find S' such that $\Gamma_{\bullet}(S')$ contains S as an induced subgraph.

Consider the case v_1 (or v_n) has opposite behavior in both the cycles. Let as assume v_1 (or v_n) is of ***type B*** when we consider the first cycle only and is of ***type A*** when we consider the second cycle also. Then the edges of the first cycle incident to it have opposite sign, otherwise it will contradict our assumption. Since the edges incident to v_1 is of opposite sign, in ***$\Gamma_{\bullet}$–semi root*** of first cycle the end vertices of e_1 already receive opposite signs. So by using any one of the above said methods, we can find a ***$\Gamma_{\bullet}$–semi root*** for S. If we assume v_1 (or v_n) is of ***type A*** when we consider the first cycle only and is of ***type B*** when we consider the second cycle also. Then the edges of the second cycle incident to it have opposite sign, otherwise it will contradict our assumption. Since the vertex v_1 is of ***type A***, in ***$\Gamma_{\bullet}$–semi root*** of first cycle the end vertices of e_1 already receive opposite signs. So by using any one of the above said methods, we can find a ***$\Gamma_{\bullet}$–semi root*** for S.

Corollary 2.3.9. If S consists of more than two permissible cycles with common intersections then S is not forbidden for dot-Gallai sigraph.

Theorem 2.3.10. The only forbidden subgraphs of Dot-Gallai Sigraph are the signed graphs discussed in lemma 2.3.1, theorems 2.3.3, 2.3.5 and 2.3.7.

Proof: Consider an arbitrary signed graph $S = (G,\sigma)$. By using the fact that, any graph is the union of cycles and trees we find S' with $\Gamma_{\bullet}(S')$ contains S as an induced subgraph. First of all consider all the signed cycles in S and by using corollary 2.3.9 find a signed graph whose dot-Gallai sigraph contains all the cycles in S as an induced subgraph. Now we can extend this signed graph such that its dot-Gallai contains the given signed graph. For that consider the vertices common to the cycles and trees. Let v be an arbitrary vertex common to a cycle and a tree. It may be a ***type A*** vertex or ***type B*** vertex. If it is a ***type A*** vertex corresponding to this vertex we have already a positive edge with one end vertex as positive and other as negative. (This is applicable if 'v' is a vertex common to more than one cycle, for one cycle it is of ***type A*** and for some other it is of ***type B***). Then starting from this vertex v by using the construction in lemma 2.3.2(3), we can extend the signed graph such that its dot-Gallai also contains the tree. If v is a ***type B*** vertex and if the edges of the cycles incident to it is of opposite signs the end vertices of the edge corresponding to v in the ***$\Gamma_{\bullet}$–semi root*** already receive opposite signs. So starting from v by using the construction in lemma 2.3.2(3) we can find the extension. If v is a ***type B*** vertex and if the edges of the cycles incident to it is of same sign then the end vertices of the edge corresponding to 'v' receive the same sign. In this case by the definition of ***type B*** vertex, the edges incident to v not in the cycle also have the same sign. So starting from v by using the construction in lemma 2.3.2(3), we can find the extension. Considering the common vertices of cycles and trees and by using the construction in lemma 2.3.2(3) again we can find S' with $\Gamma_{\bullet}(S')$ contains S.

3. Conclusion and open problems

Gallai graphs do not admit forbidden subgraph characterization but in this paper we characterized Gallai signed graph, product Gallai signed graph and dot-Gallai signed graph using forbidden subgraphs. Though Krause-type characterization of dot-line sigraph is discussed in [1] characterization of dot-line sigraph using forbidden subgraphs still remain open.

Acknowledgements. The first author thank University Grants Commission for granting fellowship under Faculty Development Programme (F.No.FIP/12th plan/KLCA059 TF 06). The second author thank Kerala State Council for Science, Technology and Environment for sanctioning major project (No.005/KSYSARG/2014/KSCSTE)

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