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All the Solutions of the Diophantine Equation $p^4 + q^2 = z^2$ when p is Prime

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Abstract. In this paper, we consider the title equation in the particular case when p is prime. It is established that the equation has exactly two distinct solutions. One solution for each and every prime $p \geq 3$, the other solution for each and every prime $p \geq 2$. The solutions are demonstrated for each prime p in the form of identities. Furthermore, the connection between the equation and the Pythagorean triples is also discussed when the prime p is replaced by any odd value $A \geq 3$.

Keywords: Diophantine equations

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 4, 5, 6, 7, 8]. The title equation stems from the equation $p^x + q^y = z^2$.

Whereas in most articles, the values x, y are investigated for the solutions of the equation, in this paper these values are fixed positive integers. In the equation $p^4 + q^2 = z^2$ we consider all primes $p \geq 2$ and $q > 1$. We are mainly interested in how many solutions exist for any given prime p. This is established in Section 2. In Section 3, we discuss the connection between the equation and the Pythagorean triple, i.e., $a^2 + b^2 = c^2$.

2. All the solutions of $p^4 + q^2 = z^2$ when p is prime

In this section we consider the equation $p^4 + q^2 = z^2$ for all primes $p \geq 2$. For $p = 2$ it will be shown that the equation has exactly one solution. For each and every $p \geq 3$, we shall establish the existence of two distinct solutions demonstrated in the form of two identities. This is done in the following Theorem 2.1.

Theorem 2.1. Suppose that $p \geq 2$ is prime, and

$$p^4 + q^2 = z^2. \tag{1}$$

The values q, z in **(a)** and in **(b)**

(a) $q = \frac{p^4 - 1}{2}$, $z = \frac{p^4 + 1}{2}$, $p \ge 3$,

(b) $q = \frac{(p-1)p(p+1)}{2}$, $z = \frac{p(p^2+1)}{2}$, $p \ge 2$,

form two distinct solutions of equation (1).

Proof: The equation $p^4 + q^2 = z^2$ yields

$$p^4 = z^2 - q^2 = (z - q)(z + q). \tag{2}$$

Since p *is* prime, it follows that the values $z - q$ and $z + q$ in (2) satisfy five possibilities, three of which are a priori impossible. Hence, we have

(i) $z - q = 1$ and $z + q = p^4$,
(ii) $z - q = p$ and $z + q = p^3$.

(i) Suppose $z - q = 1$ and $z + q = p^4$. The value $z - q = 1$ yields $z = q + 1$ implying $2q + 1 = p^4$ and $q = \frac{p^4 - 1}{2}$. The sum of $z - q = 1$ and $z + q = p^4$ is equal to $2z = p^4 + 1$ or $z = \frac{p^4 + 1}{2}$. Hence, the values $q = \frac{p^4 - 1}{2}$ and $z = \frac{p^4 + 1}{2}$ satisfy equation (1) for all primes $p \ge 3$ and **(a)** is established.

(ii) Suppose $z - q = p$ and $z + q = p^3$. The sum of $z - q = p$ and $z + q = p^3$ implies that $2z = p(p^2 + 1)$ or $z = \frac{p(p^2+1)}{2}$. The difference of $z + q = p^3$ and $z - q = p$ yields $2q = p(p^2 - 1)$ or $q = \frac{p(p^2-1)}{2}$. Thus, the values $q = \frac{(p-1)p(p+1)}{2}$ and $z = \frac{p(p^2+1)}{2}$ satisfy equation (1) for each and every prime $p \ge 2$ and **(b)** has been established.

The two distinct solutions **(a)** and **(b)** are identities valid for each and every designated value p. Equation (1) has therefore infinitely many solutions.

This completes the proof of Theorem 2.1. □

3. The equation $p^4 + q^2 = z^2$ and the Pythagorean triples

In Section 2 we have considered equation (1) for all primes p. In this section, we omit the condition that $p \ge 3$ is prime, and use instead every odd value $A \ge 3$. We show that for every odd composite equal to A^2, equation (1) has a solution.

A set of positive integers a, b, c is called a "pythagorean triple" (abbreviated triple) denoted (a, b, c) if $a^2 + b^2 = c^2$. Let $a^2 + b^2 = c^2$ be a triple. For every integer $M > 1$, $Ma^2 + Mb^2 = Mc^2$ is also a triple. For example: The triple (7, 24, 25) yields the triples $(2 \cdot 7,\ 2 \cdot 24,\ 2 \cdot 25)$, $(3 \cdot 7,\ 3 \cdot 24,\ 3 \cdot 25)$ where respectively $M = 2, 3,$ and so on for $M > 3$. Suppose that A^2 is any odd value.

If $A^2 + b^2 = c^2$ is a triple, set $b = \frac{A^2 - 1}{2}$, $c = \frac{A^2 + 1}{2}$, and the triple is

$$A^2 + \frac{(A^2-1)^2}{4} = \frac{(A^2+1)^2}{4}.$$

The above triple with $M = A^2$ yields the triple

$$A^4 + \frac{(A^2-1)^2}{4}A^2 = \frac{(A^2+1)^2}{4}A^2.$$

Substituting the values $p^2 = A^2$, $q = \frac{(A^2-1)A}{2}$, $z = \frac{(A^2+1)A}{2}$ in $p^4 + q^2 = z^2$ results in $A^4 + \frac{((A^2-1)A)^2}{4} = \frac{((A^2+1)A)^2}{4}$. Thus, equation (1) has a solution for each and every odd composite equal to A^2. The equation has therefore infinitely many solutions.

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