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A Note on the Diophantine Equation $p^3 + q^2 = z^4$ when p is Prime

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Abstract. In this note, the conditions for solutions in positive integers q and z of the title equation are investigated and established. The first 200 consecutive primes $p_1 = 2$ and $p_{200} = 1223$ are considered. For $p_9 = 23$, the equation has a unique solution which is exhibited. For each of the remaining 199 primes, it has been verified that the equation has no solution. It is conjectured that for each and every prime p_i where $i > 200$, the equation has no solution in positive integers.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 3, 4, 6, 7, 8]. The title equation stems from the equation $p^x + q^y = z^2$.

Whereas in most articles, the values x, y are investigated for the solutions of the equation, in this paper these values are fixed positive integers. In the equation

$$p^3 + q^2 = z^4 \tag{1}$$

we consider all primes $p \geq 2$ and $q > 1$. We are mainly interested in how many solutions exist for any given prime p. This is established in Section 2.

2. Some results on $p^3 + q^2 = z^4$ when p is prime

In this section, some results on the solutions of the equation are established.

From the equation $p^3 + q^2 = z^4$ with prime $p \geq 2$ we have

$$p^3 = z^4 - q^2 = (z^2 - q)(z^2 + q). \tag{2}$$

The values $z^2 - q$ and $z^2 + q$ satisfy four possibilities, two of which are a priori impossible ($z^2 - q = p^2$ and $z^2 + q = p$, $z^2 - q = p^3$ and $z^2 + q = 1$.). We have

(i) $z^2 - q = 1$ and $z^2 + q = p^3$,
(ii) $z^2 - q = p$ and $z^2 + q = p^2$.

First, we show that when $p = 2$, the equation has no solutions. If $p = 2$, then (i) and (ii) respectively yield $2z^2 = 9$, and $2z^2 = 6$, each of which is impossible. Thus, $p \neq 2$ and $p > 2$ in (2).

(ii) The sum $z^2 - q = p$ and $z^2 + q = p^2$ yields

$$2z^2 = p(p + 1). \tag{3}$$

It follows from (3) that $p \mid z^2$ and therefore $p \mid z$. This is impossible in (3), and hence possibility (ii) does not exist.

The equation $p^3 + q^2 = z^4$ has no solutions in positive integers satisfying (ii).

(i) The sum $z^2 - q = 1$ and $z^2 + q = p^3$ yields

$$z^2 = \frac{p^3+1}{2} = \frac{(p+1)}{2}(p^2 - p + 1), \tag{4}$$

but z as an integer is yet undetermined. Let M, N be positive integers. Either $\frac{p+1}{2} = M^2$, $(p^2 - p + 1) = N^2$ so that $z^2 = M^2 N^2 = (MN)^2$, then $z = MN$ is an integer, or $\frac{p+1}{2} = M$, $(p^2 - p + 1) = N$ so that $z^2 = MN$, and the decomposition of (MN) consists of primes each of which has an even exponent, then $z = \sqrt{MN}$ is an integer. We now show that in (4) the value $(p^2 - p + 1)$ is never equal to a square.

Let p be fixed. We shall assume that there exists an integer A satisfying

$$p^2 - p + 1 = A^2 \tag{5}$$

and reach a contradiction.

From (5) we have $p(p - 1) = A^2 - 1 = (A - 1)(A + 1)$. Thus, $p \mid (A - 1)$ or $p \mid (A + 1)$. If $p \mid (A - 1)$, denote $pB = A - 1$ where $B \geq 1$ is an integer. Then $p(p - 1) = pB(pB + 2)$ is clearly impossible for all values B. If $p \mid (A + 1)$, denote $pC = A + 1$ where $C \geq 1$ is an integer. When $C = 1$ we have $p = A + 1$ and $p^2 - p + 1 = (A + 1)^2 - (A + 1) + 1 = A^2 + A + 1$ contrary to (5). For all values $C > 1$, we obtain that $p(p - 1) = (pC - 2)(pC)$ which is impossible too. Thus, for all $p > 2$, $p^2 - p + 1$ never equals a square. Our assumption (5) is therefore false. Hence, for all values p, $p^2 - p + 1 \neq N^2$.

If $z^2 = MN$, and (MN) satisfies the above criterion, then equation (1) has a solution in positive integers $q = z^2 - 1 = MN - 1$ and $z = \sqrt{MN}$.

For the first 200 consecutive primes p_i where $p_{200} = 1223$, it has been verified $p_9 = 23$ yields a solution, whereas for the remaining 199 primes no solutions of equation (1) exist since z is not an integer.

When $p = 23$, set $\frac{p+1}{2} = M$, $p^2 - p + 1 = N$ and $z^2 = MN$. We have accordingly $M = 12$ and $N = 507$. Hence

$$z^2 = \frac{(p+1)}{2}(p^2 - p + 1) = 12 \cdot 507 = (2^2 \cdot 3)(3 \cdot 13^2) = 2^2 \cdot 3^2 \cdot 13^2 = (2 \cdot 3 \cdot 13)^2$$

and

$$23^3 + (7 \cdot 11 \cdot 79)^2 = (2 \cdot 3 \cdot 13)^4 \quad \text{or} \quad 12167 + 37002889 = 37015056$$

is a solution of $p^3 + q^2 = z^4$.

Remark. Mathematicians, among them the very famous John Nash, Nobel Prize – winner in Economics viewed the prime 23 as a unique and fascinating number. Many interesting facts including the so-called "The 23 Enigma" may be found in the literature.

3. Conclusion

In our study, mentioned in Section 2, we have considered each of the first 200 primes up to $p_{200} = 1223$. For $p_9 = 23$, we have demonstrated the unique solution. For each of the other 199 primes, it was verified that equation (1) has no solution. This leads us to the following conjecture.

Conjecture. In the equation $p^3 + q^2 = z^4$ let p be prime. Then, for each and every prime $p > 1223$, the equation has no solution in positive integers q and z.

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