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On Weak Convergence of Filters and Nets

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Abstract. In this paper, we introduced an equivalence between weak convergence of filters and weak convergence of nets and show that, in Uryson spaces, as was done for filters in [1], weak limits of nets are unique. Moreover, we show that, in a regular space X, with $E \subseteq X$, $x \in \overline{E}$ if and only if there is a net in X which converges weakly to x. We also prove that closure continuous maps preserve weak convergence of nets. As a main result, we prove that, in regular spaces, weak convergence of nets is equivalent to their usual convergence, once again, a mimic of filters in [1].

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1. Introduction and preliminaries

The concept of weak convergence of sequences and nets was studied not very long time ago [2,3]. In this paper, we further study weak convergence of nets and obtain some useful results. In particular, and among other results, we prove, that in regular spaces, convergence and weak convergence of nets are equivalent.

We would like, in this introduction, to note that topologists nowadays prefer to insert their applications in generalized, or enlarged settings. This might help escape tight limits and specifications. For this we urge interested readers to compare with [6,7]. But to be more specific, One can consult [8,9] for the general setting of rough set theory.

For the sake of completeness, we would like to put forth the following definitions followed by a remark.

Definition 1.1. [3] If (x_λ) is a net in a topological space X and $x \in X$, then (x_λ) is said to converge weakly to x (written $x_\lambda \xrightarrow{w} x$) if x_λ *is eventually in* $\overline{U}$.

Definition 1.2. [1] (i) A filter $\Im^{/}$ is said to be finer than the filter $\Im$ if *for all* $F \in \Im, \exists F^{/} \in \Im^{/}$ *such that* $F^{/} \subseteq F$. This is written $\Im \subseteq \Im^{/}$ *or*, $\Im^{/} \geq \Im$.

(ii) If $\Im$ is a filter on a topological space X and $x \in X$, then $\Im$ is said to converge weakly to x (written $\Im \xrightarrow{w} x$) if $\Im$ *is finer than* $\overline{U_x}$. That is,

$\Im \geq \overline{U_x}$ *where* $\overline{U_x}$ *is the filter generated by the collection* $\{\overline{U} : U \in U_x\}$.

We should note here that it is an easy check that *If* $\Im \to x$ *then* $\Im \xrightarrow{w} x$ but not conversely [1].

It is common practice to let U_x stand for the neighborhood system of x in a topological space X on which a filter $\Im$ is given. Of course, U_x is a filter on X. We say that the filter $\Im$ converges to x, and we write $\Im \to x$ *if* $\Im \geq U_x$.

If β is a filter base on X, then the family $\Im = \{F : F \supseteq B \text{ for some } B \in \beta\}$ is the filter generated by β. If $\Im$ is a filter on a topological space X then, by $\overline{\Im}$ we mean the filter generated by the filter base $\beta = \{\overline{F} : F \in \Im\}$, where $\overline{F}$ stands for the closure of F. It is clear that $\overline{\Im} \leq \Im$.

Remark 1.3. [3] One can easily check that *if* $x_\lambda \to x$ *then* $x_\lambda \xrightarrow{w} x$.

The fact that the foregoing implication can not be reversed will be an immediate consequence of theorem (2.4) together with example (2.2) of [1].

It remains to mention that some mathematicians take the other side of open sets and mappings. For example, some of them work in the framework of pre-γ-open sets and mappings, as in [4]. Others do their implantation in pre generalized pre regular weakly open sets and neighborhoods, as in [5] .

2. Main results

The interaction between filters and nets, in terms of their evolution, from each other, or in terms of their convergence, was settled much earlier by the fifties of the last century, [10]. Now, the issue is being concerned with the recently introduced weak convergence of both filters and nets, and the possibility of finding the same interaction. Our results gave affirmative answers and here is a list of the results we had so far.

Theorem 2.1. A net (x_λ) in a topological space X converges weakly to $x \in X$ if and only if the filter generated by (x_λ) converges weakly to x.

Proof: Suppose, first, that $x_\lambda \xrightarrow{w} x$, and let $\Im$ be the filter generated by (x_λ). In specific, let D be the directed set on which the net (x_λ) is defined, and let for $\alpha \in D$, $B_\alpha = \{x_\beta : \beta \geq \alpha\}$. This is the $\alpha - tail$ of the net (x_λ). So, the filter $\Im$ generated by the net (x_λ) is the filter generated by the collection $\{B_\alpha : \alpha \in D\}$. We will show that $\Im \xrightarrow{w} x$. i.e. we need $\Im \geq \overline{U_x}$.

Let $\overline{U} \in \overline{U_x}$ where U is a neighborhood of x.

Since $x_\lambda \xrightarrow{w} x$, there is $\lambda_\circ \in D$ *such that* $x_\beta \in \overline{U}$ *for all* $\beta \geq \lambda_\circ$. Now, it follows that $B_{\lambda_\circ} = \{x_\beta : \beta \geq \lambda_\circ\} \in \Im$ *and* $B_{\lambda_\circ} \subseteq \overline{U}$. Therefore, $\Im \geq \overline{U_x}$, *i.e.* $\Im \xrightarrow{w} x$.

Conversely, suppose that the filter $\Im$ generated by the net (x_λ) converges weakly to x . Let $\overline{U} \in \overline{U_x}$ where U is a neighborhood of x .

Since $\Im \geq \overline{U_x}$, *there is* $F \in \Im$ *such that* $F \subseteq \overline{U}$, and since $\Im$ is generated by $\{B_\alpha : \alpha \in D\}$, *there is* $\alpha_\circ \in D$ *such that* $B_{\alpha_\circ} \subseteq F$. Thus $B_{\alpha_\circ} \subseteq \overline{U}$. By the very definition of $B_{\alpha_\circ}$, *we have that* (x_λ) *is in* $\overline{U}$ *eventually*. *Hence*, $x_\lambda \xrightarrow{w} x$.

The dual of the foregoing theorem should be investigated, and here it is:

Theorem 2.2. Let $\Im$ be a filter on a space X . Then,

$\Im \xrightarrow{w} x$ *if and only if* the net generated by $\Im$ converges weakly to x .

Proof: Suppose that $\Im \xrightarrow{w} x$, so $\Im \geq \overline{U_x}$.

Let (x_λ) be the net generated by $\Im$. In specific, let $D = \{(x,F) : x \in F \text{ and } F \in \Im\}$ being ordered as follows: $(x_1, F_1) \leq (x_2, F_2)$ *if and only if* $F_2 \subseteq F_1$.
This, of course gives a partial order for D on which we, now define the net (x_λ) *as*: $x_\lambda((x,F)) = x$.

To show that $x_\lambda \xrightarrow{w} x$, let U be a neighborhood of x .So, $\overline{U} \subseteq \overline{U_x}$.
Since $\Im \geq \overline{U_x}$, $\overline{U} \in \Im$. By the definition of D, *we have* $(x, \overline{U}) \in D$.
If $(y,F) \geq (x, \overline{U})$, *then* $F \subseteq \overline{U}$ *and so* $x_\lambda((y,F)) = y \in F \subseteq \overline{U}$.
Therefore, $x_\lambda \xrightarrow{w} x$.
Conversely, assume that the net (x_λ) *based on* $\Im$ *converges weakly to x.* So $x_\lambda \xrightarrow{w} x$. We will show that $\Im \geq \overline{U_x}$. Let $\overline{U} \in \overline{U_x}$ where U is any neighborhood of x. Since $x_\lambda \xrightarrow{w} x$ there is $(x_\circ, F_\circ) \in D$ such that: if $(y,F) \geq (x_\circ, F_\circ)$ then $x_\lambda((y,F)) = y \in \overline{U}$.
For any $y \in F_\circ$, *and because* $(y, F_\circ) \geq (x_\circ, F_\circ)$, *we have*: $y \in \overline{U}$. Thus, there is $F_\circ \in \Im$ *such that* $F_\circ \subseteq \overline{U}$, *so* $\Im \geq \overline{U_x}$ *which means that* $\Im \xrightarrow{w} x$.
Now, we characterize closure points of open sets in terms of weak convergence. In particular we give:

Theorem 2.3. Let E be an open set in a topological space X. Then:
$x \in \overline{E}$ *if and only if there exists a filter* $\Im$ *in* X *with* $\Im \xrightarrow{w} x$ *and* $F \cap E \neq \phi$ *for all* $F \in \Im$.

Proof: Just combine Theorem (2.2) with theorem (2.3) of [3]. In specific, suppose that $x \in \overline{E}$ .Then, by (2.3) of [3], there is a net $(x_\lambda)\, in\ E\ with\ x_\lambda \xrightarrow{w} x.$ Let $\Im$ be the filter in X generated by (x_λ). By theorem (2.2), $\Im \xrightarrow{w} x.$
Remains to show that $F \cap E \neq \phi$ *for every* $F \in \Im.$ For this, let $F \in \Im$ be arbitrary.
By the very construction of $\Im$, *there is* $\beta_\alpha \subseteq E$ *for some* $\alpha \in D$ *such that* $\beta_\alpha \subseteq F.$
Hence, $F \cap E \neq \phi$ *for all* $F \in \Im.$

Conversely, suppose that $\Im$ is a filter in X *which converges weakly to* x and that $F \cap E \neq \phi$ *for every* $F \in \Im.$ Assume that $x \notin \overline{E}.$ Then there is a neighborhood U *of* x such that $U \cap E = \phi$. Since E *is open* $\overline{U} \cap E = \phi.$
But $\Im \geq \overline{U_x}$ *and* $F \cap E \neq \phi$ *for all* $F \in \Im,$ we have, in particular, $\overline{U} \cap E \neq \phi$ *for all* $U \in U_x$ which is a contradiction. Therefore $x \in \overline{E}.$

Having done weak convergence, one has to consider weak closures. The following is the definition:

Definition 2.4. Let $E \subseteq X$ *and* $x \in X$. Then x is a weak closure point of E if for every neighborhood W *of* x, $\overline{W} \cap E \neq \phi.$ The set of all weak closure points of E *is denoted* $\overline{E}^w.$

We characterize weak closure points as follows:

Theorem 2.5. [Theorem (3.3) of [1]**]** Let $E \subseteq X$ *and* $x \in X$.Then

$x \in \overline{E}^w$ *if and only if there is a filter* $\Im$ *in* X *with*
$\Im \xrightarrow{w} x$ *and* $F \cap E \neq \phi$ *for all* $F \in \Im.$

Our characterization is the following:

Theorem 2.6. Let $E \subseteq X$ *and* $x \in X$. Then

$x \in \overline{E}^w$ *if and only if there is a net* (x_λ) *in* E *with* $x_\lambda \xrightarrow{w} x.$

Proof: Suppose that $x \in \overline{E}^w$. By theorem (3.3) of [1], choose a filter $\Im$ *in* X *which converges weakly to* x, *and* $E \cap F \neq \phi$ *for all* $F \in \Im.$
Let T *be the filter in* E *generated by the collection* $\{E \cap F \text{ with } F \in \Im\}.$
Clearly $\mathrm{T} \geq \Im,$ *so* $T \xrightarrow{w} x.$
Now, let (x_λ) be the net generated by $\mathrm{T}.$ It is clear that (x_λ) is a net in $E.$
By Theorem (2.1), $x_\lambda \xrightarrow{w} x.$

Conversely, suppose that there is a net $(x_\lambda)\, in\ E\ with\ x_\lambda \xrightarrow{w} x$. Let $\Im$ be the filter in X generated by the net $(x_\lambda).$ By Theorem (2.2), $\Im \xrightarrow{w} x.$

To show that $F \bigcap E \neq \phi$ *for all* $F \in \Im$, *let* $F \in \Im$ *be arbitrary.* There is $\alpha_{\circ} \in D$ *such that* $\beta_{\alpha_{\circ}} \subseteq F$. But $\beta_{\alpha_{\circ}} \subseteq E$ *so* $F \bigcap E \pm \phi$ *for all* $F \in \Im$.

Hence, by Theorem (2.3), $x \in \overline{E}^{w}$.

By Theorem (2.2) and [Theorem (3.6) of [1]], we get:

Theorem 2.7. If X is a regular space, and (x_λ) is a net on X, then

$x_\lambda \xrightarrow{w} x$ *if and only if* $x_\lambda \to x$.

We conclude this article with a few notes on subnets, among which the next fact is straight forward check.

Fact 2.8. (i) If (x_{λ_γ}) is a subnet of a net (x_λ) then the filter generated by (x_{λ_γ}) is finer than that generated by (x_λ).

(ii) *If* $\Im$ *and* $\Im'$ *are filters on* X *with* $\Im' \geq \Im$ *and if* $\Im \xrightarrow{w} x$ *then* $\Im' \xrightarrow{w} x$ *also*.

Proposition 2.9. Let X be a topological space, and (x_λ) be a net in X. If $x_\lambda \xrightarrow{w} x$ then any subnet (x_{λ_γ}) of (x_λ) also converges weakly to x.

Proof: Let $\Im$ be the filter generated by (x_λ), and $\Im'$ be that generated by the subnet (x_{λ_γ}) Then by Theorem (2.1) $\Im \xrightarrow{w} x$. By Fact (2.8)(i) $\Im' \geq \Im$ therefore by Fact (2.8) (ii) $\Im' \xrightarrow{w} x$.

3. Conclusion

In conclusion, most of what we had in the ordinary setting, the interaction between filters and nets remains valid in this new setting of weak convergence, as for example, in characterizations of weak closures. Moreover, in regular spaces, nothing changes at all in terms of convergence. The reason however, is simply because the collection of all closed neighborhoods of a point x makes a local base for the topology at x [10].

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