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The Number of *r*-Relatively Prime Subsets and *Phi* Functions for $\{m, m+1, \ldots, n\}$- its *r*-Generalization

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Abstract. A nonempty finite set A of positive integers is relatively prime if $\gcd(A)=1$ and it is relatively prime to n if $\gcd(A\cup\{n\})=1$. For $k\geq 1$, the number of nonempty subsets A of {1,2,…,n} which are relatively prime is $f(n)$ and the number of such subsets of cardinality k, is $f_k(n)$. The number of nonempty subsets A of {1,2,…,n} which are relatively prime to n is $\Phi(n)$ and the number of such subsets of cardinality k is $\Phi_k(n)$. A nonempty finite set A of positive integers is r-relatively prime if greatest r th power common divisor of elements of A is 1. In this case we write $\gcd_r(A)=1$. The number of r-relatively prime subsets of {1,2,…,n} is $f^{(r)}(n)$ and the number of such subsets of cardinality k is $f_k^{(r)}(n)$. The number of nonempty subsets A of {1,2,…,n} which are r-relatively prime to n is $\Phi^{(r)}(n)$ and the number of such subsets of cardinality k is $\Phi_k^{(r)}(n)$. In this paper we generalize the four functions $f^{(r)}(n)$, $f_k^{(r)}(n)$, $\Phi^{(r)}(n)$ and $\Phi_k^{(r)}(n)$ for the set $\{m, m+1, \ldots, n\}$ where $m\leq n$, as follows **:**

$$f^{(r)}(m,n)=\#\{A\subseteq\{m, m+1, \ldots, n\} : A\neq\phi, \gcd_r(A)=1\}$$
$$f_k^{(r)}(m,n)=\#\{A\subseteq\{m, m+1, \ldots, n\} : \#A=k, \gcd_r(A)=1\}$$
$$\Phi^{(r)}(m,n)=\#\{A\subseteq\{m, m+1, \ldots, n\} : A\neq\phi, \gcd_r(A\cup\{n\})=1\}$$

$\Phi_k^{(r)}(m, n) = \#\{A \subseteq \{m, m+1, ..., n\} : \# A = k , \ \gcd_r(A \cup \{n\}) = 1\}$.

We obtain the exact formulae for the above four functions. We use r-Generalized Mobius inversion formula for two variables.

Keywords: *r*-relatively prime sets, *r*-Generalization of *Phi* functions, *r*-Generalized Mobius inversion for two variables.

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1. Introduction

A nonempty finite set A of positive integers is relatively prime if $\gcd(A) = 1$ and it is relatively prime to n if $\gcd(A \cup \{n\}) = 1$. The number of nonempty subsets of {1,2,…,n} which are relatively prime is $f(n)$ and the number of such subsets of cardinality k is $f_k(n)$. The number of nonempty subsets A of {1, 2 ,…, n} which are relatively prime to n is $\Phi(n)$ and the number of such subsets of cardinality k is $\Phi_k(n)$. A nonempty finite set A of positive integers {1, 2 ,…, n} is r-relatively prime if greatest r$^{\text{th}}$ power common divisor of elements of A is 1. In this case, we write $\gcd_r(A) = 1$. The number of r-relatively prime subsets of {1, 2,…, n} is $f^{(r)}(n)$ and the number of such subsets of cardinality k is $f_k^{(r)}(n)$. The number of nonempty subsets *A* of {1, 2 ,…, n} which are r-relatively prime to n is $\Phi^{(r)}(n)$ and the number of such subsets of cardinality k is $\Phi_k^{(r)}(n)$. Nathanson [2] obtained the exact formulae for the functions $f(n)$, $f_k(n)$, $\Phi(n)$ and $\Phi_k(n)$. Inspired by the work of Nathanson [2] , we obtained the exact formulae for $f^{(r)}(n)$, $f_k^{(r)}(n)$, $\Phi^{(r)}(n)$ and $\Phi_k^{(r)}(n)$ in [1]. Bachraoui [3] generalized the results of Nathanson [2]. In this paper we generalize the results obtained in [1] and derive the explicit formulae for the functions $f^{(r)}(m, n)$, $f_k^{(r)}(m, n)$, $\Phi^{(r)}(m, n)$ and $\Phi_k^{(r)}(m, n)$ by using r-Generalized Mobius inversion formula for two variables.

Definition 1. Let r ≥ 1 be a positive integer. $F^{(r)}$ and $G^{(r)}$ are generalized arithmetical functions of two variables with domain $(0, \infty)^2$ such that

$G^{(r)}(y_1, y_2) = 0 = F^{(r)}(y_1, y_2)$ whenever $0 < y_2 < 1$.

Definition 2. *r* - Relatively prime subsets of {m, m+1, …, n}. Let $r \geq 1,\ k \geq 1$ be fixed integers .Then we define,

$$g^{(r)}(m,\ n) = \#\{A \subseteq \{m,\ m+1,\ \ldots,\ n\}\ :\ m \in A,\ \gcd_r(A) = 1\}$$

$$g_k^{(r)}(m,\ n) = \#\{A \subseteq \{m,\ m+1,\ \ldots,\ n\}\ :\ m \in A,\ \#A = k,\ \gcd_r(A) = 1\}$$

$$f^{(r)}(m,\ n) = \#\{A \subseteq \{m,\ m+1,\ \ldots,\ n\}\ :\ A \neq \phi,\ \gcd_r(A) = 1\}$$

$$f_k^{(r)}(m,\ n) = \#\{A \subseteq \{m,\ m+1,\ \ldots,\ n\}\ :\ A \neq \phi,\ \#A = k,\ \gcd_r(A) = 1\}.$$

Note that $f^{(r)},\ f_k^{(r)},\ g^{(r)},\ g_k^{(r)}$ are counting sets on nonempty subsets.

$f^{(r)}(n) =$ Number of r - relatively prime subsets of $\{1,\ 2,\ \ldots,\ n\} = f^{(r)}(1,\ n)$.

Definition 3. Let $n \geq m$ be positive integers. We define

$$\psi^{(r)}(m,\ n) = \#\{A \subseteq \{m,\ m+1,\ \ldots,\ n\}\ :\ m \in A,\ \gcd_r(A \cup \{n\}) = 1\}$$

$$\psi_k^{(r)}(m,\ n) = \#\{A \subseteq \{m,\ m+1,\ \ldots,\ n\}\ :\ m \in A,\ \#A = k,\ \gcd_r(A \cup \{n\}) = 1\}$$

$$\Phi^{(r)}(m,\ n) = \#\{A \subseteq \{m,\ m+1,\ \ldots,\ n\}\ :\ A \neq \phi,\ \gcd_r(A \cup \{n\}) = 1\}$$

$$\Phi_k^{(r)}(m,\ n) = \#\{A \subseteq \{m,\ m+1,\ \ldots,\ n\}\ : \#A = k\ ,\ \gcd_r(A \cup \{n\}) = 1\}.$$

Note that the four functions defined above count the sets that are nonempty.

$\Phi^{(r)}(\ n)$ =Number of non empty subsets of $\{1,\ 2,\ \ldots,\ n\}$ that are r-relatively prime to n

$= \quad \Phi^{(r)}(1,n)$

2. r-Generalized Mobius inversion formula for two variables

Theorem 1. Let $F^{(r)}$ and $G^{(r)}$ be generalized arithmetical functions with domain $(0,\ \infty)^2$ then

$$G^{(r)}(m,\ n) = \sum_{1 \leq d^r \leq m} F^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right) \Leftrightarrow F^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right) = \sum_{1 \leq d^r \leq m} \mu_r\left(d^r\right) G^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right) \quad (1)$$

where the summation is over all integers d such that $d^r \leq m$.

Proof: Assume $G^{(r)}(m,\ n) = \sum_{1 \leq d^r \leq m} F^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right)$

Consider

$$\sum_{1\le d^r\le m}\mu_r\left(d^r\right)G^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right)=\sum_{1\le d^r\le m}\mu_r\left(d^r\right)\left[\sum_{1\le t^r\le \frac{m}{d^r}}F^{(r)}\left(\frac{m}{t^r d^r},\ \frac{n}{t^r d^r}\right)\right]$$

$$=\sum_{1\le u^r\le m}F^{(r)}\left(\frac{m}{u^r},\ \frac{n}{u^r}\right)\left[\sum_{d^r/u^r}\mu_r\left(d^r\right)\right]$$

$$=F^{(r)}(m,\ n).$$

Conversely, assume $F^{(r)}(m,\ n)=\sum_{1\le d^r\le m}\mu_r\left(d^r\right)G^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right).$

Consider

$$\sum_{1\le d^r\le m}F^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right)=\sum_{1\le d^r\le m}\left[\sum_{1\le t^r\le \frac{m}{d^r}}\mu_r\left(d^r\right)G^{(r)}\left(\frac{m}{t^r d^r},\ \frac{n}{t^r d^r}\right)\right]$$

$$=\sum_{1\le u^r\le m}G^{(r)}\left(\frac{m}{u^r},\ \frac{n}{u^r}\right)\left[\sum_{t^r/u^r}\mu_r\left(t^r\right)\right]$$

$$=G^{(r)}(m,\ n).$$

Corollary 2. Suppose $F^{(r)}(x,\ y)$ and $G^{(r)}(x,\ y)$ are generalized arithmetical functions such that $F^{(r)}(x,\ y)=0=G^{(r)}(x,\ y)$ if x is not an integer and $F^{(r)}(x,\ y)=0=G^{(r)}(x,\ y)$ if $0<y<1$, then Mobius inversion formula for such functions reduces to the following form

$$G^{(r)}(m,\ n)=\sum_{d^r|m}F^{(r)}\left(\frac{m}{d^r},\ \left[\frac{n}{d^r}\right]\right)\Leftrightarrow F^{(r)}(m,\ n)=\sum_{d^r|m}\mu_r\left(d^r\right)\left(\frac{m}{d^r},\ \left[\frac{n}{d^r}\right]\right)\quad(2)$$

Proof: Suppose $G^{(r)}(m,\ n)=\sum_{d^r|m}F^{(r)}\left(\frac{m}{d^r},\ \left[\frac{n}{d^r}\right]\right)$ holds.

Consider

$$\sum_{d^r|m} \mu_r\left(d^r\right)G^{(r)}\left(\frac{m}{d^r},\ \left[\frac{n}{d^r}\right]\right) = \sum_{d^r|m} \mu_r\left(d^r\right)\left(\sum_{t^r|\frac{m}{d^r}} F^{(r)}\left(\frac{m}{t^r d^r},\ \left[\frac{n}{t^r d^r}\right]\right)\right)$$

$$= \sum_{u^r|m} F^{(r)}\left(\frac{m}{u^r},\ \left[\frac{n}{u^r}\right]\right)\left(\sum_{d^r|u^r} \mu_r\left(d^r\right)\right)$$

$$= F^{(r)}(m,\ n).$$

Conversely, suppose $F^{(r)}(m,\ n) = \sum_{d^r|m} \mu_r\left(d^r\right)G^{(r)}\left(\frac{m}{d^r},\ \left[\frac{n}{d^r}\right]\right)$

Consider $$\sum_{d^r|m} F^{(r)}\left(\frac{m}{d^r},\ \left[\frac{n}{d^r}\right]\right) = \sum_{d^r|m}\left(\sum_{t^r|\frac{m}{d^r}} \mu_r\left(t^r\right)G^{(r)}\left(\frac{m}{t^r d^r},\ \left[\frac{n}{t^r d^r}\right]\right)\right)$$

$$= \sum_{u^r|m} G^{(r)}\left(\frac{m}{u^r},\ \left[\frac{n}{u^r}\right]\right)\left(\sum_{t^r|u^r} \mu_r\left(t^r\right)\right)$$

$$= G^{(r)}(m,\ n).$$

3. Exact formulae

To find the exact formulas we use the following theorems proved in [1].

Theorem 3. For all positive integers $n \geq 2^r$ $r \geq 1$, *k*,

(i) $$f^{(r)}(n) = \sum_{1 \leq d^r \leq n} \mu_r\left(d^r\right)\left(2^{\left[\frac{n}{d^r}\right]} - 1\right) = f^{(r)}(1,\ n) \tag{3}$$

(ii) $$f_k^{(r)}(n) = \sum_{1 \leq d^r \leq n} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]}{k} = f_k^{(r)}(1,\ n). \tag{4}$$

Theorem 4. For all positive integers n, $r \geq 1$, *k*,

(i) $\Phi^{(r)}(n) = \sum_{d^r/n} \mu^r\left(d^r\right)\left(2^{\frac{n}{d^r}} - 1\right) = \Phi^{(r)}(1, n).$ (5)

(ii) $\Phi_k^{(r)}(n) = \sum_{d^r/n} \mu^r\left(d^r\right)\binom{\frac{n}{d^r}}{k} = \Phi_k^{(r)}(1, n).$ (6)

Lemma 5. If $n \geq m,$ we have

(i) $g^{(r)}(m, n) = \sum_{d^r | m} \mu_r\left(d^r\right) 2^{\left[\frac{n}{d^r}\right] - \frac{m}{d^r}}$ (7)

(ii) $g_k^{(r)}(m, n) = \sum_{d^r | m} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right] - \frac{m}{d^r}}{k-1}.$ (8)

Proof: (i) Let $P(m, n)$ denotes the set of subsets of $\{m, m+1, ..., n\}$ containing *m*. Then # $P(m, n) = 2^{n-m}$. We define a relation on $P(m, n)$ as follows:

For $A, B \in P(m, n)$ we say that $A \cong B$ if and only if $\gcd_r(A) = d^r = \gcd_r(B)$. Note that $d^r \big| m$ and $1 \leq d^r \leq m.$ Then $\cong$ is an equivalence relation on $P(m, n)$. If *m* is *r*-free then $\gcd_r(A) = 1 = \gcd_r(B)$. Hence the elements of $P(m, n)$ can be partitioned using the relation "$\cong$" of having the same r^{th} power gcd d^r. Also the mapping $A \to \frac{1}{d^r}A$ is a 1-1 correspondence between the subsets of $P(m, n)$ having r^{th} power gcd d^r and *r*-relatively prime subsets of $\left\{\frac{m}{d^r}, ..., \frac{n}{d^r}\right\}$ which contain $\frac{m}{d^r}$. Then we have the following identity $2^{n-m} = \sum_{d^r | m} g^{(r)}\left(\frac{m}{d^r}, \left[\frac{n}{d^r}\right]\right).$

By Corollary (2) $g^{(r)}(m, n) = \sum_{d^r | m} \mu_r\left(d^r\right) 2^{\left[\frac{n}{d^r}\right] - \frac{m}{d^r}}$ which proves (7).

(ii) Note that the correspondence $A \to \frac{1}{d^r}A$ preserves the Cardinality and hence

$$\binom{n-m}{k-1} = \sum_{d^r \mid m} g_k^{(r)}\left(\frac{m}{d^r}, \left[\frac{n}{d^r}\right]\right).$$

By Corollary (2), we have

$$g_k^{(r)}(m, n) = \sum_{d^r \mid m} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right] - \frac{m}{d^r}}{k-1}.$$ which proves (8).

Theorem 6. If $n \geq m$, then the following two identities are true.

(i) $$f^{(r)}(m, n) = \sum_{1 \leq d^r \leq n} \mu_r\left(d^r\right)\left(2^{\left[\frac{n}{d^r}\right]} - 1\right) - \sum_{i=1}^{m-1} \sum_{d^r \mid i} \mu_r\left(d^r\right) 2^{\left[\frac{n}{d^r}\right] - \frac{i}{d^r}} \tag{9}$$

(ii) $$f_k^{(r)}(m, n) = \sum_{1 \leq d^r \leq n} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]}{k} - \sum_{i=1}^{m-1} \sum_{d^r \mid i} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right] - \frac{i}{d^r}}{k-1}. \tag{10}$$

Proof: (i) Repeatedly applying Lemma (5) and equations (7) and (3)

$$\begin{aligned} f^{(r)}(m, n) &= f^{(r)}(m-1, n) - g^{(r)}(m-1, n) \\ &= f^{(r)}(m-2, n) - \left[g^{(r)}(m-2, n) + g^{(r)}(m-1, n)\right] \\ &\vdots \\ &= f^{(r)}(1, n) - \sum_{i=1}^{m-1} g^r(i, n) \\ &= \sum_{1 \leq d^r \leq n} \mu_r\left(d^r\right)\left(2^{\left[\frac{n}{d^r}\right]} - 1\right) - \sum_{i=1}^{m-1}\left(\sum_{d^r \mid i} \mu_r\left(d^r\right) 2^{\left[\frac{n}{d^r}\right] - \frac{i}{d^r}}\right) \end{aligned}$$

(ii) From equation (4), we have $f_k^{(r)}(1, n) = \sum_{1 \leq d^r \leq n} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]}{k}$.

By equation (8), we have

$$g_k^{(r)}(m,\ n) = \sum_{d^r \mid m} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]-\frac{m}{d^r}}{k-1}$$

Therefore $f_k^{(r)}(m,\ n) = f_k^{(r)}(m-1,\ n) - g_k^{(r)}(m-1,\ n)$

$$= f_k^{(r)}(m-2,\ n) - \left[g_k^{(r)}(m-2,\ n) + g_k^{(r)}(m-1,\ n)\right]$$

$$\vdots$$

$$= f_k^{(r)}(1,\ n) - \sum_{i=1}^{m-1} g_k^{(r)}(i,\ n)$$

$$= \sum_{1\le d^r \le n} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]}{k} - \sum_{i=1}^{m-1}\sum_{d^r \mid i} \mu_r\left(d^r\right)\binom{\left[\frac{n}{d^r}\right]-\frac{i}{d^r}}{k-1}.$$

Note : $\gcd_r\left(A\cup\{n\}\right)=1 \iff \left(\gcd_r(A),\ n\right)_r = 1.$

Proof: Assume $\gcd_r\left(A\cup\{n\}\right)=1.$

Let $\left(\gcd_r(A),\ n\right)_r = d^r$

$\Rightarrow d^r \mid n$ and $d^r \mid \gcd_r(A)$ (since $\gcd_r(A) \mid a$ for all $a\in A$)

$\Rightarrow d^r \mid n$ and $d^r \mid a$ for all $a \in A$

$\Rightarrow d^r \mid \gcd_r\left(A\cup\{n\}\right) \Rightarrow d^r \mid 1 \Rightarrow d=1.$

Conversely, let $\left(\gcd_r(A),\ n\right)_r = 1$ and $\gcd_r\left(A\cup\{n\}\right)=d^r$

$\Rightarrow d^r \mid a$ for all $a\in A$ and $d^r \mid n$

$\Rightarrow d^r \mid a$ and $d^r \mid n$ for all $a \in A$

$\Rightarrow d^r \mid \gcd_r(A)$ and $d^r \mid n$

$\Rightarrow d^r \mid \left(\gcd_r(A),\ n\right)_r$

$\Rightarrow d^r \mid 1 \Rightarrow d=1.$

Lemma 7. If $n_2 \ge n_1$ are natural numbers and $G^{(r)},\ F^{(r)}$ are generalized arithmetical functions with domain $(0,\ \infty)^2$. Then

$$G^{(r)}(n_1,\ n_2) = \sum_{d^r \mid \gcd_r(n_1,\, n_2)} F^{(r)}\left(\frac{n_1}{d^r},\ \frac{n_2}{d^r}\right)$$

$$\Leftrightarrow F^{(r)}(n_1,\ n_2) = \sum_{d^r \mid \gcd_r(n_1,\, n_2)} \mu_r\left(d^r\right) G^{(r)}\left(\frac{n_1}{d^r},\ \frac{n_2}{d^r}\right)$$

Proof: Assume $G^{(r)}(n_1,\ n_2) = \sum_{d^r \mid \gcd_r(n_1,\, n_2)} F^{(r)}\left(\frac{n_1}{d^r},\ \frac{n_2}{d^r}\right)$

Consider $\sum_{d^r \mid \gcd_r(n_1,\, n_2)} \mu_r\left(d^r\right) G^{(r)}\left(\frac{n_1}{d^r},\ \frac{n_2}{d^r}\right)$

$$= \sum_{d^r \mid \gcd_r(n_1,\, n_2)} \mu_r\left(d^r\right)\left(\sum_{t^r \mid \gcd_r\left(\frac{n_1}{d^r},\, \frac{n_2}{d^r}\right)} F^{(r)}\left(\frac{n_1}{t^r d^r},\ \frac{n_2}{t^r d^r}\right)\right)$$

$$= \sum_{u^r \mid \gcd_r(n_1,\, n_2)} F^{(r)}\left(\frac{n_1}{u^r},\ \frac{n_2}{u^r}\right)\left(\sum_{d^r \mid u^r} \mu_r\left(d^r\right)\right) = F^{(r)}(n_1,\ n_2).$$

Conversely, assume $F^{(r)}(n_1,\ n_2) = \sum_{d^r \mid \gcd_r(n_1,\, n_2)} \mu_r\left(d^r\right) G^{(r)}\left(\frac{n_1}{d^r},\ \frac{n_2}{d^r}\right)$

Consider

$$\sum_{d^r \mid \gcd_r(n_1,\, n_2)} F^{(r)}\left(\frac{n_1}{d^r},\ \frac{n_2}{d^r}\right) = \sum_{d^r \mid \gcd_r(n_1,\, n_2)}\left(\sum_{t^r \mid \gcd_r\left(\frac{n_1}{d^r},\, \frac{n_2}{d^r}\right)} \mu_r\left(t^r\right) G^{(r)}\left(\frac{n_1}{t^r d^r},\ \frac{n_2}{t^r d^r}\right)\right)$$

$$= \sum_{u^r \mid \gcd_r(n_1,\, n_2)} G^{(r)}\left(\frac{n_1}{u^r},\ \frac{n_2}{u^r}\right)\left(\sum_{t^r \mid u^r} \mu_r\left(t^r\right)\right) = G^{(r)}(n_1,\ n_2).$$

Theorem 8. If $n \geq m$, then we have the following two identities

(i) $\psi^{(r)}(m,\ n) = \sum_{d^r \mid \gcd_r(m,\ n)} \mu_r\left(d^r\right) 2^{\frac{n-m}{d^r}}$ (11)

(ii) $\psi_k^{(r)}(m,\ n) = \sum_{d^r \mid \gcd_r(m,\ n)} \mu_r\left(d^r\right) \binom{\frac{n-m}{d^r}}{k}.$ (12)

Proof: (i) Let $P(m,\ n)$ denotes the set of subsets of $\{m,\ m+1,\ ...,\ n\}$ containing *m*. Then $\#\ P(m,\ n) = 2^{n-m}$. We define the set

$P\left(m,\ n,\ d^r\right) = \left\{A \subseteq \{m,\ m+1,\ ...,\ n\} : m \in A \text{ and } \gcd_r\left(A \cup \{n\}\right) = d^r\right\}$

The set $P(m,\ n)$ can be partitioned using the equivalence relation $"\cong"$ for having the same r^{th} power gcd, that is , $A \cong B \Leftrightarrow A,\ B \in P\left(m,\ n,\ d^r\right)$ for some $d^r \mid \gcd_r(m,\ n)$. Further, the mapping $A \to \frac{1}{d^r} A$ is a $1-1$ correspondence between $P\left(m,\ n,\ d^r\right)$ and the set of subsets *B* of $\left\{\frac{m}{d^r},\ \frac{m}{d^r}+1,\ \frac{m}{d^r}+2,\ ...,\ \frac{n}{d^r}\right\}$ such that $\frac{m}{d^r} \in B$ and $\gcd_r\left(B \cup \left\{\frac{n}{d^r}\right\}\right) = 1.$

Then we have $\# P\left(m,\ n,\ d^r\right) = \psi^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right).$

Thus we have $2^{n-m} = \sum_{d^r \mid \gcd_r(m,\ n)} \# P\left(m,\ n,\ d^r\right) = \sum_{d^r \mid \gcd_r(m,\ n)} \psi^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right)$

Therefore $2^{n-m} = \sum_{d^r \mid \gcd_r(m,\ n)} \psi^{(r)}\left(\frac{m}{d^r},\ \frac{n}{d^r}\right).$

Applying Lemma (7) we have

$$\psi^{(r)}(m,\ n) = \sum_{d^r \mid \gcd_r(m,\ n)} \mu_r\left(d^r\right) 2^{\frac{n-m}{d^r}} \quad \text{which proves (11).}$$

(ii) Note that the correspondence $A \to \frac{1}{d^r} A$ defined above preserves the Cardinality and hence using an argument similar to the one in part (i) , we have

$$\binom{n-m}{k-1} = \sum_{d^r \mid \gcd_r(m,\, n)} \psi_k^{(r)}\left(\frac{m}{d^r},\, \frac{n}{d^r}\right)$$

is equivalent to

$$\psi_k^{(r)}(m,\, n) = \sum_{d^r \mid \gcd_r(m,\, n)} \mu_r\left(d^r\right)\binom{\frac{n-m}{d^r}}{k-1}$$

which proves (12).

Theorem 9. If $n \geq m$, then the following two identities are true.

(i) $$\Phi^{(r)}(m,\, n) = \sum_{d^r \mid n} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r}} - 1\right) - \sum_{i=1}^{m-1} \sum_{d^r \mid \gcd_r(i,\, n)} \mu_r\left(d^r\right) 2^{\frac{n-i}{d^r}} \tag{13}$$

(ii) $$\Phi_k^{(r)}(m,\, n) = \sum_{d^r \mid n} \mu_r\left(d^r\right)\binom{\frac{n}{d^r}}{k} - \sum_{i=1}^{m-1} \sum_{d^r \mid \gcd_r(i,\, n)} \mu_r\left(d^r\right)\binom{\frac{n-i}{d^r}}{k-1}. \tag{14}$$

Proof: (i) Applying theorem (8) repeatedly and using the equation (5) we have

$$\begin{aligned}
\Phi^{(r)}(m,\, n) &= \Phi^{(r)}(m-1,\, n) - \psi^{(r)}(m-1,\, n) \\
&= \Phi^{(r)}(m-2,\, n) - \left[\psi^{(r)}(m-2,\, n) + \psi^{(r)}(m-1,\, n)\right] \\
&\;\;\vdots \\
&= \Phi^{(r)}(1,\, n) - \sum_{i=1}^{m-1} \psi^{(r)}(i,\, n) \\
&= \sum_{d^r \mid n} \mu_r\left(d^r\right)\left(2^{\frac{n}{d^r}} - 1\right) - \sum_{i=1}^{m-1}\left(\sum_{d^r \mid \gcd_r(i,\, n)} \mu_r\left(d^r\right) 2^{\frac{n-i}{d^r}}\right)
\end{aligned}$$

which proves (13).

(ii) Applying theorem (8) repeatedly and using equation (6)

$$\begin{aligned}
\Phi_k^{(r)}(m,\, n) &= \Phi_k^{(r)}(m-1,\, n) - \psi_k^{(r)}(m-1,\, n) \\
&= \Phi_k^{(r)}(m-2,\, n) - \left[\psi_k^{(r)}(m-2,\, n) + \psi_k^{(r)}(m-1,\, n)\right] \\
&\;\;\vdots
\end{aligned}$$

$$= \Phi_k^{(r)}(1,\ n) - \sum_{i=1}^{m-1} \psi_k^{(r)}(i,\ n)$$

$$= \sum_{d^r \mid n} \mu_r\left(d^r\right)\binom{\frac{n}{d^r}}{k} - \sum_{i=1}^{m-1}\left(\sum_{d^r \mid \gcd_r(i,\ n)} \mu_r\left(d^r\right)\binom{\frac{n-i}{d^r}}{k-1}\right).$$

which proves (14).

4. Conclusion

We count the exact number of nonempty subsets of { m , m+1 ,…, n} which are

(i) r-relatively prime
(ii) r-relatively prime of cardinality k.
(iii) r-relatively prime to n
(iv) r-relatively prime to n of cardinality k.

by using r-Generalized version of Mobius inversion formula for two variables.

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