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Prime Cordial Labeling for Theta Graph

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Abstract. A prime cordial labeling of a graph $G=(V(G),E(G))$ is a bijection f from V(G) to $\{1,2,\ldots|V(G)|\}$ such that for each edge uv is assigned the label 1 if $\gcd(f(u),f(v))=1$ and 0 if $\gcd(f(u),f(v))>1$; then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits a prime cordial labeling is called a prime cordial graph. In this paper, we investigate the prime cordial labeling of Theta graph. We also discuss prime cordial labeling in the context of some graph operations namely duplication, switching, fusion, path union and the graph obtained by joining two copies of Theta graph by a path of arbitrary length .

Keywords: Prime cordial labeling, fusion, switching, contraction, path union.

***AMS Mathematics Subject Classification (2010)*:** 05C78

1. Introduction

Labeling of vertices and edges play a vital role in graph theory. Graph labeling is an active area of research in graph theory which has rigorous applications in coding theory, communication networks, optimal circuits layouts and crystallography. For a dynamic survey on graph labeling we refer to Gallian [3]. We provide a brief summary of results which will be useful for the present investigations.

Definition 1.1. *If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.*

A detailed study on variety of applications of graph labeling is reported in Bloom [1].

Definition 1.2. *Let G be a graph. A mapping $f:V(G)\to\{0,1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

Pradhan and Kumar [4] have analyzed the L(2, 1)-Labeling on α-Product of Graphs.

For an edge $e=uv$, the induced edge labeling $f^*:E(G)\to\{0,1\}$ is given by $f^*(e)=|f(u)-f(v)|$. Let $v_f(0),v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f while $e_f(0),e_f(1)$ be the number of edges of

G having labels 0 and 1 respectively under f^*.

Vaidya and Vyas [7] have proved antimagic labeling of some path and cycle related graphs.

Definition 1.3. *A binary vertex labeling f of a graph G is called a cordial labeling if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. A graph G is called cordial if it admits cordial labeling.*

The concept of cordial labeling was introduced by Cahit [2]. Thirusangu et al. [6] proved some results on cordial labeling of duplicate graph of ladder graph.

Definition 1.4. *A prime cordial labeling of a graph G with vertex set $V(G)$ is a bijection $f: V(G) \to \{1,2,3,\ldots,|V(G)|\}$ defined by*

$$f(e = uv) = \begin{cases} 1, if \text{ gcd } (f(u), f(v)) = 1 \\ 0, \quad \text{otherwise} \end{cases}$$

further $|e_f(0)-e_f(1)| \leq 1$.

A graph which admits prime cordial labeling is called a *prime cordial graph*. The concept of prime cordial labeling was introduced by Sundaram et al. [5].

Now let us recall the definition of Theta graph and the graph operations such as duplication, switching, fusion and the path union of a graph.

Definition 1.5. *A Theta graph is a block with two non-adjacent vertices of degree 3 and all other vertices of degree 2 is called a Theta graph.*

Definition 1.6. *Duplication of a vertex v_k of a graph G produces a new graph G_1 by adding a vertex $v_{k'}$ with $N(v_{k'}) = N(v_k)$. In other words, a vertex $v_{k'}$ is said to be a duplication of v_k if all the vertices which are adjacent to v_k are now adjacent to $v_{k'}$.*

Definition 1.7. *A vertex switching of a graph G is a graph G_v obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.*

Definition 1.8. *Let u and v be any two distinct vertices of a graph G. A new graph G_1 is constructed by fusing (identifying) two vertices u and v by a single vertex x in G_1 such that every edge which was incident with either u or v in G now incident with x in G_1.*

Definition 1.9. *Let $G_1, G_2, G_3, \ldots G_n$, $n \geq 2$ be n copies of a fixed graph G. The graph obtained by adding an edge between G_i and G_i+1 for $i = 1,2,\ldots n-1$ is called the path union of G.*

2. Main results

Theorem 2.1. *The Theta graph T_α is not a prime cordial graph.*

Proof: If $v_0, v_1, v_2, \ldots, v_6$ are the vertices of the Theta graph $T\alpha$ with centre v_0 and $E(T\alpha) = \{v_i v_i + 1/1 \le i \le 5\} \bigcup \{v_0 v_1, v_0 v_4, v_1 v_6\}$, then $|V(T_\alpha)| = 7$ and $|E(T_\alpha)| = 8$.

We define vertex labeling $f : V(T_\alpha) \to \{1,2,3,\ldots,7\}$ as follows.

For the graph T_α the possible pairs of labels of adjacent vertices are $(1,2),(1,3),(1,4),(1,5),(1,6),(1,7)$, $(2,3),(2,4),(2,5),(2,6),(2,7)$, $(3,4),(3,5),(3,6),(3,7)$, $(4,5),(4,6),(4,7)$, $(5,6),(5,7),(6,7)$.

Out of these pairs, only the pairs $(2,4),(2,6),(3,6)$ and $(4,6)$ yields the edge label value as 0 and the remaining possible labeling of pairs yields the edge label value as 1. But any labeling of vertices in T_α must contain at most any three pairs only from $(2,4),(2,6),(3,6)$ and $(4,6)$. This implies that $e_f(0) \le 3$ and so $e_f(1) \ge 5$.

Then $|e_f(0) - e_f(1)| \ge 2$.

Therefore, T_α is not a prime cordial graph.

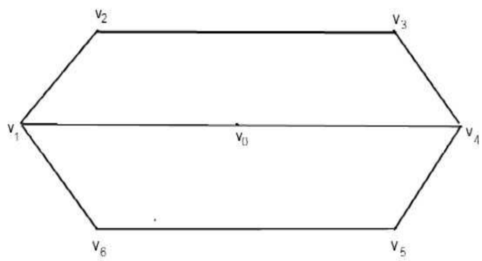

Figure 1: Theta graph

Theorem 2.2. *The duplication of any vertex of degree 3 in the Theta graph T_α is a prime cordial graph.*

Proof: If $v_0, v_1, v_2, \ldots, v_6$ are the vertices of the Theta graph $T\alpha$ with centre v_0 and $E(T\alpha) = \{v_i v_i + 1/1 \le i \le 5\} \bigcup \{v_0 v_1, v_0 v_4, v_1 v_6\}$, then $|V(T_\alpha)| = 7$ and $|E(T_\alpha)| = 8$.

Let G_i be a graph obtained from $T\alpha$ after duplication vertex of the vertex v_i of degree 3 in T_α and $v_{i'}$ be the duplication vertex of the vertex v_i of degree 3. In T_α, only two vertices are of degree 3. Let it be v_1 and v_4.

Clearly $|V(G_i)| = 8$. We define vertex labeling $f : V(G_i) \to \{1,2,3,\ldots,8\}$ as follows.

Let $f(v_0) = 4$.

Case (i): Duplication of the vertex v_4.

We define $f(v_{4'}) = 8$ where $v_{4'}$ is the duplicating vertex of v_4.

Further, $f(v_1)=5,\ f(v_2)=1,\ f(v_3)=3,\ f(v_4)=6,\ f(v_5)=2,\ f(v_6)=7$.

Case (ii): Duplication of the vertex v_1.

We define $f(v_{1'})=8$ where $v_{1'}$ is the duplicating vertex of v_1.

Further, $f(v_1)=6,\ f(v_2)=3,\ f(v_3)=1,\ f(v_4)=5,\ f(v_5)=7,\ f(v_6)=2$.

In view of the labeling pattern defined above we have $e_f(0)=5, e_f(1)=6.$

Thus in both cases we have $|e_f(0)-e_f(1)| \le 1$.

Hence the graph obtained by the duplication of any vertex v_i of degree 3 in the Theta graph T_α is a prime cordial graph.

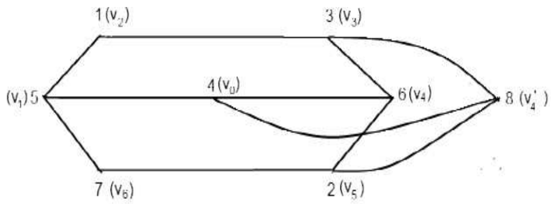

Figure 2: The duplication of the vertex v_4 in T_α is a prime cordial graph

Theorem 2.3. *The switching of any vertex of degree* 3 *in the Theta graph* T_α *is a prime cordial graph.*

Proof: If $v_0, v_1, v_2, \ldots, v_6$ are the vertices of the Theta graph $T\alpha$ with centre v_0 and $E(T\alpha)=\{v_i v_i+1/1\le i\le 5\}\bigcup\{v_0v_1, v_0v_4, v_1v_6\}$, then $|V(T_\alpha)|=7$ and $|E(T_\alpha)|=8$.

Let G_s be the graph obtained from $T\alpha$ after switching the vertex v_i of degree 3. In T_α, only two vertices are of degree 3. Let it be v_1 and v_4. Clearly $|V(G_s)|=7$. We define vertex labeling $f:V(G_s)\to\{1,2,3,\ldots,8\}$ as follows.

Let $f(v_0)=3$.

Case (i): switching of the vertex v_1

We define, $f(v_1)=2,\ f(v_2)=1,\ f(v_3)=4,\ f(v_4)=6,\ f(v_5)=5,\ f(v_6)=7$.

Case (ii): switching of the vertex v_4

We define, $f(v_1)=6,\ f(v_2)=2,\ f(v_3)=1,\ f(v_4)=4,\ f(v_5)=5,\ f(v_6)=7$.

In view of the labeling pattern defined above we have $e_f(0)=e_f(1)=4.$

Thus in both cases we have $|e_f(0)-e_f(1)| \le 1$.

Hence the graph G_s admits prime cordial graph.

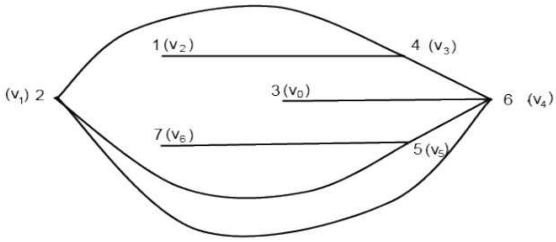

Figure 3: The switching of the vertex v_1 in T_α is a prime cordial graph

Theorem 2.4. *The fusion of any two vertices in the Theta graph T_α is a prime cordial graph.*

Proof: If $v_0, v_1, v_2, \ldots, v_7$ be the vertices of the Theta graph $T\alpha$ with centre v_4

$E(T\alpha) = \{v_i v_i + 1/1 \leq i \leq 6\}$, then $|V(T_\alpha)| = 7$ and $|E(T_\alpha)| = 8$.

Let G be a graph obtained by fusion of any two vertices in T_α. Then $|V(G)| = 6$ and $|E(G)| = 7$. We define vertex labeling $f : V(G) \to \{1,2,3,\ldots,|V(G)|\}$ as follows.

For the graph G the possible pairs of labels of adjacent vertices are $(1,2),(1,3),(1,4),(1,5),(1,6)$, $(2,3),(2,4),(2,5),(2,6)$, $(3,4),(3,5),(3,6)$, $(4,5),(4,6)$, $(5,6)$.

Out of these pairs, only the pairs $(2,4),(2,6),(3,6)$ and $(4,6)$ yields the edge label value as 0 and the remaining possible labeling of pairs yields the edge label value as 1. We choose the labeling of vertices in G contains any three pairs only from $(2,4),(2,6),(3,6)$ and $(4,6)$.

In view of the labeling pattern defined above we have $e_f(0) = 3, e_f(1) = 4.$

Then $|e_f(0) - e_f(1)| \leq 1$.

Hence the graph G admits prime cordial graph.

Lemma 2.5. *The graph G obtained by path union of two copies of theta graph T_α is a prime cordial graph.*

Proof: Let G be the graph obtained by path union of two copies of Theta graph T_α and $T_{\alpha'}$ respectively.

Let $u_1, u_2 \ldots, u_7$ be the vertices of first copy T_α and $v_1, v_2, \ldots, v_7$ be the vertices of second copy $T_{\alpha'}$.

$V(G) = V(T_\alpha) \bigcup V(T_{\alpha'})$ and $E(G) = E(T_\alpha) \bigcup E(T_{\alpha'}) \bigcup \{u_k v_k\}$.

Then $|V(G)| = 14$ and $|E(G)| = 17$.

We define vertex labeling $f : V(G_s) \to \{1,2,3,\ldots,|V(G)|\}$ such that

$f(u_i) = 2i$ for $1 \leq i \leq 7$

$f(v_i) = 2i - 1$ for $1 \le i \le 7$

This implies that $e_f(0) = 8$ in T_α , $e_f(1) = 8$ in $T_{\alpha'}$ and the edge connecting T_α and $T_{\alpha'}$ gets a label 1.

In view of the labeling pattern defined above we have $e_f(0) = 8, e_f(1) = 9$.

Thus we have $|e_f(0) - e_f(1)| \le 1$.

Hence G is a prime cordial graph.

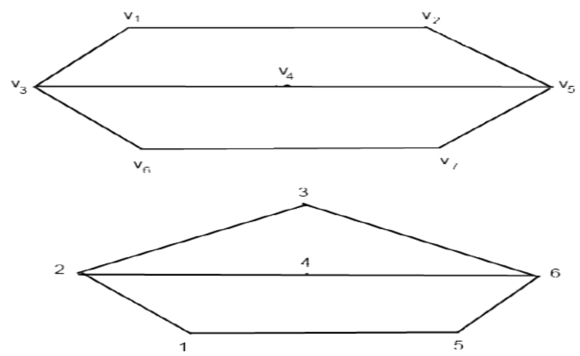

Figure 4: The fusion of the vertex v_1 and v_3 in T_α is a prime cordial graph

We extend this result to the path of any arbitrary length connecting two copies of Theta graph in the following theorem.

Theorem 2.6. *The graph G obtained by joining two copies of theta graph by a path of arbitrary length is prime cordial.*

Proof: Let G be the graph obtained by joining two copies of Theta graph by a path P_k, where k is arbitrary.

Let $u_1, u_2 \ldots, u_7$ be the vertices of first copy of Theta graph, $w_1, w_2, \ldots, w_7$ be the vertices of second copy of Theta graph and $v_1, v_2, \ldots, v_k$ be the vertices of path P_k and we choose $v_1 = u_1$ and $v_k = w_1$

Then $|V(G)| = 14 + k - 2$ and $|E(G)| = 16 + k - 1$.

To define vertex labeling $f : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$, we consider the following two cases.

Case 1: k is odd.

Let $k = 2t + 1$, $t \in N$.

We define, $f(v_1) = f(u_2) = 3$ and $f(v_k) = f(w_1)$.

Further,

$f(u_i) = 2i - 1$ *if* $1 \le i \le 7$

$$f(v_i) = \begin{cases} 11 + 2i, & if\ 2 \le i \le t + 1 \\ 2\{i - (t + 1)\}, & if\ t + 2 \le i \le 2t + 1 \end{cases}$$

and $f(w_i) = 2t + 2(i-1)$ if $1 \le i \le 7$.

In view of the labeling pattern defined above we have $e_f(0) = e_f(1) = \dfrac{15+k}{2}$.

Thus we have $|e_f(0) - e_f(1)| \le 1$.

Case 2: k is even.

Let $k = 2t+1$, $t \in N$.

We define, $f(v_1) = f(u_2) = 3$ and $f(v_k) = f(w_1)$.

Further,

$f(u_i) = 2i - 1$ if $1 \le i \le 7$

$$f(v_i) = \begin{cases} 11 + 2i, & if\ 2 \le i \le t \\ 2\{i - t\}, & if\ t + 1 \le i \le 2t \end{cases}$$

and $f(w_i) = 2t + 2(i-1)$ if $1 \le i \le 7$.

In view of the labeling pattern defined above we have when $k = 2$, $e_f(0) = 8, e_f(1) = 9$.

Also, $e_f(0) = 8 + \dfrac{k}{2} and e_f(1) = 7 + \dfrac{k}{2}$ for $k = 4,6,\ldots$.

Thus we have $|e_f(0) - e_f(1)| \le 1$.

Hence G is a prime cordial graph.

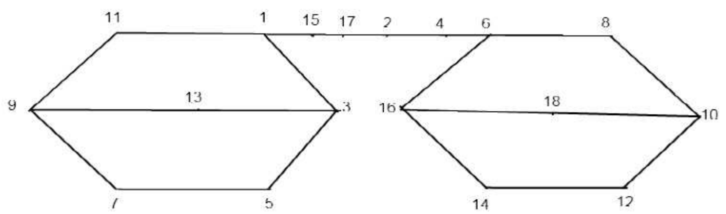

Figure 5: Prime cordial labeling of graph obtained by joining 2 copies of T_α by path P_6

3. Concluding remarks

In this paper, we investigated the prime cordial labeling of theta graph. We also proved that the prime cordial labeling in the context of some graph operations namely duplication, witching, fusion, path union and the graph obtained by joining two copies of Theta graph by a path of arbitrary length.

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