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Common Tripled Fixed Point Theorems forWeakly Compatible Mappings in $\mathcal{M}$ - Fuzzy Metric Spaces

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Abstract. In this paper, we present a common tripled fixed point theorem for weakly compatible mappings under ϕ-contractive condition in $\mathcal{M}$-fuzzy metric spaces. The result generalizes, extends and improves several classical and very recent related results of Sedghi, Altun and Shobe.

Keywords: Common fixed point theorem, tripled fixed point, generalized fuzzy metric spaces.

AMS Mathematics Subject Classification (2010): 47H10, 54H25

1. Introduction

The notion of fuzzy sets was introduced by Zadeh [16]. Since then to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. The concept of fuzzy metric space was introduced by Kramosil and Michalek [8]. George and Veeramani [5] studied the properties of fuzzy metric space. Grabiec [6] obtained the Banach contraction principle for this space. In 2006, Sedghi and Shobe [16] defined $\mathcal{M}$- fuzzy metric spaces and proved a common fixed point theorem for four weakly compatible mappings in this space.

The concept of tripled fixed point was introduced by Berinde and Borcut in [2]. Later, in 2012, Borcut and Berinde [3] presented the concept of tripled coincidence point for a pair of nonlinear contractive mappings F: $X \to X$ and g: $X \to X$ and obtained tripled coincidence point theorems which generalized the results of [2]. In 2013, Roldan and Martinez - Moreno etc., gave a slight modification of the concept of a tripled fixed point introduced by Berinde and Borcut [2] for nonlinear mappings, and established a common tripled fixed point theorem for contractive type mappings in $\mathcal{M}$- fuzzy metric spaces.

The aim of this paper is to introduce the concepts of weakly compatible mappings in $\mathcal{M}$ -fuzzy metric spaces. Based on this notion, a common tripled fixed point for mappings

$$F: X\times X \times X \to X \text{ and } g: X \to X$$

are obtained. The presented theorem generalizes, extends and improves several well known comparable results in the literature.

2. Preliminaries

Definition 2.1. [14] A binary operation $*: [0, 1] \times [0,1] \to [0, 1]$ is a continuous t - norm if it satisfies the following conditions:
(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$, for all $a \in [0, 1]$,
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.
Example of continuous t - norm is $a * b = \min\{a, b\}$.

Definition 2.2. [16] A 3 triple $(X, \mathcal{M}, *)$ is $\mathcal{M}$ - fuzzy metric space X is an arbitrary non - empty set, $*$ is a continuous t - norm and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions: For each x, y, z, a ∈ X and t, s > 0.
(FM - 1) $\mathcal{M}(x, y, z, t) > 0$,
(FM- 2) $\mathcal{M}(x, y, z, t) = 1$ if $x = y = z$,
(FM- 3) $\mathcal{M}(x, y, z, t) = M(p\{x, y, z\}, t)$, where p is a permutation function,
(FM - 4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \le \mathcal{M}(x, y, z, t + s)$,
(FM - 5) $\mathcal{M}(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous,
(FM - 6) $\lim_{t\to\infty} \mathcal{M}(x, y, z, t) = 1$.

Example 2.3. Let X be a non- empty set and D* is defined as D* - metric on X. Denote $a * b = ab$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define $\mathcal{M}(x, y, z, t) = \dfrac{t}{t + D^*(x, y, z)}$ for all $x, y, z \in X$, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$ - fuzzy metric space. We call this $\mathcal{M}$ - fuzzy metric induced by D* - metric space. Thus, every D* - metric induces a $\mathcal{M}$ - fuzzy metric.

Definition 2.4. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ - fuzzy metric space. $\mathcal{M}$ is said to satisfy the n - property on $X \times X \times X \times (0, \infty)$ if $\lim_{n \to \infty} [\mathcal{M}(x,y,z,l^{n}(t))]^{n^p} = 1$ whenever $x, y, z \in X$, $l > 1$ and $p > 0$. Define $\Phi = \{\phi: \mathbb{R}^+ \to \mathbb{R}^+\}$, where $\mathbb{R}^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfies the following conditions:
(ϕ -1) ϕ is non – decreasing,
(ϕ -2) ϕ is upper semi - continuous from the right,
(ϕ -3) $\sum_{n=0}^{\infty} \phi^n(t_0) < +\infty$ for all $t > 0$, where $\phi^{(n+1)}(t) = \phi(\phi^n(t))$, $n \in N$ It is easy to prove that if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

Lemma 2.5. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ - fuzzy metric space, where * is a continuous t - norm of H - type. If there exists $\phi \in \Phi$ such that $\mathcal{M}(x, y, z, \phi(t)) \ge \mathcal{M}(x, y, z, t)$ for all $t > 0$, then $x = y$.

Definition 2.6. An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of

$F : X \times X \times X \to X$ if $F(x, y, z) = x$; $F(y, z, x) = y$ and $F(z, x, y) = z$.

Definition 2.7. An element $(x, y, z) \in X \times X \times X$ is called a tripled coincidence point of the mappings $F : X \times X \times X \to X$ and $g : X \to X$ if $F(x, y, z) = g(x)$; $F(y, z, x) = g(y)$; $F(z, x, y) = g(z)$. Moreover, (x, y, z) is called a common tripled fixed point of F and g if $F(x, y, z) = g(x) = x$, $F(y, z, x) = g(y) = y$, and $F(z, x, y) = g(z) = z$.

Definition 2.8. Let X be a non-empty set, $F : X \times X \times X \to X$ and $g : X \to X$ be two mappings. F is said to be commutative with g, if $gF(x, y, z) = F(g(x), g(y), g(z))$ for all x, y, $z \in X$.

In [1], Abbas et al., introduced the concept of Weakly compatible mappings. Here we give a similar concept in fuzzy metric spaces as follows.

Definition 2.9. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ - fuzzy metric space, and let $F : X\times X \times X \to X$ and $g : X \to X$ be two mappings. F and g are said to be weakly compatible (W-compatible) if they commute at their coupled coincidence points, i.e. if (x, y, z) is a tripled coincidence point of g and F, then $gF(x, y, z) = F(g(x), g(y), g(z))$.

3. Main results

In this section, the $\mathcal{M}$ - fuzzy metric space $(X, \mathcal{M}, *)$ is in the sense of Sedghi and Shobe and the $\mathcal{M}$ - fuzzy metric $\mathcal{M}$ is assumed to satisfy the condition:

$\lim_{t\to\infty} \mathcal{M}(x, y, z, t) = 1$ for all $x, y, z \in X$. For simplicity, denote

$[\mathcal{M}(x, y, z, t)]^n = \underbrace{\mathcal{M}(x, y, z, t)*\mathcal{M}(x, y, z, t)*\ldots*\mathcal{M}(x, y, z, t)}_{n}$ for all $n \in \mathbb{N}$.

Theorem 3.1. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ - fuzzy metric space such that $*$ is a t - norm of H - type. Let $F: X \times X \times X \to X$ and $g: X \to X$ be two mappings, and there exists $\phi \in \Phi$ satisfying

$$\mathcal{M}(F(x, y, z), F(u, v, w), F(u, v, w), \phi(t)) \geq \{\mathcal{M}(g(x), g(u), g(u), t) * \mathcal{M}(g(y), g(v), g(v), t) * \mathcal{M}(g(z), g(w), g(w), t)\} \quad (3.1.1)$$

for all $x, y, z, u, v, w \in X$, $t > 0$. Suppose that $F(X \times X \times X) \subseteq g(x)$, $g(x)$ is complete, F and g are weakly compatible. Then F and g have a unique common tripled fixed point in X.

Proof: Let $x_0, y_0, z_0 \in X$ be three arbitrary points in X. Since $F(X \times X \times X) \subseteq g(x)$.

We can choose $x_1, y_1, z_1 \in X$ such that $g(x_1) = F(x_0, y_0, z_0)$, $g(y_1) = F(y_0, z_0, x_0)$ and $g(z_1) = F(z_0, x_0, y_0)$. Continuing this process, we can construct three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that $g(x_{n+1}) = F(x_n, y_n, z_n)$, $g(y_{n+1}) = F(y_n, z_n, x_n)$, $g(z_{n+1}) = F(z_n, x_n, y_n)$ for all $n \geq 0$. We shall do the proof in four steps.

Step I: We shall show that $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences.

Since $*$ is a t - norm of H - type, for any $\lambda > 0$, there exists an $\mu > 0$ such that

$\underbrace{(1-\mu)*(1-\mu)*....*(1-\mu)}_{k} \geq 1 - \lambda$ for all $k \in \mathbb{N}$. Since $\mathcal{M}$ (x, y, z, .) is continuous and $\lim_{t\to\infty} \mathcal{M}$ (x, y, z, t) = 1 for all x, y, z ∈ X, so there exists $t_0 > 0$ such that

$$\left.\begin{array}{l} \mathcal{M}(g(x_0), g(x_1), g(x_1), t_0) \geq 1-\mu \\ \mathcal{M}(g(y_0), g(y_1), g(y_1), t_0) \geq 1-\mu \\ \mathcal{M}(g(z_0), g(z_1), g(z_1), t_0) \geq 1-\mu \end{array}\right\} \quad (3.1.2)$$

On the other hand, since $\phi \in \Phi$, by condition (ϕ -3), we have $\sum \phi^n(t_0) < \infty$.

Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$.such that $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$. (3.1.3)

By use of condition (3.1.1), we get

$$\begin{aligned} \mathcal{M}(g(x_1), g(x_2), g(x_2), \phi(t_0)) &= \mathcal{M}(F(x_0, y_0, z_0), F(x_1, y_1, z_1), F(x_1, y_1, z_1), \phi(t_0)) \\ &\geq \{\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0) * \mathcal{M}(g(y_0), g(y_1), g(y_1), t_0) \\ &\quad * \mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)\} \\ \mathcal{M}(g(y_1), g(y_2), g(y_2), \phi(t_0)) &= \mathcal{M}(F(y_0, z_0, x_0), F(y_1, z_1, x_1), F(y_1, z_1, x_1), \phi(t_0)) \\ &\geq \{\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0) * \mathcal{M}(g(z_0), g(z_1), g(z_1), t_0) \\ &\quad * \mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)\} \quad \text{and} \\ \mathcal{M}(g(z_1), g(z_2), g(z_2), \phi(t_0)) &= \mathcal{M}(F(z_0, x_0, y_0), F(z_1, x_1, y_1), F(z_1, x_1, y_1), \phi(t_0)) \\ &\geq \{\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0) * \mathcal{M}(g(x_0), g(x_1), g(x_1), t_0) \\ &\quad * \mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathcal{M}(g(x_2), g(x_3), g(x_3), \phi^2(t_0)) &= \mathcal{M}(F(x_1, y_1, z_1), F(x_2, y_2, z_2), F(x_2, y_2, z_2), \phi^2(t_0)) \\ &\geq \{\mathcal{M}(g(x_1), g(x_2), g(x_2), \phi(t_0)) \\ &\quad * \mathcal{M}(g(y_1), g(y_2), g(y_2), \phi(t_0)) * \mathcal{M}(g(z_1), g(z_2), g(z_2), \phi(t_0))\} \\ &\geq \{[\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^3 * [\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)]^3 \\ &\quad * [\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^3\} \\ \mathcal{M}(g(y_2), g(y_3), g(y_3), \phi^2(t_0)) &= \mathcal{M}(F(y_1, z_1, x_1), F(y_2, z_2, x_2), F(y_2, z_2, x_2), \phi^2(t_0)) \\ &\geq \{[\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)]^3 * [\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^3 \\ &\quad * [\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^3\} \quad \text{and} \\ \mathcal{M}(g(z_2), g(z_3), g(z_3), \phi^2(t_0)) &= \mathcal{M}(F(z_1, x_1, y_1), F(z_2, x_2, y_2), F(z_2, x_2, y_2), \phi^2(t_0)) \\ &\geq \{[\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^3 * [\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^3 \\ &\quad * [\mathcal{M} g(y_0), g(y_1), g(y_1), t_0)]^3\}. \end{aligned}$$

From the inequalities above and by induction, it is easy to find that

$$\begin{aligned} \mathcal{M}(g(x_n), g(x_{n+1}), g(x_{n+1}), \phi^n(t_0)) &= \{[\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^{3n-1} \\ &\quad * [\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)]^{3n-1} * [\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^{3n-1}\} \quad (3.1.4) \\ \mathcal{M}(g(y_n), g(y_{n+1}), g(y_{n+1}), \phi^n(t_0)) &= \{[\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)]^{3n-1} \\ &\quad * [\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^{3n-1} * [\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^{3n-1}\} \quad (3.1.5) \\ \mathcal{M}(g(z_n), g(z_{n+1}), g(z_{n+1}), \phi^n(t_0)) &= \{[\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^{3n-1} \\ &\quad * [\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^{3n-1} * [\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)]^{3n-1}\}. \quad (3.1.6) \end{aligned}$$

So, from (3.1.2), (3.1.3) and (3.1.4) for $m > n \geq n_0$, we have

$$\mathcal{M}(g(x_n), g(x_m), g(x_m), t) \geq \mathcal{M}(g(x_n), g(x_m), g(x_m), \sum_{k=n_0}^{\infty} \phi^k(t_0))$$

$$\geq \mathcal{M}(g(x_n), g(x_m), g(x_m), \sum_{k=n}^{m=1} \phi^k(t_0))$$

$$\geq \{\mathcal{M}(g(x_n), g(x_{n+1}), g(x_{n+1}), \phi^n(t_0)) * \mathcal{M}(g(x_{n+1}), g(x_{n+2}), g(x_{n+2}), \phi^{n+1}(t_0))$$

$$* \ldots\ldots * \mathcal{M}(g(x_{m-1}), g(x_m), g(x_m), \phi^{m-1}(t_0))\}$$

$$\geq \{[\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^{3^{n-1}} * [\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)]^{3^{n-1}}$$

$$* [\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^{3^{n-1}} * [\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^{3^n}$$

$$* [\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)]^{3^n} * [\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^{3^n} * \ldots$$

$$* [\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^{3^{m-2}} * [\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)]^{3^{m-2}}$$

$$* [\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^{3^{m-2}}\}$$

$$= \left\{ [\mathcal{M}(g(x_0), g(x_1), g(x_1), t_0)]^{\frac{3^{(m-1)} - 2^{(n-1)}}{2}} * [\mathcal{M}(g(y_0), g(y_1), g(y_1), t_0)]^{\frac{3^{(m-1)} - 2^{(n-1)}}{2}} \right.$$

$$\left. * [\mathcal{M}(g(z_0), g(z_1), g(z_1), t_0)]^{\frac{3^{(m-1)} - 2^{(n-1)}}{2}} \right\}$$

$$\geq \underbrace{(1-\mu) * (1-\mu) * \ldots . * (1-\mu)}_{\frac{3(3^{(m-1)} - 2^{(n-1)})}{2}} \geq 1 - \lambda$$

which implies that $\mathcal{M}(g(x_n), g(x_m), g(x_m), t) > 1 - \lambda$, (3.1.7)
for all $m, n \in \mathbb{N}$, with $m > n \geq n_0$ and $t > 0$.
So $\{g(x_n)\}$ is a Cauchy sequence. Similarly, we can prove that $\{g(y_n)\}$ and $\{g(z_n)\}$ are also Cauchy sequences.

Step II: We shall show that g and F have tripled coincidence point.
Since g(X) is complete, there exist $\hat{x}, \hat{y}, \hat{z} \in g(X)$, and exist a, b, c $\in$ X such that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} F(x_n, y_n, z_n) = g(a) = \hat{x},$$
$$\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} F(y_n, z_n, x_n) = g(b) = \hat{y} \quad and \tag{3.1.8}$$
$$\lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} F(z_n, x_n, y_n) = g(c) = \hat{z}.$$

By use of condition (3.1.1) , we get

$$\mathcal{M}(F(x_n, y_n, z_n), F(a, b, c), F(a, b, c), \phi(t)) \geq \{\mathcal{M}(g(x_n), g(a), g(a), t)$$
$$* \mathcal{M}(g(y_n), g(b), g(b), t) * \mathcal{M}(g(z_n), g(c), g(c), t)\}.$$

Taking the limit as n $\to \infty$ in the above inequality, by continuity of $\mathcal{M}$ and using (3.1.8),

we have $\mathcal{M}$(g(a), F(a, b, c), F(a,b, c), ϕ(t)) = 1, which implies that $F(a, b, c) = g(a) = \hat{x}$.
In a similar way, we can prove that $F(b, c, a) = g(b) = \hat{y}$ and $F(c, a, b) = g(c) = \hat{z}$.
Since F and g are weakly compatible, we get that $gF(a, b, c) = F(g(a), g(b), g(c))$,
$gF(b, c, a) = F(g(b), g(c), g(a))$ and $gF(c, a, b) = F(g(c), g(a), g(b))$, which implies that
$g(\hat{x}) = F(\hat{x}, \hat{y}, \hat{z})$, $g(\hat{y}) = F(\hat{y}, \hat{z}, \hat{x})$ and $g(\hat{z}) = F(\hat{z}, \hat{x}, \hat{y})$.

Step III: We shall show that $g(\hat{x}) = \hat{y}$, $g(\hat{y}) = \hat{z}$ and $g(\hat{z}) = \hat{x}$.
Since * is a t - norm of H - type for any $\lambda > 0$ there exists an $\mu > 0$ such that
$\underbrace{(1-\mu)(1-\mu)....(1-\mu)}_{k} \geq 1 - \lambda$, for all k $\in \mathbb{N}$.

Since $\mathcal{M}$(x, y, z, .) is continuous and $\lim_{n\to\infty} \mathcal{M}$(x, y, z, t) = 1 for all x, y, z $\in$ X, there exists $t_0 > 0$ such that $\mathcal{M}\ (g(\hat{x}), \hat{y}, \hat{y},\ t_0) \geq 1 - \mu$, $\mathcal{M}\ (g(\hat{y}), \hat{z}, \hat{z},\ t_0) \ \geq\ 1 - \mu$ and
$\mathcal{M}\ (g(\hat{z}), \hat{x}, \hat{x},\ t_0) \ \geq\ 1 - \mu$.

On the other hand, Since $\phi \in \Phi$, by condition (ϕ - 3), we have $\sum_{n=1}^{\infty} \phi^n(t_0) < 0$.

Thus, for any t > 0, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$. Since

$$\mathcal{M}(g(\hat{x}), g(y_{n+1}), g(y_{n+1}), \phi(t_0)) = \mathcal{M}(F(\hat{x}, \hat{y}, \hat{z}), F(y_n, z_n, x_n), F(y_n, z_n, x_n), \phi(t_0)).$$
$$\geq \{\mathcal{M}(g(\hat{x}), g(y_n), g(y_n), t_0) * \mathcal{M}(g(\hat{y}), g(z_n), g(z_n), t_0)$$
$$* \mathcal{M}(g(\hat{z}), g(x_n), g(x_n), t_0)\},$$

$$\mathcal{M}(g(\hat{y}), g(z_{n+1}), g(z_{n+1}), \phi(t_0)) = \mathcal{M}(F(\hat{y}, \hat{z}, \hat{x}), F(z_n, x_n, y_n), F(z_n, x_n, y_n), \phi(t_0))$$
$$\geq \{\mathcal{M}(g(\hat{y}), g(z_n), g(z_n), t_0) * \mathcal{M}(g(\hat{z}), g(x_n), g(x_n), t_0)$$
$$* \mathcal{M}(g(\hat{x}), g(y_n), g(y_n), t_0)\}.$$

$$\mathcal{M}(g(\hat{z}), g(x_{n+1}), g(x_{n+1}), \phi(t_0)) = \mathcal{M}(F(\hat{z}, \hat{x}, \hat{y}), F(x_n, y_n, z_n), F(x_n, y_n, z_n), \phi(t_0))$$
$$\geq \{\mathcal{M}(g(\hat{z}), g(x_n), g(x_n), t_0) * \mathcal{M}(g(\hat{x}), g(y_n), g(y_n), t_0)$$
$$* \mathcal{M}(g(\hat{y}), g(z_n), g(z_n), t_0)\}.$$

Letting $n \to \infty$ in the above inequalities, we obtain

$$\mathcal{M}\ (g(\hat{x}), \hat{y}, \hat{y}, \phi(t_0)) \ \geq \{\mathcal{M}\ (g(\hat{x}), \hat{y}, \hat{y},\ t_0) * \mathcal{M}\ (g(\hat{y}), \hat{z}, \hat{z},\ t_0)$$
$$* \mathcal{M}\ (g(\hat{z}), \hat{x}, \hat{x}, t_0)\} \tag{3.1.9}$$

$$\mathcal{M}\ (g(\hat{y}), \hat{z}, \hat{z}, \phi(t_0)) \ \geq \{\ \mathcal{M}\ g(\hat{y}), \hat{z},\ \hat{z},\ t_0) * \mathcal{M}\ (g(\hat{z}), \hat{x}, \hat{x},\ t_0)$$
$$* \mathcal{M}\ (g(\hat{x}), \hat{y}, \hat{y}, t_0)\} \tag{3.1.10}$$

$$\mathcal{M}\ (g(\hat{z}), \hat{x}, \hat{x}, \phi(t_0)) \geq \{\mathcal{M}\ (g(\hat{z}), \hat{x},\ \hat{x},\ t_0) * \mathcal{M}\ (g(\hat{x}), \hat{y}, \hat{y},\ t_0)$$
$$* \mathcal{M}(g(\hat{y}), \hat{z}, \hat{z}, t_0)\}\ . \tag{3.1.11}$$

According to (3.1.9), (3.1.10) and (3.1.11) , we get that

$$\left\{\begin{array}{l}\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},\phi(t_0)) \\ *\mathcal{M}(g(\hat{y}),\hat{z},\hat{z},\phi(t_0)) \\ *\mathcal{M}(g(\hat{z}),\hat{x},\hat{x},\phi(t_0))\end{array}\right\} \geq \left\{\begin{array}{l}[\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},t_0)]^3 \\ *[\mathcal{M}(g(\hat{y}),\hat{z},\hat{z},t_0)]^3 \\ *[\mathcal{M}(g(\hat{z}),\hat{x},\hat{x},t_0)]^3\end{array}\right\}.$$

Therefore, from this inequality and by induction, we obtain that

$$\left\{\begin{array}{l}\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},\phi^n(t_0)) \\ *\mathcal{M}(g(\hat{y}),\hat{z},\hat{z},\phi^n(t_0)) \\ *\mathcal{M}(g(\hat{z}),\hat{x},\hat{x},\phi^n(t_0))\end{array}\right\} \geq \left\{\begin{array}{l}[\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},\phi^{n-1}(t_0))]^3 \\ *[\mathcal{M}(g(\hat{y}),\hat{z},\hat{z},\phi^{n-1}(t_0))]^3 \\ *[\mathcal{M}(g(\hat{z}),\hat{x},\hat{x},\phi^{n-1}(t_0))]^3\end{array}\right\} \geq \left\{\begin{array}{l}[\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},t_0)]^{3^n} \\ *[\mathcal{M}(g(\hat{y}),\hat{z},\hat{z},t_0)]^{3^n} \\ *[\mathcal{M}(g(\hat{z}),\hat{x},\hat{x},t_0)]^{3^n}\end{array}\right\}$$

for all $n \in \mathbb{N}$. Since $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$, we get

$$\left\{\begin{array}{l}[\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},t)] \\ *[\mathcal{M}(g(\hat{y}),\hat{z},\hat{z},t)] \\ *[\mathcal{M}(g(\hat{z}),\hat{x},\hat{x},t)]\end{array}\right\} \geq \left\{\begin{array}{l}\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},\sum_{k=n_0}^{\infty}\phi^k(t_0)) \\ *\mathcal{M}(g(\hat{y}),\hat{z},\hat{z},\sum_{k=n_0}^{\infty}\phi^k(t_0)) \\ *\mathcal{M}(g(\hat{z}),\hat{x},\hat{x},\sum_{k=n_0}^{\infty}\phi^k(t_0))\end{array}\right\} \geq \left\{\begin{array}{l}[\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},\phi^{n_0}(t_0))] \\ *[\mathcal{M}(g(\hat{y}),\hat{z},\hat{z},\phi^{n_0}(t_0))] \\ *[\mathcal{M}(g(\hat{z}),\hat{x},\hat{x},\phi^{n_0}(t_0))]\end{array}\right\}$$

$$\geq \left\{\begin{array}{l}[\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},t_0)]^{3^{n_0}} \\ *[\mathcal{M}(g(\hat{y}),\hat{z},\hat{z},t_0)]^{3^{n_0}} \\ *[\mathcal{M}(g(\hat{z}),\hat{x},\hat{x},t_0)]^{3^{n_0}}\end{array}\right\} \geq \underbrace{(1-\mu)*(1-\mu)*\cdots*(1-\mu)}_{3^{n_0}+1} \geq 1-\lambda.$$

Thus, for any $\lambda > 0$, we have $\mathcal{M}(g(\hat{x}),\hat{y},\hat{y},t) * \mathcal{M}(g(\hat{y}),\hat{z},\hat{z},t) * \mathcal{M}(g(\hat{z}),\hat{x},\hat{x},t) \geq 1-\lambda$ for all $t > 0$, which implies that $g(\hat{x}) = \hat{y}$, $g(\hat{y}) = \hat{z}$, and $g(\hat{z}) = \hat{x}$.

Step IV: We shall prove that $\hat{x} = \hat{y} = \hat{z}$. Since $*$ is a t- norm of H - type, for any $\lambda > 0$, there exists an $\mu > 0$ such that $\underbrace{(1-\mu)*(1-\mu)*\cdots*(1-\mu)}_{k} \geq 1-\lambda$ for all $k \in \mathbb{N}$.

Since $\mathcal{M}(x, y, z, .)$ is continuous and $\lim_{t\to\infty}\mathcal{M}(x, y, z, t) = 1$ for all $x, y, z \in X$, there exists $t_0 > 0$ such that $\mathcal{M}(\hat{x},\hat{y},\hat{z},t_0) = 1-\mu$, $\mathcal{M}(\hat{y},\hat{z},\hat{x},t_0) = 1-\mu$ and $\mathcal{M}(\hat{z},\hat{x},\hat{y},t_0) = 1-\mu$. On the other hand, since $\phi \in \mathbb{N}$, by condition (ϕ - 3), we have $\sum_{n=1}^{\infty}\phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{k=n_0}^{\infty}\phi^k(t_0)$

By use of condition (3.1.1), we have

$$\mathcal{M}(g(x_{n+1}),g(y_{n+1}),g(y_{n+1}),(t_0)) = \mathcal{M}(F(x_n,y_n,z_n), F(y_n,z_n,x_n), F(y_n,z_n,x_n),\phi(t_0))$$
$$\geq \{\mathcal{M}(g(x_n),g(y_n),g(y_n), t_0)*\mathcal{M}(g(y_n),g(z_n),g(z_n), t_0)$$
$$*\mathcal{M}(g(z_n), g(x_n), g(x_n), t_0)\},$$

$$\mathcal{M}(g(y_{n+1}), g(z_{n+1}), g(z_{n+1}),(t_0)) = \mathcal{M}(F(y_n, z_n, x_n), F(z_n, x_n, y_n), F(z_n, x_n, y_n),\phi(t_0))$$
$$\geq \{\mathcal{M}(g(y_n), g(z_n), g(z_n), t_0) *\mathcal{M}(g(z_n), g(x_n), g(x_n), t_0)$$
$$*\mathcal{M}(g(x_n), g(y_n), g(y_n), t_0)\} \text{ and}$$

$$\mathcal{M}(g(z_{n+1}), g(x_{n+1}), g(x_{n+1}),\phi(t_0)) = \mathcal{M}(F(z_n, x_n, y_n), F(x_n, y_n, z_n), F(x_n, y_n, z_n),\phi(t_0))$$
$$\geq \{\mathcal{M}(g(z_n), g(x_n), g(x_n), t_0)*\mathcal{M}(g(x_n), g(y_n), g(y_n), t_0)$$
$$*\mathcal{M}(y_n), g(z_n), g(z_n), t_0)\}.$$

Letting $n \to \infty$ in the above inequalities, we have

$$\mathcal{M}(\hat{x},\hat{y},\hat{y},\phi(t_0)) \geq \mathcal{M}(\hat{x},\hat{y},\hat{y},t_0) * \mathcal{M}(\hat{y},\hat{z},\hat{z},t_0) * \mathcal{M}(\hat{z},\hat{x},\hat{x},t_0)$$
$$\mathcal{M}(\hat{y},\hat{z},\hat{z},\phi(t_0)) \geq \mathcal{M}(\hat{y},\hat{z},\hat{z},t_0) * \mathcal{M}(\hat{z},\hat{x},\hat{x},t_0) * \mathcal{M}(\hat{x},\hat{y},\hat{y},t_0) \text{ and}$$
$$\mathcal{M}(\hat{z},\hat{x},\hat{x},\phi(t_0)) \geq \mathcal{M}(\hat{z},\hat{x},\hat{x},t_0) * \mathcal{M}(\hat{x},\hat{y},\hat{y},t_0) * \mathcal{M}(\hat{y},\hat{z},\hat{z},t_0).$$

Therefore, we obtain that

$$(\mathcal{M}(\hat{x},\hat{y},\hat{y},\phi(t_0))* \mathcal{M}(\hat{y},\hat{z},\hat{z},\phi(t_0)) *\mathcal{M}(\hat{z},\hat{x},\hat{x},\phi(t_0)) \geq \{ [\mathcal{M}(\hat{x},\hat{y},\hat{y},t_0)]^3$$
$$*[\mathcal{M}(\hat{y},\hat{z},\hat{z}. t_0)]^3*[\mathcal{M}(\hat{z},\hat{x},\hat{x}, t_0)]^3\}$$

From this inequality and by induction, we can obtain

$$\left\{\begin{array}{l}\mathcal{M}(\hat{x},\hat{y},\hat{y},\phi^n(t_0)) \\ *\mathcal{M}(\hat{y},\hat{z},\hat{z},\phi^n(t_0)) \\ *\mathcal{M}(\hat{z},\hat{x},\hat{x},\phi^n(t_0))\end{array}\right\} \geq \left\{\begin{array}{l}[\mathcal{M}(\hat{x},\hat{y},\hat{y},\phi^{n-1}(t_0))]^3 \\ *[\mathcal{M}(\hat{y},\hat{z},\hat{z},\phi^{n-1}(t_0))]^3 \\ *[\mathcal{M}(\hat{z},\hat{x},\hat{x},\phi^{n-1}(t_0))]^3\end{array}\right\} \geq \left\{\begin{array}{l}[\mathcal{M}(\hat{x},\hat{y},\hat{y},t_0)]^{3^n} \\ *[\mathcal{M}(\hat{y},\hat{z},\hat{z},t_0)]^{3^n} \\ *[\mathcal{M}(\hat{z},\hat{x},\hat{x},t_0)]^{3^n}\end{array}\right\}$$

for all $n \in \mathbb{N}$. Since $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$, we have

$$\left\{\begin{array}{l}\mathcal{M}(\hat{x},\hat{y},\hat{y},t) \\ *\mathcal{M}(\hat{y},\hat{z},\hat{z},t) \\ *\mathcal{M}(\hat{z},\hat{x},\hat{x},t)\end{array}\right\} \geq \left\{\begin{array}{l}\mathcal{M}(\hat{x},\hat{y},\hat{y},\sum_{k=n_0}^{\infty}\phi^k(t_0)) \\ *\mathcal{M}(\hat{y},\hat{z},\hat{z},\sum_{k=n_0}^{\infty}\phi^k(t_0)) \\ *\mathcal{M}(\hat{z},\hat{x},\hat{x},\sum_{k=n_0}^{\infty}\phi^k(t_0))\end{array}\right\} \geq \left\{\begin{array}{l}\mathcal{M}(\hat{x},\hat{y},\hat{y},\phi^{n_0}(t_0)) \\ *\mathcal{M}(\hat{y},\hat{z},\hat{z},\phi^{n_0}(t_0)) \\ *\mathcal{M}(\hat{z},\hat{x},\hat{x},\phi^{n_0}(t_0))\end{array}\right\}$$

$$\geq \left\{ \begin{array}{l} [\mathcal{M}(\hat{x},\hat{y},\hat{y},t_0)]^{3^{n_0}} \\ *[\mathcal{M}(\hat{y},\hat{z},\hat{z},t_0)]^{3^{n_0}} \\ *[\mathcal{M}(\hat{z},\hat{x},\hat{x},t_0)]^{3^{n_0}} \end{array} \right\} \geq \frac{(1-\mu)*(1-\mu)*\ldots*(1-\mu)}{3^{n_0+1}} \geq 1-\lambda$$

which implies that $\hat{x}=\hat{y}=\hat{z}$. Thus we proved that F and g have a common tripled fixed point in X. The uniqueness of the fixed point can be easily proved in the same way as above.

This completes the proof of Theorem.

Corollary 3.2. Let (X, $\mathcal{M}$, $*$) be a $\mathcal{M}$ - fuzzy metric spaces with a H- type t - norm. Let F: X×X×X → X and g : X →X be two mappings such that

$$\mathcal{M}(F(x, y, z), F(u, v, w), F(u, v, w), kt) \geq \{\mathcal{M}(g(x), g(u), g(u), t) * \mathcal{M}(g(y), g(v), g(v),t) * \mathcal{M}(g(z), g(w), g(w), t)\}$$

for all x, y, z, u, v, w ∈ X, t > 0, 0 < k < 1. Suppose that F(X×X×X) ⊆ g(X), g(X) is complete, F and g are weakly compatible. Then F and g have a unique common tripled fixed point in X.

Corollary 3.3. Let (X, $\mathcal{M}$, $*$) be a $\mathcal{M}$ - fuzzy metric space with a H-type t- norm. Let F: X×X×X → X, is complete, and there exists $\phi \in \Phi$ such that

$$\mathcal{M}(F(x, y, z), F(u, v, w), F(u, v, w), \phi(t)) \geq \mathcal{M}(x, u, u, t) * \mathcal{M}(y, v, v, t) * \mathcal{M}(z, w, w, t)$$

for all x, y, z, u, v, w ∈X, t > 0. Then exists x ∈ X such that x = F(x, x, x), that is, F admits a unique fixed point in X.

Corollary 3.4. Let a, b, c ∈ [0, 1] be real numbers such that a+b+ c ≤ 1. Let (X, $\mathcal{M}$,$*$) be a $\mathcal{M}$ - fuzzy metric space with a H - type t- norm. Let F: X×X×X → X is complete, and there exists $\phi \in \Phi$ such that

$$\mathcal{M}(F(x, y, z), F(u, v, w), F(u, v, w), \phi(t)) \geq \{ [\mathcal{M}(x, u, u, t)]^{a} * [\mathcal{M}(y, v, v, t)]^{b} * [\mathcal{M}(z, w, w, t)]^{c} \}$$

for all x, y, z, u, v, w ∈ X, t > 0.

Then there exist x ∈ X such that x = F(x, x, x), that is, F admits a unique fixed point in X.

4. Conclusion

Efforts have been taken in generalizing the concepts of fixed point theorems for weakly compatible mappings in fuzzy metric spaces and the existence and uniqueness of common fixed points in abstract spaces like generalized fuzzy metric space are proved.

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