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Fixed Point Theorems in 2-Fuzzy Metric Space

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Abstract. The main purpose of this paper is to develop fixed point theorems for self mappings and for the pair of weakly compatible on 2- fuzzy metric space.

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1. Introduction

The concept of fuzzy sets introduced by Zadeh (1965) became active in the field of research. Kramosil and Michalek [5] in 1975 coined fuzzy metric space which was further modified by George and Veeramani [2] using t-norm. Singh and Chauhan [3] introduced the concepts of compatibility of fuzzy metric and proved fixed point theorems in fuzzy metric space.

In this paper fixed point theorems are established for self mappings and the pair of weakly compatible mappings on 2- fuzzy metric space.

2. Preliminaries

Definition 2.1. A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a t-norm if for all $a, b, c, d \in [0,1]$ the following conditions are satisfied

(i) $a * 1 = a$
(ii) $a * b = b * a$
(iii) $a * b \leq c * d$ whenever $a \leq c, b \leq d$
(iv) $a * (b * c) = (a * b) * c$

Definition 2.2. The 3 tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty]$ satisfying the conditions.

(M1) $M(x, y, t) = 0$
(M2) $M(x, y, t) = 1, \quad \forall t > 0$ if and only if $x = y$
(M3) $M(x, y, t) = M(y, x, t)$
(M4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(M5) $M(x, y, \cdot): [0, \infty] \to [0,1]$ is left continuous.
(M6) $\lim_{t\to\infty} M(x, y, t) = 1$

Definition 2.3. The 3 tuple $(X, M,*)$ is called a 2- fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty]$ satisfying the conditions.

(2-M1) $M(f,g,t) = 0$

(2-M2) $M(f,g,t) = 1, \quad \forall t > 0$ if and only if $f = g$

(2-M3) $M(f,g,t) = M(g,f,t)$

(2-M4) $M(f,g,t) * M(g,h,s) \leq M(f,h,t+s)$

(2-M5) $M(f,g,\cdot)\colon [0,\infty] \to [0,1]$ is left continuous.

(2-M6) $\lim_{t\to\infty} M(f,g,t) = 1$

3. Fixed point theorems in 2-fuzzy metric space

Lemma 3.1. Let M be a 2-fuzzy metric on $\mathcal{F}(X)$ then $M(f,g,\ \cdot)$ is nondecreasing for all $f,g \in \mathcal{F}(X)$

Proof: Let $s,t > 0$ and for all $f,g \in \mathcal{F}(X)$.

By (2-M4), M(f,g,t+s)) $\geq M(f,h,t) * M(h,g,s)$

we have $M(f,g,t+s) \geq M(f,h,t) * M(h,g,s)$

If we set $h = g$ then $M(f,h,t+s) \geq M(f,h,t) * M(h,h,s)$

(i.e) $M(f,h,t+s) \geq M(f,h,t)$

Theorem 3.1. Let $(X,M,*)$ be a 2-fuzzy metric space. If there exists $q \in (0,1)$ such that $M(f,g,qt) \geq M(f,g,t)$ for all $f,g \in \mathcal{F}(X)$ and $t > 0$then $f = g$.

Proof: Since $M(f,g,t) \geq M(f,g,qt)$

$\geq M(f,g,t)$for all $t > 0$,

implies$M(f,g,t)$ is constant

Since $\lim_{t\to\infty} M(f,g,t) = 1$ we get $M(f,g,t) = 1$ for all $t > 1$

And hence $f = g$.

Definition 3.1. Let $(X,M,*)$ be a 2-fuzzy metric space

(i) A sequence $\{f_n\}$ is said to be convergent to $f \in \mathcal{F}(X)$ if for any $r \in (0,1)$ and $t > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0, M(f_n,f,t) > 1 - r$.
That is $\lim_{t\to\infty} M(f_n,f,t) = 1$ for all $t > 0$.

(ii) A sequence $\{f_n\}$ is said to be a Cauchy sequence if for any $r \in (0,1)$ and $t > 0$ there exists $n_0 \in N$ such that for all $m,n \geq n_0, M(f_n,f_m,t) > 1 - r$.

(iii) A 2-fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 3.2. A function M is continuous in the 2-fuzzy metric space if and only if whenever $f_n \to f, g_n \to g$then $\lim_{n\to\infty} M(f_n,g_n,t) = M(f,g,t)$.

Definition 3.3. Two mappings A and B on 2-fuzzy metric space are weekly commuting if and only if

$M(ABf,BAf,t) \geq M(Af,Bf,t)$for all $f \in \mathcal{F}(X)$ and $t > 0$.

Definition 3.4. Self mappingsA and B of a 2-fuzzy metric space $(X, M,*)$ is said to be compatible if $\lim_{t\to\infty} M(ABf_n, BAf_n, t) = 1$ for all $t > 0$ whenever $\{f_n\}$ is a sequence in $\mathcal{F}(X)$ such that

$$\lim_{n\to\infty} Af_n = \lim_{n\to\infty} Bf_n = g \text{for some } g \in \mathcal{F}(X).$$

Definition 3.5. Suppose A and B be self mappings of 2-fuzzy metric $(X, M,*)$. A point f in $\mathcal{F}(X)$ is called the coincidence point of A and B if and only if$Af = Bf$. Then $g = Af = Bf$ is called the point of coincidence of Aand B.

Definition 3.6. Self maps A and B are said to be weakly compatible if they commute at their coincidence points that is if $Af = Bf$ for some $f \in \mathcal{F}(X)$then $ABf = BAf$.

Theorem 3.2. Let $(X, M,*)$ be a 2-fuzzy metric space. Let A, B be weakly commuting mappings of $\mathcal{F}(X)$. Let $\{f_n\}$ in $\mathcal{F}(X)$. converge to f, such that $Af_n \to f$ and $Bf_n \to f$ then A and B has a coincidence point which is unique.

Proof: Given $\{f_n\}$ in $\mathcal{F}(X)$.such that $f_n \to f$ and $Af_n \to f$ and $Bf_n \to f$

Since $[A, B]$ is weakly commuting.

$$M(ABf_n, BAf_n, t) \geq M(Af_n, Bf_n, t) \text{for all } t > 0$$

Then $M(Af, Bf, t) \geq M(f, f, t)$ for all $t > 0$ ……………………(1)

Implies $Bf = Af = g$ thus we have a coincidence point for A and B.

For uniqueness let h be another common fixed point of A, B.

Then for all $t > 0$, $M(f, h, t) = M(Af, Bh, t) \geq M(f, h, t)$ by (1)

which implies that $f = g$.This completes the proof of the theorem.

Theorem 3.3. Let A, B, P, Q be self maps of a 2-fuzzy metric space $(X, M,*)$. If the pairs (AB, P) and (PQ, B) are compatible maps having the same point of coincidence and $AB = BA, BP = PB, AP = PA, PQ = QP, PB = BP, QB = BQ$ then for all $f, g \in \mathcal{F}(X), q \in (0,1), t > 0$

$$M(Pf, Bf, qt) \geq M(Pf, PQg, t) * M(Pf, ABf, t) * M(PQg, Bg, t) * M(ABf, PQg, t) * M(ABf, Bg, t) \text{…………(1)}$$

Then A, B, P, Q have a unique common fixed point in $F(X)$.

Proof: Given the pairs (AB, P) and (PQ, B) are compatible. $(AB, P)$is compatible iff $\lim_{t\to\infty} M(ABPf_n , PABf_n, t) = 1$ whenever $\{f_n\}$ is a sequence in $F(X)$ such that

$\lim ABf_n = u$ and $\lim Pf_n = u$

Therefore $\lim_{n\to\infty} M(ABu , Pu, t) = 1$

(PQ, B) is compatible if and only if $\lim_{n\to\infty} M(PQBg_n , BPQg_n, t) = 1$ whenever $\{g_n\}$ is a sequence in $F(X)$ such that $\lim Bg_n = \lim PQg_n = v$. Therefore

$\lim_{n\to\infty} M(PQv , Bv, t) = 1$

Hence $ABu = Pu$ and $PQv = Bv$

Thus 'u' is the coincidence point of $AB\&P$ and 'v' is the coincidence point of $PQ\&B$. If asserts to prove that $u = v$.

Take $f = f_n$ and $g = g_n$ in (1)

$$M(Pf_n, Bg_n, qt) \geq M(Pf_n, PQg_n, t) * M(Pf_n, ABf_n, t) * M(PQg_n, Bg_n, t) * M(ABf_n, PQg_n, t) * M(ABf_n, Bg_n, t)$$

$$M(u, v, kt) \geq M(u, v, t) * M(u, u, t) * M(v, v, t) * M(u, v, t) * M(u, v, t) \geq M(u, v, t)$$

Thus by theorem (2) $u = v$ which implies that AB, P, PQ, B have the same coincidence point.
Finally it is to prove that $Au = Bu = Pu = Qu$.
Take $f = u$, $g = g_n$.

$$M(Pu, Bg_n, qt) \geq M(Pu, PQg_n, t) * M(Pu, ABu, t) * \; M(PQg_n, Bg_n, t) * M(ABu, PQg_n, t) * \; M(ABu, Bg_n, t)$$

Taking a limit as $n \to \infty$ we get

$$M(Pu, v, qt) \geq M(Pu, v, t) * M(Pu, u, t) * M(v, v, t) * \; M(u, v, t) * \; M(u, v, t)$$

As $u = v$ it implies $M(Pu, u, qt) \geq M(Pu, u, t)$
Therefore $Pu = u$. Take $f = f_n$ and $g = u$

$$M(Pf_n, Bu, qt) \geq M(Pf_n, PQu, t) * M(Pf_n, ABf_n, t) * M(PQu, Bu, t) * M(ABf_n, PQu, t) * M(ABf_n, Bu, t)$$

Taking a limit as $n \to \infty$

$$M(u, Bu, qt) \geq M(u, PQu, t) * M(u, u, t) * M(PQu, Bu, t) * M(u, u, t) * \; M(u, Bu, t)$$
$$\geq M(u, Bu, t) * M(u, u, t) * M(Bu, Bu, t) * M(u, u, t) * \; M(u, Bu, t)$$

Therefore$M(u, Bu, qt) \geq M(u, Bu, t)$ which gives $Bu = u$. Take $f = Bu$ and $g = g_n$

$$M(PBu, Bg_n, qt) \geq M(PBu, PQg_n, t) * M(PBu, ABBu, t) * M(ABBu, PQg_n, t) * M(ABBu, Bg_n, t)$$

As A, B and P commutes, $ABBu = BABu = BPu = Bu$ and $PBu = BPu = Bu$

$$M(Bu, u, qt) \geq M(Bu, u, t) * M(Bu, Bu, t) * M(Bu, u, t) * M(Bu, u, t)$$
$$M(Bu, u, qt) \geq M(Bu, u, t)$$

Therefore $Bu = u$
Put $f = Au$ and $g = g_n$

$$M(Pau, Bg_n, qt) \geq M(PAu, PQg_n, t) * M(PAu, ABAu, t) * M(PQg_n, Bg_n, t) * M(ABAu, Bg_n, t) * M(ABAu, PQg_n, t)$$

As A, B, P commutes $ABAu = APu = Au$ and $PAu = APu = Au$

$$M(Au, u, qt) \geq M(Au, u, t) * M(Au, Au, t) * M(u, u, t) * M(Au, u, t) * \; M(Au, u, t)$$

Hence $M(Au, u, qt) \geq M(Au, u, t)$
Therefore $Au = u$ Hence $Au = Bu = Pu = u$ On taking $f = f_n$ and $g = Qf$, We get $Qu = u$Thus $Au = Bu = Pu = Qu = u$.

Theorem 3.4. Let $(X, M, *)$ be a complete 2-fuzzy metric space and let A, B, S, T be mappings from $\mathcal{F}(X)$ into itself such that the following conditions holds.

(i) $AY \subset TY$, $BY \subset SY$. where $Y = \mathcal{F}(X)$
(ii) A or B or S or T is continuous
(iii) The pair $[A, S]$ and $[B, T]$ are weakly compatible. Then A, B, S, T have a unique common fixed point in X
(iv) $M(Af, Bg. qt) \geq \min\{M(sf, Tg, t), M(Af, Tg, t), M(Sf, Bg, 2t)\} \ldots\ldots (1)$

Proof: Let f_0 be an arbitrary point in $\mathcal{F}(X)$ as $AY \subset TY$ and $BY \subset SY$. there exists $f_1, f_2 \in Y$ such that $Af_0 = Tf_1, Bf_1 = Sf_2$
Construct a sequence $\{g_n\}$ in Y such that $g_{2n+1} = Tf_{2n+1} = Af_{2n}$ and $g_{2n} = Sf_{2n} = Bf_{2n-1}forn = 1,2, \ldots$To prove $\{g_n\}$ is a Cauchy sequence it is enough to show that

$$M(g_{2n+1}, g_{2n+2}, kt) \geq M(g_{2n}, g_{2n+1}, t), \forall\, t > 0$$

Assume as a contrary

$$M(g_{2n+1}, g_{2n+2}, kt) < M(g_{2n}, g_{2n+1}, t), \forall\ t > 0$$

Consider

$$M(g_{2n+1}, g_{2n+2}, kt) = M(Af_{2n}, Bf_{2n+1}, kt)$$

$$\geq \min\{M(Sf_{2n}, Tf_{2n+1}, t), M(Af_{2n}, Tf_{2n+1}, t), M(Sf_{2n}, Bf_{2n+1}, 2t)\}$$
$$= \min\{M(g_{2n}, g_{2n+1}, t), M(g_{2n+1}, g_{2n+1}, t), M(g_{2n}, g_{2n+2}, 2t)\}$$
$$= \min\{M(g_{2n}, g_{2n+1}, t), M(g_{2n}, g_{2n+2}, 2t)\}$$
$$\geq M(g_{2n+1}, g_{2n+2}, kt)$$

which is a contradiction.

Hence $\{g_n\}$ is a Cauchy sequence by completeness of $\mathcal{F}(X)$. It converges to some uin $\mathcal{F}(X)$.

So the subsequence $\{Af_{2n}\}, \{Bf_{2n-1}\}, \{Sf_{2n}\}, \{Tf_{2n+1}\}$ of $\{g_n\}$ also converges to uin $\mathcal{F}(X)$.

Since $Af_{2n} \to u$ and $Sf_{2n} \to u$, and $ASf_{2n} \to Su, S$ is continuous $SAf_{2n} \to Su$ and $SSf_{2n} \to Su$.

Put $f = Sf_{2n}$ and $g = f_{2n-1}$ in (1) then

$$M(ASf_{2n}, Bf_{2n-1}, qt) \geq \min\{M(SSf_{2n}, Tf_{2n-1}, t), M(ASf_{2n-1}, Tf_{2n-1}, t), M(SSf_{2n}, Bf_{2n}, t)\}$$

As $n \to \infty M(Su, u, qt) \geq \min\{M(Su, u, t), M(Su, u, t), M(Su, u, t)\}$

$$= M(Su, u, t)$$

which implies $Su = u$

Again put $f = u$ and $g = f_{2n-1}$ then (1) implies

$$M(Au, Bf_{2n-1}, qt) \geq \min\{M(Su, Tf_{2n-1}, t), M(Au, Tf_{2n-1}, t), M(Su, Bf_{2n-1}, t)\}$$

Allow $n \to \infty$ then $M(Au, u, qt) \geq M(Au, u, t)$

Therefore $Au = u$.

Since $A(\mathcal{F}(X)) \subset T(\mathcal{F}(X))$ there exists $h \in \mathcal{F}(X)$

such that $u = Au = Th$ Put $f = u$ and $g = h$ in (1) then

$$M(Au, Bh, qt) \geq \min\{M(Su, Th, t), M(Au, Th, t), M(Su, Bh, t)\}$$

then$M(f, Bh, qt) \geq M(u, Bh, t)$ which implies $u = Bh$

Since B and T are weakly compatible they commute at their coincidence points that if $Bh = Th$ for some $P \in \mathcal{F}(X)$ then $BTh = TBh$.

Therefore $Bg = BTh = TBh = Tu$. Finally put $f = u$ and $g = u$ in (1)

Then $M(Au, Bu, qt) \geq \min\{M(Su, Tu, t), M(Au, Tu, t), M(Su, Bu, t)\}$

Allow $n \to \infty$ then $M(u, Bu, qt) \geq M(u, Bu, t)$

Therefore $Bu = u$ hence $Au = Bu = Su = Tu = u$.

Thus 'u' is a common fixed point of A, B, S, T similarly we can prove when A is continuous and (A, S) is weakly compatible.

Similarly we can prove when B is continuous.

Now to prove the uniqueness of the fixed point assume that 'l' is another common fixed point of A, B, S, T

Then $l = Al = Bl = Sl = Tl$ Put $f = u$ and $g = l$ in (1) then

$$M(Au, Bl, qt) \geq \min\{M(Su, Tl, t), M(Au, Tl, t), M(Su, Bl, t)\}$$

$M(u, l, qt) \geq M(u, l, t)$ which implies $u = l$.

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