Annals of Pure and Applied Mathematics
Vol. 15, No. 1, 2017, 25-40
ISSN: 2279-087X (P), 2279-0888(online)
Published on 11 December 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v15n1a3

Annals of Pure and Applied Mathematics

Solution and Generalized Ulam-Hyers Stability of a *n*-Dimensional Additive Functional Equation in Banach Space and Banach Algebra: Direct and Fixed Point Methods

***M.Arunkumar*[1], *P.Agilan*[2] and *S.Ramamoorthy*[3]**

[1]Department of Mathematics, Government Arts College
Tiruvannamalai - 606 603, TamilNadu, India.
e-mail:annarun2002@yahoo.co.in
[2]Department of Mathematics, Jeppiaar Institute of Technology
Sriperumbudur, Chennai - 631 604, Tamil Nadu, India.
e-mail:agilram@gmail.com
[3]Department of Mathematics, Arunai Engineering College,
Tiruvannamalai - 606 603, TamilNadu, India.
e-mail:ramsdmaths@yahoo.com

Received 1 November 2017; accepted 8 December 2017

Abstract. In this paper, the authors investigate the general solution and generalized Ulam - Hyers stability of a new type of n-dimensional additive functional equation

$$f\left(\sum_{k=1}^{n} kx_k\right)+\sum_{l=2}^{n} f\left(\sum_{k=1,k\neq l}^{n} kx_k - lx_l\right)+f\left(x_1-\sum_{k=2}^{n} kx_k\right)$$
$$=(n+1)f(x_1)+(n-3)\sum_{k=2}^{n} kf(x_k)$$

with $n>3$ in Banach space and Banach Algebra using direct and fixed point methods.

Keywords: Additive functional equation, Generalized Ulam-Hyers stability, Banach Space, Banach Algebra, Fixed point.

AMS Mathematics Subject Classification (2010): 39B52, 39B82

1. Introduction

The stability problem of functional equations originated from a question of Ulam [41] concerning the stability of group homomorphisms. Hyers [25] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [34] for linear mappings by considering an un-bounded Cauchy difference. The paper of Rassias [34] has provided a lot of influence in the development of what we call generalized Ulam stability of functional equations. In 1982, Rassias [17] followed the innovative approach of the Rassias theorem [34] in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\|y\|^q$ for $p,q\in R$

with $p+q=1$. A generalization of the Rassias theorem was obtained by Gavruta [21] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et.al., [38] by considering the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 7, 8, 9, 10, 30, 20, 38]).

The solution and stability of the following additive functional equations

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

$$f\left(nx_0+\sum_{i=1}^{n}x_i\right)=\sum_{i=1}^{n}f(x_0+x_i), \tag{1.2}$$

$$\sum_{i=1}^{n}g\left(\sum_{j=1}^{i}x_j\right)=\sum_{i=1}^{n}(n-i+1)g(x_i),\ n\geq 2 \qquad \backslash \qquad \backslash(1.3)$$

$$\sum_{i=1}^{n}p_i f(x_i)=f\left(\sum_{i=1}^{n}p_i x_i\right) \tag{1.4}$$

were discussed in (see [4, 5, 39])

In this paper, the authors investigate the general solution and generalized Ulam-Hyers stability of a new type of $n-$ dimensional additive functional equation of the form

$$f\left(\sum_{k=1}^{n}kx_k\right)+\sum_{l=2}^{n}f\left(\sum_{k=1,k\neq l}^{n}kx_k-lx_l\right)+f\left(x_1-\sum_{k=2}^{n}kx_k\right)$$
$$=(n+1)f(x_1)+(n-3)\sum_{k=2}^{n}kf(x_k) \tag{1.5}$$

With $n>3$ in Banach space and Banach Algebra using direct and fixed point methods.

Now we will recall the fundamental results in fixed point theory.

Theorem 1.1. [16] (The alternative of fixed point) Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T:X\to X$ with Lipschitz constant L. Then, for each given element $x\in X$, either

(A1) $$d\left(T^n x, \mathrm{T}^{n+1}x\right)=\infty \quad \forall n\geq 0,$$

or

(A2) there exists a natural number n_0 such that:

(i) $d\left(T^n x, T^{n+1}x\right)<\infty$ for all $n\geq n_0$;

(ii)The sequence $\left(T^n x\right)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y=\left\{y\in X: d\left(T^{n0}x,y\right)<\infty\right\}$;

(iv) $d\left(y^*,y\right)\leq\frac{1}{1-L}d(y,Ty)$ for all $y\in Y$.

2. General solution

In this section, the authors discuss the general solution of the functional equation (1.5) by considering X and Y are real vector spaces.

Theorem 2.1. If $f: X \to y$ satisfies the functional equation (1.1) for all $x, y \in X$ if and only if f satisfies the functional equation (1.5) for all $x_1, x_2, x_3, ..., x_n \in X$.

Proof: Let $f: X \to Y$ satisfy the functional equation (1.1). Setting $x = y = 0$ in (1.1), we have $f(0) = 0$. Set $x = -y$ in (1.1), we get $f(-y) = -f(y)$ for all $y \in X$. Therefore f is an odd function. Replacing y by x and y by $2x$ in (1.1), we obtain

$$f(2x) = 2f(x) \quad \text{and} \quad f(3x) = 3f(x) \tag{2.1}$$

for all $x \in X$. In general for any positive integer a, we have

$$f(ax) = af(x) \tag{2.2}$$

for all $x \in X$. Replacing x by $\frac{x}{a}$ in (2.2), we get

$$f\left(\frac{x}{a}\right) = \frac{1}{a} f(x) \tag{2.3}$$

for all $x \in X$. It is easy to verify from (1.1) that

$$f(x_1 + x_2 + ... + x_n) = f(x_1) + f(x_2) + ... + f(x_n) \tag{2.4}$$

for all $x_1, x_2, x_3, ..., x_n \in X$. Replacing $(x_1, x_2, ..., x_n)$ in $(x_1, 2x_2, ..., \mathrm{n}\, x_n)$ in (2.4) we arrive

$$f(x_1, 2x_2, 3x_3, ..., \mathrm{n}\, x_n) = f(x_1) + f(2x_2) + f(3x_3) + ... + f(nx_n) \tag{2.5}$$

for all $x_1, x_2, ..., x_n \in X$. Replacing x_2 by $-2x_2, x_3$ by $-3x_3, ...,$ and x_n by $-nx_n$ respectively in (2.4) and using the oddness of f, we get the following equations

$$\begin{cases} f(x_1 - 2x_2 + 3x_3 + ... + nx_n) = f(x_1) - f(2x_2) + f(3x_3) + ... + f(nx_n) \\ f(x_1 + 2x_2 - 3x_3 + ... + nx_n) = f(x_1) + f(2x_2) - f(3x_3) + ... + f(nx_n) \\ f(x_1 + 2x_2 + 3x_3 + ... - nx_n) = f(x_1) + f(2x_2) + f(3x_3) + ... - f(nx_n) \end{cases} \tag{2.6}$$

for all $x_1, x_2, ..., x_n \in X$. Replacing $(x_1, x_2, ..., x_n)$ in $(x_1, -2x_2, ..., -\mathrm{n}\, x_n)$ in (2.4), we have

$$f(x_1 - 2x_2 + 3x_3 + ... + nx_n) = f(x_1) - f(2x_2) - f(3x_3) + ... - f(nx_n) \tag{2.7}$$

for all $x_1, x_2, ..., x_n \in X$. Adding (2.5),(2.6), (2.7) and using (2.1) (2.2), we have demonstrated our result.

Conversely, Let $f: X \to Y$ satisfy the functional equation (1.5). Setting $x_1, x_2, ..., x_n \in X$. by $(0, 0, ..., 0)$ in (1.5), we get $f(0) = 0$. Replacing $(x_1, x_2, ..., x_n)$ by $(0, x, ..., 0)$ and $(x, x, ..., 0)$ in (1.5), we obtain

$$f(2x) = 2f(x) \quad \text{and} \quad f(3x) = 3f(x) \tag{2.8}$$

for all $x \in X$. Replacing $(x_1, x_2, ..., x_n)$ by $(0, -\mathrm{x}, ..., 0)$ and using (2.8) in (1.5), we get

$f(-x) = -f(x)$ for all $x \in X$. Therefore f is an odd function. Replacing $(x_1, x_2, ..., x_n)$ by $\left(x_1, \frac{x_2}{2}, ..., 0\right)$ and using (2.8) in (1.5), we have

$$(n-1)\, f(x_1 + x_2) + 2f(x_1 - x_2) = (n+1)\, f(x_1) + (n-3)\, f(x_2) \tag{2.9}$$

for all $x_1, x_2, ..., x_n \in X$. Replacing x_1 by x_2, x_2 by x_1 in (2.9) and using the oddness of f, we get

$$(n-1)\, f(x_1 + x_2) - 2f(x_1 - x_2) = (n-3)\, f(x_1) + (n+1)\, f(x_2) \tag{2.10}$$

for all $x_1, x_2, \in X$. Adding (2.9) (2.10) and replacing x_1 by x, x_2 by y, and since $n > 3$, we arrive our result. Hence the proof is completed.

3. Generalized ulam-hyers stability in Banach space

In this section, let Y_1 be a normed space and Y_2 be a Banach space. The authors investigate the generalized Ulam-Hyers stability of the $n-$ Dimensional Additive Functional equation (1.5). Define a mapping $D\ f : X^n \to Y$ by

$$Df(x_1, x_2, ..., x_n) = f\left(\sum_{k=1}^{n} kx_k\right) + \sum_{l=2}^{n} f\left(\sum_{k=1, k\neq 1}^{n} kx_k - lx_l\right) + f\left(x_1 - \sum_{k=2}^{n} kx_k\right)$$
$$-(n+1)\, f(x_1) - (n-3)\sum_{k=2}^{n} kf(x_k)$$

for all $x_1, x_2, ..., x_n \in X$ with $n > 3$

3.1. Direct method

Theorem 3.1. Let $j \pm 1$. Let $\psi : X^n \to [0, \infty)$ be a function such that

$$\lim_{l\to\infty} \frac{\psi\left(n^{lj}x_1, n^{lj}x_2, ..., n^{lj}x_n\right)}{n^{lj}} = 0 \tag{3.1}$$

for all $x_1, x_2, ..., x_n \in X$ with $n > 3$ let $f : X \to Y$ be a function satisfying the inequality

$$\left\|D\ f(x_1, x_2, ..., x_n)\right\| \leq \psi(x_1, x_2, ..., x_n) \tag{3.2}$$

for all $x_1, x_2, ..., x_n \in X$. Then there exists a unique function $\mathrm{A} : X \to Y$ such that

$$\left\| f(x) - A(x) \right\| \leq \frac{1}{n(n-3)} \sum_{l=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(n^{lj}x\right)}{n^{lj}} \tag{3.3}$$

Where $\Psi\left(n^{lj}x\right) = (\underbrace{\psi(0, ..., 0)}_{n-1\ times}, n^{lj}x)$ for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{t\to\infty} \frac{f\left(n^{lj}x\right)}{n^{lj}} \tag{3.4}$$

for all $x \in X$.

Proof. Assume $j = 1$. Replacing $(x_1, x_2, ..., x_n)$ by $(\underbrace{0, ..., 0}_{n-1\ times}, x)$ in (3.2) and using oddness of f, we get

$$\left\|(n-3)\, f(nx) - n(n-3)\, f(x)\right\| \leq \psi(\underbrace{0, ..., 0}_{n-1\ times}, x)$$

for all $x \in X$. Dividing the above inequality by $n(n-3)$ we obtain

$$\left\| \frac{f(nx)}{n} - f(x) \right\| \leq \frac{1}{n(n-3)} \psi(\underbrace{0,...,0}_{n-1\ times}, x)$$

for all $x \in X$. Letting $\Psi(x) = \psi(\underbrace{0,...,0}_{n-1\ times}, x)$ in (3.6), we arrive

$$\left\| \frac{f(nx)}{n} - f(x) \right\| \leq \frac{\Psi(x)}{n(n-3)}$$

for all $x \in X$. Now replacing x by nx and dividing by n in (3.7), we obtain

$$\left\| \frac{f(n^2x)}{n^2} - \frac{f(nx)}{n} \right\| \leq \frac{\Psi(nx)}{n^2(n-3)}$$

for all $x \in X$. Combining (3.7) and (3.8), we obtain

$$\left\| \frac{f(n^2x)}{n^2} - f(x) \right\| \leq \frac{1}{n(n-3)} \left[\Psi(x) + \frac{\Psi(nx)}{n} \right]$$

for all $x \in X$. In general for any positive integer l , we obtain that

$$\begin{aligned} \left\| \frac{f(n^l x)}{n^l} - f(x) \right\| &\leq \frac{1}{n(n-3)} \sum_{k=0}^{l-1} \frac{\Psi(n^k x)}{n^k} \\ &\leq \frac{1}{n(n-3)} \sum_{k=0}^{\infty} \frac{\Psi(n^k x)}{n^k} \end{aligned}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{f(n^l x)}{n^l} \right\}$, replace x by $n^m x$ and divided by n^m in (3.10), for any $m, l > 0,$ we arrive

$$\begin{aligned} \left\| \frac{f(n^{l+m})x}{n^{l+m}} - \frac{f(n^m x)}{n^m} \right\| &= \frac{1}{n^m} \left\| \frac{f(n^{l+m})x}{n^{l+m}} - f(n^m x) \right\| \\ &\leq \frac{1}{n(n-3)} \sum_{k=0}^{\infty} \frac{\Psi(n^{l+m} x)}{n^{l+m}} \qquad (3.11) \\ &\to 0 \ \ as\ m \to \infty \end{aligned}$$

For all $x \in X$.Hence the sequence $\left\{ \frac{f(n^l x)}{n^l} \right\}$ is a chauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{l \to \infty} \frac{f(n^l x)}{n^l} \forall x \in X.$$

Letting $l \to \infty$ in (3.10), we see that (3.3) holds for all $x \in X$. Now we need to prove A satisfies (1.5), replacing $(x_1, x_2, ..., x_n)$ by $(n^l x_1, n^l x_2, ..., n^l x_n)$ and divide by n^l in (3.2), we arrive

$$\frac{1}{n^l}\left\|Df\left(n^l x_1, n^l x_2, ..., n^l x_n\right)\right\| \le \frac{1}{n^l}\psi\left(n^l x_1, n^l x_2, ..., n^l x_n\right)$$

for all $x_1, x_2, ..., x_n \in X$. Letting $l \to \infty$ in the above inequality, we see that

$$A\left(\sum_{k=1}^{n} kx_k\right) + \sum_{l=2}^{n} A\left(\sum_{k=1,k\neq 1}^{n} kx_k - lx_l\right) + A\left(x_1 - \sum_{k=2}^{n} kx_k\right)$$
$$= (n+1)A(x_1) + (n-3)\sum_{k=2}^{n} kA(x_k)$$

Hence A satisfies (1.5) for all $x_1, x_2, ..., x_n \in X$ with $n > 3$. To prove A is unique, let $R(x)$ be another additive mapping satisfying (1.5) and (3.3). Then

$$\begin{aligned}\|A(x) - R(x)\| &\le \frac{1}{n^l}\left\{\left\|A(n^l x) - f(n^l x)\right\| + \left\|f(n^l x) - R(n^l x)\right\|\right\} \\ &\le \frac{2}{n(n-3)}\sum_{k=0}^{\infty}\frac{\Psi\left(n^{k+l}x\right)}{n^{k+l}} \\ &\to 0 \ as \ l \to \infty\end{aligned}$$

for all $x \in X$. Hence A is unique.
For $j = -1$, we can prove the similar stability result. This completes the proof of the theorem.
The following corollary is a immediate consequence of Theorem 3.1 concerning the stability of (1.5).

Corollary 3.2. Let λ and s be nonnegative real numbers. If a function $f : X \to Y$ satisfies the inequality

$$\left\|D\ f(x_1, x_2, ..., x_n)\right\| \le \begin{cases} \lambda, & \\ \lambda\sum_{i=1}^{n}\|x_i\|^s, & s \neq 1; \\ \lambda\left\{\prod_{i=1}^{n}\|x_i\|^s + \prod_{i=1}^{n}\|x_i\|^{ns}\right\}, & s \neq \frac{1}{n}; \end{cases} \quad (3.12)$$

for all $x_1, x_2, ..., x_n \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \begin{cases} \dfrac{\lambda}{(n-3)|n-1|}, \\ \dfrac{\lambda\|x\|^s}{(n-3)|n-n^s|}, \\ \dfrac{\lambda\|x\|^{ns}}{(n-3)|n-n^{ns}|}, \end{cases} \quad (3.13)$$

for all $x \in X$.

3.2. Fixed Point Method

In this section, the authors present the generalized Ulam-Hyers stability of the functional equation (1.5) in Banach space using fixed point method.

Here after through out this section, let V be a vector space and B Banach space respectively. Define a mapping

$$D\,f(x_1,x_2,...,x_n) = f\left(\sum_{k=1}^{n} kx_k\right)+\sum_{l=2}^{n} f\left(\sum_{k=1,k\neq 1}^{n} kx_k - lx_l\right) + f\left(x_1-\sum_{k=2}^{n} kx_k\right)-(n+1)\,f(x_1)-(n-3)\sum_{k=2}^{n} kf(x_k)$$

for all $x_1,x_2,...,x_n \in X$. with $n>3$.

For to prove the stability result we define the following:
μ_i is a constant such that

$$\mu_i^k = \begin{cases} n, & \text{if} \quad i=0, \\ \frac{1}{n}, & \text{if} \quad i=1 \end{cases}$$

and Ω is the set such that

$$\Omega = \{ g \,|\, g : X \to Y, g(0)=0 \}$$

Theorem 3.3. Let $f:V\to B$ be a mapping for which there exist a function $\psi,\Psi,\Upsilon:V^n\to[0,\infty)$ with the condition

$$\lim_{k\to\infty}\frac{1}{\mu_i^k}\psi\left(\mu_i^k x_1,\mu_i^k x_2,...,\mu_i^k x_n,\right)=0 \tag{3.1}$$

Such that the functiona inequality with

$$\left\|D\ f(x_1,x_2,...,x_n)\right\| \le \psi(x_1,x_2,...,x_n) \tag{3.2}$$

for all $x_1,x_2,...,x_n \in V$. If there exists $L=L(i)<1$. Such that the function

$$x\to\Upsilon(x)=\frac{1}{(n-3)}\Psi\left(\frac{x}{n}\right) \tag{3.3}$$

has the property

$$\Upsilon(x)=L\mu_i\Upsilon\left(\frac{x}{\mu_i}\right) \tag{3.4}$$

for all $x\in V$. Then there exists a unique additive mapping $A:V\to B$ satisfying the functional equation (1.5) and

$$\|f(x)-A(x)\| \le \frac{L^{1-i}}{1-L}\Upsilon(x) \tag{3.5}$$

for all $x\in V$.

Proof. Consider the set $\Omega=\{p\,/\,p:\mathrm{X}\to Y,\ p(0)=0\}$ and introduce the generalized metric on Ω,

$$d(p,q)=d(p,q)=\inf\left\{K\in(0,\infty):\|p(x)-q(x)\|\le K\Upsilon(x), x\in V\right\}$$

It is easy to see that (Ω,d) is complete.

Define $T:\Omega\to\Omega$ by

$$Tp(x)=\frac{1}{\mu_i}\,p(\mu_i x),$$

For all $x\in V$. Now, $p,q\in\Omega$, we have

$$\begin{aligned}
d(p,q)\le K&\\
&\Rightarrow \|p(x)-q(x)\|\le K\Upsilon(x), x\in V\\
&\Rightarrow \left\|\frac{1}{\mu_i}p(\mu_i x)-\frac{1}{\mu_i}q(\mu_i x)\right\|\le \frac{1}{\mu_i}K\Upsilon(\mu_i x), x\in V,\\
&\Rightarrow \left\|\frac{1}{\mu_i}p(\mu_i x)-\frac{1}{\mu_i}q(\mu_i x)\right\|\le LK\Upsilon(x), x\in V,\\
&\Rightarrow \|Tp(x)-Tq(x)\|\le LK\Upsilon(x), x\in V,\\
&\Rightarrow d(p,q)\le LK.
\end{aligned}$$

This implies $d(Tp,Tq)\le Ld(p,q)$, for all $p,q\in\Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L. From (3.6), we arrive

$$\left\|\frac{f(nx)}{n}-f(x)\right\|\le\frac{1}{n(n-3)}\psi(\underbrace{0,\dots,0}_{n-1\ times},x) \tag{3.6}$$

for all $x\in V$. Using (3.4) for the case $i=0$ it reduces to

$$\|f(nx)-f(x)\|\le\frac{1}{n}\Upsilon(x)$$

for all $x\in V$.

$$\text{i.e.,}\qquad d(Tf,f)\le\frac{1}{n}=L=L^{1-0}=L^{1-i}<\infty$$

Again replacing $x=\frac{x}{n}$ in (3.6), we get

$$\left\|f(x)-nf\left(\frac{x}{n}\right)\right\|\le\frac{1}{n(n-3)}\psi\left(\underbrace{0,\dots,0}_{n-1\ times},\frac{x}{n}\right) \tag{3.7}$$

for all $x\in V$. Using (3.4) for the case $i=1$ it reduces to

$$\left\|f(x)-nf\left(\frac{x}{n}\right)\right\|\le\Upsilon(x)$$

for all $x\in V$.

$$\text{i.e.,}\qquad d(f,Tf)\le 1=L^0=L^{1-1}=L^{1-i}<\infty$$

in both cases, we arrive

$$d(f,Tf)\le L^{1-i}$$

Therefore (A1) holds.
By (A2), it follows that there exists a fixed point A of T in Ω such that

$$A(x)=\lim_{k\to\infty}\frac{1}{\mu_i^k}f_a\left(\mu_i^k x\right) \tag{3.8}$$

For all $x\in V$.

To prove $A:V \to B$ is additive. Replacing $(x_1, x_2, ..., x_n)$ by $\left(\mu_i^k x_1, \mu_i^k x_2, ..., \mu_i^k x_n,\right)$ in (3.2) and dividing by μ_i^k, it follows from (3.1) that

$$\left\|A\left(x_1, x_2, ..., x_n\right)\right\| = \lim_{k\to\infty} \frac{\left\|D\ f\left(\mu_i^k x_1, \mu_i^k x_2, ..., \mu_i^k x_n\right)\right\|}{\mu_i^k}$$
$$\leq \lim_{k\to\infty} \frac{\left\|\psi\left(\mu_i^k x_1, \mu_i^k x_2, ..., \mu_i^k x_n\right)\right\|}{\mu_i^k} = 0$$

For all $x_1, x_2, ..., x_n \in V$. i.e., A satisfies the functional equation (1.5)

By (A3), A is the unique fixed point of T in the set $y = \{A \in \Omega : d\left(f, A\right) < \infty\}, A$ is the unique function such that

$$\| f(x) - A(x)\| \leq K\Upsilon(x)$$

for all $x \in V$ and $K > 0$. Finally by (A4), we obtain

$$d(f, A) \leq \frac{1}{1-L} d(f, T, F)$$

this implies

$$d(f, A) \leq \frac{L^{1-i}}{1-L}$$

Which yields

$$\| f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Upsilon(x)$$

for all $x \in V$. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem (3.3) concerning the stability of (1.5)

Corollary 3.4. Let $f : V \to B$ be a mapping and there exists real numbers λ and s such that

$$\left\|D\ f\left(x_1, x_2, ..., x_n\right)\right\| \leq \begin{cases} \lambda, & \\ \lambda\left\{\sum_{i=1}^{n}\|x_i\|^s\right\}, & s \neq 1; \\ \lambda\left\{\prod_{i=1}^{n}\|x_i\|^s + \sum_{i=1}^{n}\|x_i\|^{ns}\right\}, & s \neq \frac{1}{n}; \end{cases} \tag{3.9}$$

for all $x_1, x_2, ..., x_n \in V$. Then there exists a unique additive function $A : V \to B$ such that

$$\left\| f(x) - A(x) \right\| \leq \begin{cases} \dfrac{\lambda}{(n-3)\left|n-1\right|}, \\ \dfrac{\lambda \|x\|^s}{(n-3)\left|n-n^s\right|}, \\ \dfrac{\lambda \|x\|^{ns}}{(n-3)\left|n-n^{ns}\right|}, \end{cases} \tag{3.10}$$

for all $x \in V$.

Proof: Setting

$$\psi\left(x_{1,} x_2, ..., x_n\right) = \begin{cases} \lambda, \\ \lambda\left\{\sum_{i=1}^{n} \|x_i\|^s\right\}, \\ \lambda\left\{\prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns}\right\}, \end{cases}$$

for all $x_1, x_2, ..., x_n \in V$. Now,

$$\psi\left(\mu_i^k x_{1,} \mu_i^k x_2, ..., \mu_i^k x_n\right) = \begin{cases} \dfrac{\lambda}{\mu_i^k}, \\ \dfrac{\lambda}{\mu_i^k}\left\{\sum_{i=1}^{n} \left\|\mu_i^k x_i\right\|^s\right\}, \\ \dfrac{\lambda}{\mu_i^k}\left\{\prod_{i=1}^{n} \left\|\mu_i^k x_i\right\|^s + \sum_{i=1}^{n} \left\|\mu_i^k x_i\right\|^{ns}\right\}, \end{cases}$$

$$= \begin{cases} \lambda \mu_i^{-k}, \\ \lambda \mu_i^{k(s-1)} \left\{\sum_{i=1}^{n} \|x_i\|^s\right\}, \\ \lambda \mu_i^{k(ns-1)} \left\{\prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns}\right\}, \end{cases}$$

$$= \begin{cases} \to 0 \ as \ n \to \infty, \\ \to 0 \ as \ n \to \infty, \\ \to 0 \ as \ n \to \infty. \end{cases}$$

Thus, (3.1) is holds.

But, we have $\Upsilon(x) = \dfrac{1}{(n-3)} \Psi\left(\dfrac{x}{n}\right) = \dfrac{1}{(n-3)} \phi\left(\underbrace{0, ..., 0,}_{n-1\, times} \dfrac{x}{n}\right)$

Hence

$$\Upsilon(x)=\frac{1}{(n-3)}\phi\left(\underbrace{0,...,0,}_{n-1\ times}\frac{x}{n}\right)=\begin{cases}\dfrac{\lambda}{(n-3)},\\ \dfrac{\lambda}{(n-3)n^{s}}\|x\|^{s},\\ \dfrac{\lambda}{(n-3)n^{ns}}\|x\|^{ns}.\end{cases}$$

Now,

$$\frac{1}{\mu_i}\Upsilon(\mu_i x)=\begin{cases}\dfrac{\lambda}{\mu_i(n-3)},\\ \dfrac{\lambda}{\mu_i}\dfrac{1}{(n-3)n^{s}}\|\mu_i x\|^{s},\\ \dfrac{\lambda}{\mu_i}\dfrac{1}{(n-3)n^{ns}}\|\mu_i x\|^{ns}\end{cases}=\begin{cases}\mu_i^{-1}\gamma(x),\\ \mu_i^{s-1}\gamma(x),\\ \mu_i^{ns-1}\gamma(x).\end{cases}$$

Now from (3.5), we prove the following cases for condition (i).

Case : 1 $L=n^{-1}$ if $i=0$

$$\|f(x)-A(x)\|\le\frac{\lambda}{n-3}\frac{\left(n^{-1}\right)^{1-0}}{1-n^{(-1)}}=\frac{\lambda}{(n-1)(n-3)}$$

Case : 2 $L=n^{1}$ if $i=1$

$$\|f(x)-A(x)\|\le\frac{\lambda}{n-3}\left(\frac{\left(n^{1}\right)^{1-1}}{1-n^{1}}\right)=\frac{-\lambda}{(1-n)(n-3)}$$

Case : 3 $L=n^{s-1}$ for $s<1$ if $i=0$

$$\|f(x)-A(x)\|\le\frac{\lambda}{(n-3)n^{s}}\frac{\left(n^{s-1}\right)^{1-0}}{1-n^{(s-1)}}\|x\|^{s}=\frac{\lambda}{(n-3)(n-n^{s})}\|x\|^{s}.$$

Case : 4 $L=\dfrac{1}{n^{s-1}}$ for $s>1$ if $i=1$

$$\|f(x)-A(x)\|\le\frac{\lambda}{n^{s}}\left(\frac{\frac{1}{n^{s-1}}}{1-\frac{1}{n^{s-1}}}\right)\|x\|^{s}=\frac{\lambda}{(n-3)(n^{s}-n)}\|x\|^{s}.$$

Case : 5 $L=n^{ns-1}$ for $s<\dfrac{1}{n}$ if $i=0$

$$\|f(x)-A(x)\|\le\frac{\lambda}{(n-3)n^{ns}}\left(\frac{\left(n^{(ns-1)}\right)^{1-0}}{1-n^{(ns-1)}}\right)\|x\|^{ns}=\frac{\lambda}{(n-3)(n-n^{ns})}\|x\|^{ns}.$$

Case : 6 $L=\dfrac{1}{n^{ns-1}}$ for $s>\dfrac{1}{n}$ if $i=1$

$$\| f(x)-A(x)\| \le \frac{\lambda}{n^{ns}}\left(\frac{\left(\frac{1}{n^{ns-1}}\right)^{1-1}}{1-\frac{1}{n^{ns-1}}}\right)\|x\|^{ns} = \frac{\lambda}{(n-3)(n^{ns}-n)}\|x\|^{ns}.$$

Hence the proof is complete.

4. Basic results in Banach algebra

Here after, through out this paper, let us consider B_1 and B_2 to be a normed Algebra and a Banach Algebra, respectively.

Definition 4.1. A C-linear mapping $A: X \to X$ is called **Additive Derivation** on X if A satisfies.

$$A(xy) = A(x)y + xA(y) \tag{4.1}$$

For all $x, y \in X$.

Definition 4.2. A C-linear mapping $A: X \to X$ is called **Generalized Additive Derivation** on X if A satisfies.

$$A(x_1x_2...x_n) = A(x_1)(x_2...x_n) + ... + (x_1x_2...x_{n-1})A(x_n) \tag{4.2}$$

For all $x_1x_2...x_n \in X$.

5. Stability results: a direct method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (4.1)

Theorem 5.1. Let $j \pm 1$. Let $\psi : \mathrm{X}^n \to [0,\infty)$ be a function such that

$$\lim_{l\to\infty}\frac{\left\|\psi\left(n^{lj}x_1, \mathrm{n}^{lj}\, x_2, ..., \mathrm{n}^{lj}\, x_n\right)\right\|}{n^{lj}} = 0, \quad \lim_{l\to\infty}\frac{\left\|\psi\left(n^{lj}x_1, \mathrm{n}^{lj}\, x_2, ..., \mathrm{n}^{lj}\, x_n\right)\right\|}{n^{nlj}} = 0, \tag{5.1}$$

For all $x_1x_2...x_n \in X$. let $\mathrm{f} : X \to Y$ be a function satisfies the inequality

$$\left\|D \;\; f\left(x_1, x_2, ..., x_n\right)\right\| \le \psi\left(x_1, x_2, ..., x_n\right) \tag{5.2}$$

And

$$\begin{aligned} &\left\| f\left(x_1, ..., x_{n-1}, x_n\right) - f\left(x_1\right)\left(x_2, ..., x_{n-1}, x_n\right) - ... \right. \\ &\qquad \left. - \left(x_1, x_1, ..., x_{n-1}\right) f\left(x_n\right)\right\| \le \psi\left(x_2, ..., x_{n-1}, x_n\right) \end{aligned} \tag{5.3}$$

For all $x_1x_2...x_n \in X$. Then there exists a unique function $\mathrm{A} : X \to Y$ such that

$$\left\| f(x) - A(x)\right\| \le \frac{1}{n(n-3)}\sum_{l=\frac{1-j}{2}}^{\infty}\frac{\Psi\left(n^{lj}x\right)}{n^{lj}} \tag{5.4}$$

where $\Psi\left(n^{lj}x\right) = \psi\left(\underbrace{(0,...,0)}_{n-1 times}, n^{lj}x\right)$ for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x)\lim_{l\to\infty}\frac{f\left(n^{lj}x\right)}{n^{nlj}} \tag{5.5}$$

for all $x \in X$.

Corollary 5.2. Let λ and s be a nonnegative real numbers. If a function $f: V \to B$ satisfies the inequality

$$\left\| D\ f\left(x_1, x_2, ..., x_n\right)\right\| \leq \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \|x_i\|^s, & s \neq 1; \\ \lambda \left\{ \prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns} \right\}, & s \neq \frac{1}{n}; \end{cases} \tag{5.6}$$

$$\left\| f\left(x_1, ..., x_{n-1}, x_n\right) - f\left(x_1\right)\left(x_2, ..., x_{n-1}, x_n\right) - ... - \left(x_1, ..., x_{n-1}\right) f\left(x_n\right)\right\|$$

$$\leq \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \|x_i\|^s, & s \neq 1; \\ \lambda \left\{ \prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns} \right\}, & s \neq \frac{1}{n}; \end{cases} \tag{5.7}$$

for all $x_1, x_2, ..., x_n \in V$. Then there exists a unique additive function $A: \mathrm{X} \to Y$ such that

$$\left\| f(x) - A(x)\right\| \leq \begin{cases} \dfrac{\lambda}{(n-3)\left|n-1\right|}, \\ \dfrac{\lambda \|x\|^s}{(n-3)\left|n-n^s\right|}, \\ \dfrac{\lambda \|x\|^{ns}}{(n-3)\left|n-n^{ns}\right|}, \end{cases} \tag{5.8}$$

for all $x \in X$.

6. Stability results: a fixed point method

Theorem 6.1. Let $f: V \to B$ be a mapping for which there exist a function $\psi, \Psi, \gamma: V^n \to [0, \infty)$ with the condition

$$\lim_{k\to\infty} \frac{1}{\mu_i^k} \psi\left(\mu_i^k x_1, \mu_i^k x_2, ..., \mu_i^k x_n\right) = 0,\ \lim_{k\to\infty} \frac{1}{\mu_i^k} \psi\left(\mu_i^k x_1, \mu_i^k x_2, ..., \mu_i^k x_n\right) = 0 \tag{6.1}$$

Such that the functional inequality with

$$\left\| D\ f\left(x_1, x_2, ..., x_n\right)\right\| \leq \psi\left(x_1, x_2, ..., x_n\right) \tag{6.2}$$

and

$$\left\| f(x_1,\ldots,x_{n-1},x_n) - f(x_1)(x_2,\ldots,x_{n-1},x_n) - \ldots \right.$$
$$\left. -(x_1,x_1,\ldots,x_{n-1})\, f(x_n) \right\| \le \phi(x_1,\ldots,x_{n-1},x_n)$$

for all $x_1, x_2, \ldots, x_n \in V$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \frac{1}{n-3}\Psi\left(\frac{x}{n}\right),$$

Has the property $\gamma(x) = L\mu_i\gamma\left(\frac{x}{\mu_i}\right)$,

For all $x \in X$. Then there exists a unique additive mapping $A : V \to B$ satisfying the functional equation (1.5) and

$$\| f(x) - A(x) \| \le \frac{L^{1-i}}{1-L}\gamma(x)| \text{ for all } x \in X.$$

Corollary 6.2. Let λ and s be a nonnegative real numbers. If a function $f : X \to Y$ satisfies the inequality

$$\left\| D\, f(x_1, x_2, \ldots, x_n) \right\| \le \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \|x_i\|^s, & s \ne 1; \\ \lambda \left\{ \prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns} \right\}, & s \ne \frac{1}{n}; \end{cases}$$

$$\left\| f(x_1,\ldots,x_{n-1},x_n) - f(x_1)(x_2,\ldots,x_{n-1},x_n) - \ldots - (x_1,\ldots,x_{n-1})\, f(x_n) \right\|$$

$$\le \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \|x_i\|^s, & s \ne 1; \\ \lambda \left\{ \prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns} \right\}, & s \ne \frac{1}{n}; \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in V$. Then there exists a unique additive function $A : X \to Y$ such that

$$\| f(x) - A(x) \| \le \begin{cases} \dfrac{\lambda}{(n-3)|n-1|}, \\ \dfrac{\lambda \|x\|^s}{(n-3)|n-n^s|}, \\ \dfrac{\lambda \|x\|^{ns}}{(n-3)|n-n^{ns}|}, \end{cases} \qquad \text{for all } x \in X.$$

REFERENCES

1. J. Aczel and J. Dhombres, Functional Equations in Several Variables, Cambridge Univ, Press, 1989.

2. T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, 2 (1950) 64-66.
3. M. Arunkumar, Solution and Stability of Arun-Additive functional equations, *International Journal Mathematical Sciences and Engineering Applications*, 4(3) (2010) 33-46.
4. M. Arunkumar and T.Namachivayam, stability of $n-$ homomorphisms, $n-$ derivations of a $n-$ dimensional additive functional equation in c* ternary algebras , *International Journal of Advanced Computer Research*, 3(3) (2013).
5. M. Arunkumar Stability of $n-$ Dimensional Additive Functional Equation in generalized $n-$ normed space, Demonstratio Mathematica, 49(3) 2016.
6. S.Czerwik, Functional Equations and Inequalities in Several Variables, World Scientiffic, River Edge, NJ,2002.
7. G.Z.Eskandani, P.Gavruta, J.M.Rassias, R.Zarghami, Generalized Hyers-Ulam Stability for a General Mixed Functional Equation in Quasi$-\beta-$ normed Spaces, *Mediterr. J. Math.*, 8 (2011) 331-348.
8. G.Z.Eskandani, P.Gavruta, On the stability problem in quasi-Banach spaces, *Nonlinear Funct, anal. Appl.* (to appear).
9. P.Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J.Math. Anal. Appl.*, 184 (1994) 431-436.
10. P.Gavruta, An answer to a question of J.M.Rassias concerning the stability of cauchy functional equation, *Advances in Equations and Inequalities*, (1999) 67-71.
11. P.Gavruta, On a problem of G.Isac and Th.M.Rassias concerning the stability of mappings, *J.Math. Anal. Appl.*, 261 (2001) 543-553.
12. D.H.Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.*, 27 (1941) 222-224.
13. D.H.Hyers, G.Isac, Th.M.Rassias, Stability of functional equations in several variables, Birkhauser, Basel,1998.
14. S.M.Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press,Palm Harbor, 2001.
15. D.O. Lee, Hyers-Ulam stability of an addtiive type functional equation, *J. Appl. Math. and Computing*, 13 (2003) 471-477.
16. B. Margolis, J. B. Diaz, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.*, 126(74) (1968) 305-309.
17. J.M. Rassias, On approximately of approximately linear mappings by linear mappings, *J. Funct. Anal.* 46 (1982) 126-130.
18. J.M. Rassias, On approximately of approximately linear mappings by linear mappings, *Bull. Sc. Math*, 108 (1984) 445-446.
19. J.M. Rassias, H.M.Kim, Generalized Hyerslam stability for general additive functional equations in quasi$-\beta-$ normed spaces, *J.Math. Anal. Appl.*, 356 (2009) 302-309.
20. J.M. Rassias, K.W.Jun, H.M.Kim, Approximate (m,n) *J.Math. Anal. Appl.*, 356 (2009) 302-309.
21. P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J.Math. Anal. Appl.*, 184 (1994) 431-436.

22. H. Haruki and Th.M. Rassias, New characterizations of some mean values, *Journal of Mathematical Analysis and Applications*, 202 (1996) 333 - 348.
23. K.W. Jun, H.M. Kim, On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive type functional equation, *Bull. Korean Math. Soc.,* 42(1) (2005) 133-148.
24. K.W. Jun, H.M. Kim, On the stability of an n-dimensional quadratic and additive type functional equation, *Math. Ineq. Appl.,* 9(1) (2006) 153-165.
25. D.H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.*, 27 (1941) 222-224.
26. D.H. Hyers, G. Isac, Th.M. Rassias, Stability of functional equations in several variables, Birkhauser, Basel, 1998.
27. K.W. Jun and H.M. Kim, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, *J. Math. Anal. Appl.*, 274 (2002) 867-878.
28. S.M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.*, 222 (1998) 126-137.
29. S.M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.
30. Pl. Kannappan, Quadratic functional equation inner product spaces, *Results Math.,* 27(3-4) (1995) 368-372.
31. M. Moslehian and Th.M. Rassias, Stability of functional equations in non-Archimedian spaces, *Applicable Analysis and Discrete Mathematics*, 1(2) (2007) 1-10.
32. A. Najati, M.B. Moghimi, On the stability of a quadratic and additive functional equation, *J. Math. Anal. Appl*., 337 (2008) 399-415.
33. J.M. Rassias and M.J. Rassias, On the Ulam stability of Jensen and Jensen type mappings on the restricted domains, *J. Math. Anal. Appl*., 281 (2003) 516-524.
34. Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, 72 (1978) 297-300.
35. Th.M. Rassias and P. Semrl, On the behavior of mappings which do not satisfy Hyers- Ulam stability, *Proc. Amer. Math. Soc.*, 114 (1992) 989-993.
36. Th.M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta. Appl. Math.,* 62 (2000) 23-130.
37. Th.M. Rassias, Functional Equations, Inequalities and Applications, Kluwer Acedamic Publishers, Dordrecht, Bostan London, 2003.
38. K. Ravi, M. Arunkumar and J.M. Rassias, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *International Journal of Mathematical Sciences*, 3 (2008) 36-47.
39. J. Tipyan , C. Srisawat, P. Udomkavanich and P. Nakmahachalasint, The generalized stability of an $n-$ dimensional jensen type functional equation, *Thai Journal Mathematics*, 12 (2014) 265-274.
40. G. Toader and Th.M. Rassias, New properties of some mean values, *Journal of Mathematical Analysis and Applications*, 232 (1999) 376-383.
41. S.M. Ulam, Problems in Modern Mathematics, Science Editions, Wiley, NewYork, 1964.